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A NOTE ON INVARIANT MEASURES

Piotr Niemiec

Abstract. The aim of the paper is to show that if $\mathcal{F}$ is a family of continuous transformations of a nonempty compact Hausdorff space Ω, then there is no $\mathcal{F}$-invariant probabilistic regular Borel measures on Ω iff there are $\varphi_1, \ldots, \varphi_p \in \mathcal{F}$ (for some $p \geq 2$) and a continuous function $u\colon \Omega^p \to \mathbb{R}$ such that $\sum_{\sigma \in S_p} u(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) = 0$ and $\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (u \circ \Phi^k)(x_1, \ldots, x_p) \geq 1$ for each $x_1, \ldots, x_p \in \Omega$, where $\Phi\colon \Omega^p \ni (x_1, \ldots, x_p) \mapsto (\varphi_1(x_1), \ldots, \varphi_p(x_p)) \in \Omega^p$ and Φ^k is the k-th iterate of Φ. A modified version of this result in case the family $\mathcal{F}$ generates an equicontinuous semigroup is proved.

Keywords: invariant measures, equicontinuous semigroups, compact spaces.

Mathematics Subject Classification: 28C10, 54H15.

1. INTRODUCTION

Invariant measures are present in many parts of mathematics, including harmonic analysis, ergodic theory and topological dynamics. Ergodic theory deals with a single measurable transformation which preserves a fixed measure and it focuses on properties of the measure-theoretic discrete dynamical system obtained in this way. The reader interested in this subject is referred to standard textbooks such as [6] or [4]. Another approach to the aspect of invariant measures, treated in the recent paper, is, in a sense, related to (common) fixed point theory and it concentrates on the problem of the existence of a measure preserved by all transformations of a fixed family. The most classical result in this topic is the Haar measure theorem which states that on every locally compact topological group there is a unique, up to a constant factor, positive regular Borel measure invariant under the left shifts of the group (see e.g. [5, 13] or [12]; for a much more general result see [9, 10, 19]). This meaningful result plays an important role in abstract harmonic analysis and group representation theory and gave foundations to this new branch of mathematics which is still widely investigated. This includes invariant measures for both the groups as well as the semigroups of continuous or measurable transformations acting on metric spaces, compact spaces or totally arbitrary topological spaces. There is a huge range of literature concerning this subject and we mention here only a part: [1,3,7,8,11,14–17,20] or a survey article [21] and references therein.

The recent paper deals with an arbitrary semigroup $\mathcal{F}$ of continuous transformations of a compact Hausdorff space. Our aim is to give an equivalent condition for the existence of a Borel regular probabilistic measure invariant under each member of $\mathcal{F}$. The condition reduces the problem of the existence of the measure to the more friendly issue of the nonexistence of a continuous function satisfying certain explicitly stated conditions.

2. PRELIMINARIES

In this paper Ω is a nonempty compact Hausdorff space and S_n stands for the group of all permutations of $\{1,\ldots,n\}$. For $\sigma \in S_n$, let $\boldsymbol{\sigma}\colon \Omega^n \to \Omega^n$ be a function defined by

$$\boldsymbol{\sigma}(x_1,\ldots,x_n) = (x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$

Whenever Φ is a transformation of some set, Φ^k denotes the k-th iterate of Φ. The algebra of all the continuous real-valued functions on Ω is denoted by $\mathcal{C}(\Omega)$, $\mathfrak{B}(\Omega)$ denotes the σ-algebra of all the Borel subsets of Ω and $\mathcal{C}(\Omega,\Omega)$ stands for the family of continuous transformations of Ω; $\mathcal{M}(\Omega)$ is the vector space of all the (signed) real-valued regular Borel measures on Ω and $\operatorname{Prob}(\Omega)$ is its subset of probabilistic measures. The space $\mathcal{M}(\Omega)$ is equipped with the standard weak-* topology induced by linear functionals of the form $\mathcal{M}(\Omega) \ni \mu \mapsto \int_\Omega f\,\mathrm{d}\mu \in \mathbb{R}$, where $f \in \mathcal{C}(\Omega)$. For a continuous transformation $\varphi\colon \Omega \to \Omega$, let $\hat{\varphi}\colon \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$ be a transformation given by the formula $\hat{\varphi}(\mu)(A) = \mu(\varphi^{-1}(A))$ $(\mu \in \mathcal{M}(\Omega),\ A \in \mathfrak{B}(\Omega))$. (Thus $\hat{\varphi}(\mu)$ is the transport of the measure μ under the transformation φ.) The set $\operatorname{Prob}(\Omega)$ is compact and the transformation $\hat{\varphi}$ is continuous in the weak-* topology (for continuous φ). (For the proof of the second statement see e.g. [14].) For measures $\mu_1,\ldots,\mu_n \in \mathcal{M}(\Omega)$ and transformations $\varphi_1,\ldots,\varphi_n\colon \Omega \to \Omega$, we denote by $\mu_1 \otimes \ldots \otimes \mu_n$ $(\in \mathcal{M}(\Omega^n))$ and $\varphi_1 \times \ldots \times \varphi_n$ the product of $\mu_1,\ldots,\mu_n$ and of $\varphi_1,\ldots,\varphi_n$, respectively. Thus $\varphi_1 \times \ldots \times \varphi_n\colon \Omega^n \to \Omega^n$ and $(\varphi_1 \times \ldots \times \varphi_n)(x_1,\ldots,x_n) = (\varphi_1(x_1),\ldots,\varphi_n(x_n))$.

If $\mathcal{F}$ is a family of continuous transformations of Ω, we say that $\mathcal{F}$ is *equicontinuous* if and only if the closure of $\mathcal{F}$ in the compact-open topology of $\mathcal{C}(\Omega,\Omega)$ is compact (in that topology). Equivalently, $\mathcal{F}$ is equicontinuous if for any points $x, y \in \Omega$ and every open neighbourhood $V \subset \Omega$ of the point y there exist open subsets U and W of Ω such that $x \in U$, $y \in W$ and for each $\varphi \in \mathcal{F}$, $\varphi(U) \subset V$ provided $\varphi(x) \in W$. If $\mathcal{F}$ is equicontinuous and $h \in \mathcal{C}(\Omega)$, then the closure of the set $h \circ \mathcal{F} = \{h \circ \varphi\colon\ \varphi \in \mathcal{F}\}$ in the norm topology of $\mathcal{C}(\Omega)$ is compact. For proofs, details and more information on the compact-open topology the reader is referred to [2].

For a family $\mathcal{F} \subset \mathcal{C}(\Omega,\Omega)$, let $\operatorname{Inv}(\mathcal{F}) \subset \operatorname{Prob}(\Omega)$ be the set of all the $\mathcal{F}$*-invariant* measures, i.e. a measure $\mu \in \operatorname{Prob}(\Omega)$ belongs to $\operatorname{Inv}(\mathcal{F})$ if and only if $\hat{\varphi}(\mu) = \mu$ for each $\varphi \in \mathcal{F}$. Note, for example, that $\operatorname{Inv}(\varnothing) = \operatorname{Prob}(\Omega)$. If $\mathcal{F} = \{\varphi_1,\ldots,\varphi_n\}$, we shall write $\operatorname{Inv}(\varphi_1,\ldots,\varphi_n)$ instead of $\operatorname{Inv}(\{\varphi_1,\ldots,\varphi_n\})$. The set $\operatorname{Inv}(\mathcal{F})$ is always compact (in the weak-* topology) and the following result has entered folklore in ergodic theory:

Theorem 2.1. *If* $\varphi \in \mathcal{C}(\Omega, \Omega)$ *and* $\mu \in \mathrm{Prob}(\Omega)$*, then every limit point of the sequence* $(\frac{1}{n}\sum_{k=0}^{n-1}(\hat{\varphi})^k(\mu))_{n=1}^{\infty}$ *belongs to* $\mathrm{Inv}(\varphi)$*.*

A variation of the above result is the crucial key in the proof of Markov's-Kakutani's fixed point theorem – see e.g. [18]. Theorem 2.1 is in fact a special case of this variation.

For simplicity, let $\mathcal{A}_p(\Omega)$ (where $p \geq 2$) denote the family of all continuous functions $u\colon \Omega^p \to \mathbb{R}$ such that

$$\sum_{\sigma \in S_p} u \circ \boldsymbol{\sigma} \equiv 0. \tag{2.1}$$

The following result is well known and easy to prove.

Lemma 2.2. *For a family* $\mathcal{F} \subset \mathcal{C}(\Omega, \Omega)$*, the following conditions are equivalent:*

(i) *the set* $\mathrm{Inv}(\mathcal{F})$ *is empty,*
(ii) *there are a natural number* $N \geq 2$ *and* $\varphi_1, \ldots, \varphi_N \in \mathcal{F}$ *such that* $\mathrm{Inv}(\varphi_1, \ldots, \varphi_N) = \varnothing$*.*

3. MAIN RESULTS

Lemma 2.2 says that we may restrict our investigations to finite sets of transformations, which shall be done in the sequel. For simplicity, we fix the situation.

Let $\varphi_1, \ldots, \varphi_p$ ($p \geq 2$) be members of $\mathcal{C}(\Omega, \Omega)$ and $\Phi = \varphi_1 \times \ldots \times \varphi_p : \Omega^p \to \Omega^p$ (note that $\mathrm{Inv}(\Phi) \subset \mathcal{M}(\Omega^p)$). Additionally, let $\mathrm{Inv}(\boldsymbol{S}_p)$ be the collection of all signed real-valued regular Borel measures on Ω^p, invariant under all permutations of variables.

The following simple result may be interesting in itself.

Lemma 3.1. $\mathrm{Inv}(\varphi_1, \ldots, \varphi_p)$ *is nonempty iff* $\mathrm{Inv}(\Phi) \cap \mathrm{Inv}(\boldsymbol{S}_p) \neq \varnothing$*.*

Proof. It is easy to check that if $\mu \in \mathrm{Inv}(\varphi_1, \ldots, \varphi_p)$, then $\lambda \in \mathrm{Inv}(\Phi) \cap \mathrm{Inv}(\boldsymbol{S}_p)$ for $\lambda = \mu \otimes \ldots \otimes \mu \in \mathcal{M}(\Omega^p)$. Conversely, if λ belongs to both the sets $\mathrm{Inv}(\Phi)$ and $\mathrm{Inv}(\boldsymbol{S}_p)$, then it is easily verified that a measure $\mu \in \mathcal{M}(\Omega)$ defined by $\mu(A) = \lambda(A \times \Omega^{p-1})$ ($A \in \mathfrak{B}(\Omega)$) is φ_j-invariant for $j = 1, \ldots, p$. □

So, if we want to know when $\mathrm{Inv}(\varphi_1, \ldots, \varphi_p) = \varnothing$, it is enough to verify when the sets $\mathrm{Inv}(\Phi)$ and $\mathrm{Inv}(\boldsymbol{S}_p)$ are disjoint. Since the first of them is convex and compact and the latter is a closed (in the weak-* topology) linear subspace of $\mathcal{M}(\Omega^p)$, thus – by the separation theorem – they are disjoint if and only if there is $u \in \mathcal{C}(\Omega^p)$ such that $\int_{\Omega^p} u \, \mathrm{d}\mu = 0$ for $\mu \in \mathrm{Inv}(\boldsymbol{S}_p)$, but for some positive t we have

$$\int_{\Omega^p} u \, \mathrm{d}\lambda \geq t \tag{3.1}$$

for any $\lambda \in \mathrm{Inv}(\Phi)$.

The proof of the following fact is immediate.

Lemma 3.2. *Let $u \in \mathcal{C}(\Omega^p)$. Then $\int_{\Omega^p} u \,\mathrm{d}\mu = 0$ for each $\mu \in \mathrm{Inv}(\boldsymbol{S}_p)$ if and only if $u \in \mathcal{A}_p(\Omega)$.*

The property (3.1) of the function u which separates the sets $\mathrm{Inv}(\boldsymbol{S}_p)$ and $\mathrm{Inv}(\Phi)$ can be reformulated as follows:

Lemma 3.3. *For a function $u \in \mathcal{C}(\Omega^p)$ and a number $t \in \mathbb{R}$ the following conditions are equivalent:*

(i) *the inequality* (3.1) *holds for every $\lambda \in \mathrm{Inv}(\Phi)$,*
(ii) $\liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}(u \circ \Phi^k)(z) \geq t$ *for each $z \in \Omega^p$.*

Proof. The implication (ii) $\Longrightarrow$ (i) follows from Fatou's lemma. Indeed, take $m \in \mathbb{R}$ such that $u(z) \geq m$ for each $z \in \Omega^p$. Then for $\lambda \in \mathrm{Inv}(\Phi)$ we obtain:

$$\begin{aligned}\liminf_{n\to\infty} \int_{\Omega^p} \Big(\frac{1}{n}\sum_{k=0}^{n-1} u \circ \Phi^k - m\Big)\,\mathrm{d}\lambda &\geq \int_{\Omega^p} \Big(\liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} u \circ \Phi^k - m\Big)\,\mathrm{d}\lambda \geq \\ &\geq \int_{\Omega^p} (t-m)\,\mathrm{d}\lambda = t - m,\end{aligned} \tag{3.2}$$

but

$$\begin{aligned}\int_{\Omega^p} \Big(\frac{1}{n}\sum_{k=0}^{n-1} u \circ \Phi^k - m\Big)\,\mathrm{d}\lambda &= \frac{1}{n}\sum_{k=0}^{n-1}\int_{\Omega^p} u \circ \Phi^k \,\mathrm{d}\lambda - m = \frac{1}{n}\sum_{k=0}^{n-1}\int_{\Omega^p} u \,\mathrm{d}\hat{\Phi}^k(\lambda) - m = \\ &= \frac{1}{n}\sum_{k=0}^{n-1}\int_{\Omega^p} u\,\mathrm{d}\lambda - m = \int_{\Omega^p} u\,\mathrm{d}\lambda - m.\end{aligned}$$

For the converse implication, fix $z \in \Omega^p$ and take a subsequence $(s_{n_k})_k$ of the sequence $s_n = \frac{1}{n}\sum_{j=0}^{n-1} u(\Phi^j(z))$ such that

$$\lim_{k\to\infty} s_{n_k} = \liminf_{n\to\infty} s_n.$$

Let δ be the Dirac measure on Ω^p with the atom at z. Since $\mathrm{Prob}(\Omega^p)$ is compact, the sequence $\mu_k = \frac{1}{n_k}\sum_{j=0}^{n_k-1} \hat{\Phi}^j(\delta)$ has a limit point in $\mathrm{Prob}(\Omega^p)$, say λ. By Theorem 2.1, $\lambda \in \mathrm{Inv}(\Phi)$ and hence $\int_{\Omega^p} u\,\mathrm{d}\lambda \geq t$. Finally, since the function $\mathcal{M}(\Omega^p) \ni \nu \mapsto \int_{\Omega^p} u\,\mathrm{d}\nu \in \mathbb{R}$ is continuous, therefore the integral $\int_{\Omega^p} u\,\mathrm{d}\lambda$ is a limit point of the sequence $\int_{\Omega^p} u\,\mathrm{d}\mu_k$. But $\int_{\Omega^p} u\,\mathrm{d}\mu_k = s_{n_k} \to \liminf_{n\to\infty} s_n$ $(k \to +\infty)$, which finishes the proof. □

Now putting together the above facts, we obtain the main result of the paper.

Theorem 3.4. *The following conditions are equivalent:*

(i) $\mathrm{Inv}(\varphi_1, \dots, \varphi_p) = \varnothing$,
(ii) *there is* $u \in \mathcal{A}_p(\Omega)$ *such that for each* $z \in \Omega^p$,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (u \circ \Phi^k)(z) \geq 1. \tag{3.3}$$

Before we strengthen condition (ii) of the foregoing theorem in case the family $\{\varphi_1, \dots, \varphi_p\}$ generates an equicontinuous semigroup, we shall prove the following

Lemma 3.5. *Let* $v \in \mathcal{A}_p(\Omega)$. *The following conditions are equivalent:*

(i) *there are* $c > 0$ *and a number* $n_0 \geq 1$ *such that* $\sum_{k=0}^{n-1} c(v \circ \Phi^k)(z) > n$ *for each* $z \in \Omega^p$ *and* $n \geq n_0$,
(ii) *there is* $m \geq 0$ *such that the function* $\sum_{k=0}^{m} v \circ \Phi^k$ *has only positive values.*

Each of the above conditions implies that $\mathrm{Inv}(\varphi_1, \dots, \varphi_p) = \varnothing$.

Proof. Thanks to Theorem 3.4, it is enough to prove the equivalence of (i) and (ii).

To see that (ii) implies (i), put $\varepsilon = \inf_{z\in\Omega^p} \sum_{k=0}^{m} (v \circ \Phi^k)(z)$. By (ii) and the compactness of Ω, $\varepsilon > 0$. For simplicity, put $l = m + 1$ (≥ 1) and $c = \frac{2l}{\varepsilon} > 0$. Let $t = \inf\bigl\{\sum_{j=0}^{k} (v \circ \Phi^j)(z)\colon\ k \in \{0, \dots, l-1\},\ z \in \Omega^p\bigr\}$ (observe that $t \leqslant 0$, because v is either constantly equal 0 or is not nonnegative). Finally, take $n_0 \geq 1$ such that

$$\frac{t - \varepsilon}{n} > -\frac{1}{c} \tag{3.4}$$

for every $n \geq n_0$. Let n be an arbitrary natural number no less than n_0. Express n in the form $n = sl + r$, where $s \geq 1$ and $0 \leqslant r < l$. From the definition of ε it follows that $\sum_{j=0}^{l-1} (v \circ \Phi^j)(\Phi^{ql+r}(z)) \geq \varepsilon$ for every $z \in \Omega^p$ and $q = 0, \dots, s-1$. Furthermore, $\sum_{j=0}^{r-1} (v \circ \Phi^j)(z) \geq t$ (this is true also for $r = 0$, under the agreement that $\sum_{\varnothing} = 0$). Hence, by (3.4):

$$\begin{aligned} \sum_{j=0}^{n-1} c(v \circ \Phi^j)(z) &= c \sum_{j=0}^{r-1} (v \circ \Phi^j)(z) + c \sum_{q=0}^{s-1} \sum_{j=0}^{l-1} (v \circ \Phi^{ql+r+j})(z) \geq \\ &\geq c(t + s\varepsilon) = c\Bigl(t + \frac{n-r}{l}\varepsilon\Bigr) > c\Bigl(t + \frac{n-l}{l}\varepsilon\Bigr) = c\Bigl(\frac{\varepsilon}{l} n + t - \varepsilon\Bigr) > \\ &> c\Bigl(\frac{2}{c} n - \frac{n}{c}\Bigr) = n, \end{aligned}$$

which finishes the proof of the implication (ii) $\Longrightarrow$ (i). The converse implication is immediate. □

The announced strengthening of Theorem 3.4 has the following form:

Theorem 3.6. *If the family* $\{\varphi_1, \ldots, \varphi_p\}$ *generates an equicontinuous semigroup* (*with the action of composition*), *then the following conditions are equivalent:*

(i) $\mathrm{Inv}(\varphi_1, \ldots, \varphi_p) = \varnothing$,
(ii) *there are* $v \in \mathcal{A}_p(\Omega)$ *and a number* $n_0 \geq 1$ *such that* $\sum_{k=0}^{n-1}(v \circ \Phi^k)(z) > n$ *for each* $n \geq n_0$ *and* $z \in \Omega^p$.

Proof. It suffices to prove that (i) implies (ii). Suppose that the set $\mathrm{Inv}(\varphi_1, \ldots, \varphi_p)$ is empty and let $u \in \mathcal{A}_p(\Omega)$ be as in the statement of the condition (ii) of Theorem 3.4. Put $v = 2u \in \mathcal{A}_p(\Omega)$. We shall show that v is the function which we are looking for. Suppose, to the contrary, there exist an increasing sequence $(n_k)_k$ of natural numbers and a sequence $(z_k)_k$ of elements of Ω^p such that

$$\sum_{j=0}^{n_k-1}(v \circ \Phi^j)(z_k) \leqslant n_k \qquad (k \geq 1). \tag{3.5}$$

Let $v_k = \frac{1}{n_k}\sum_{j=0}^{n_k-1} v \circ \Phi^j$. Since the semigroup generated by $\mathcal{F}$ is equicontinuous, hence so is the semigroup $\{\Phi^j\colon\ j \geq 0\}$. This implies that the uniform closure of $\mathcal{V} = \{v \circ \Phi^j\colon\ j \geq 0\}$ is compact and therefore the closed convex hull of $\mathcal{V}$ in $\mathcal{C}(\Omega^p)$ is compact as well. So, replacing eventually the sequence $(v_k)_k$ by a suitable subsequence, we may assume that $(v_k)_k$ is uniformly convergent to some $v_0 \in \mathcal{C}(\Omega^p)$. Let $z_0 \in \Omega^p$ be a limit point of the sequence $(z_k)_k$. From the uniform convergence of $(v_k)_k$ to v_0 we infer that $v_0(z_0)$ is a limit point of the sequence $(v_k(z_k))_k$ and thus, by (3.5), $v_0(z_0) \leqslant 1$. But on the other hand, by (3.3):

$$v_0(z_0) = \lim_{k\to\infty} v_k(z_0) \geq \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}(v \circ \Phi^j)(z_0) = 2\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}(u \circ \Phi^j)(z_0) \geq 2,$$

which is a contradiction. □

Other conditions for the nonemptiness of $\mathrm{Inv}(\mathcal{F})$ in case the family $\mathcal{F}$ generates an equicontinuous semigroup can be found in [14].

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Piotr Niemiec
piotr.niemiec@uj.edu.pl

Jagiellonian University
Institute of Mathematics
ul. Łojasiewicza 6, 30-348 Kraków, Poland

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