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OSCILLATION THEOREMS CONCERNING NON-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Abstract. This paper concerns the oscillation of solutions of the differential eq.

$$\Big[r(t)\,\psi(x(t))\,f(\dot{x}(t))\Big]^{\cdot} + q(t)\,\varphi(g(x(t)), r(t)\,\psi(x(t))\,f(\dot{x}(t))) = 0,$$

where $u\varphi(u,v) > 0$ for all $u \neq 0$, $xg(x) > 0$, $xf(x) > 0$ for all $x \neq 0$, $\psi(x) > 0$ for all $x \in \mathbb{R}$, $r(t) > 0$ for $t \geq t_0 > 0$ and q is of arbitrary sign. Our results complement the results in [A.G. Kartsatos, *On oscillation of nonlinear equations of second order*, J. Math. Anal. Appl. 24 (1968), 665–668], and improve a number of existing oscillation criteria. Our main results are illustrated with examples.

Keywords: second order, nonlinear, differential equations, oscillation.

Mathematics Subject Classification: 34C10, 34C15.

1. INTRODUCTION

This paper is concerned with the oscillation of the solutions of the second-order nonlinear differential equation

$$\Big[r(t)\,\psi(x(t))\,f(\dot{x}(t))\Big]^{\cdot} + q(t)\,\varphi(g(x(t)), r(t)\,\psi(x(t))\,f(\dot{x}(t))) = 0, \tag{1.1}$$

where q and r are continuous functions on the interval $[t_0, \infty)$, $t_0 > 0$ and r is positive function, ψ and f are continuous functions on $\mathbb{R}$ with $\psi(x) > 0$ for all $x \in \mathbb{R}$ and $xf(x) > 0$ for all $x \neq 0$, g is continuously differentiable function on the real line $\mathbb{R}$ except possibly at 0 with $xg(x) > 0$ and $g'(x) > 0$ for all $x \neq 0$, and φ is defined and continuous on $\mathbb{R} \setminus \{0\} \times \mathbb{R}$ with $u\varphi(u,v) > 0$ for all $u \neq 0$ and $\varphi(\lambda u, \lambda v) = \lambda\varphi(u,v)$, where $\lambda \in (0, \infty)$.

Equation (1.1) is said to be superlinear if

$$0 < \int_{\pm\varepsilon}^{\pm\infty} \frac{du}{g(u)} < \infty \text{ for all } \varepsilon > 0,$$

and sublinear if

$$0 < \int_{0}^{\pm\varepsilon} \frac{du}{g(u)} < \infty \text{ for all } \varepsilon > 0,$$

and of mixed type if

$$0 < \int_{0}^{\pm\infty} \frac{du}{g(u)} < \infty.$$

We restrict our attention to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > T\} > 0$ for any $T > t_x$, where t_x depends on the particular solution x. We make a standing hypothesis that (1.1) does possess such solutions. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, otherwise it is non-oscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

In the previous two decades, there has been increasing interest in obtaining sufficient conditions for the oscillation and non-oscillation of solutions of different classes of second order differential equations, see for example [2–11, 13–30] and the references therein.

A lot of work has been done on the following particular cases of (1.1)

$$\ddot{x}(t) + q(t)\, x(t) = 0, \tag{1.2}$$

$$\left[r(t)\, \dot{x}(t)\right]^{\cdot} + q(t)\, g(x(t)) = 0, \tag{1.3}$$

and

$$\ddot{x}(t) + q(t)\, \varphi(x(t), \dot{x}(t)) = 0. \tag{1.4}$$

An important tool in the study of oscillatory behavior of solutions of these equations is the averaging technique which goes back as far as the classical result of Fite [10] which proved that (1.2) is oscillatory if $q(t) > 0$ for all $t \geq t_0$ and

$$\lim_{t\to\infty} \int_{t_0}^{t} q(s)\, ds = \infty.$$

The following theorem extends the results of Fite [10] to an equation in which q is of arbitrary sign.

Wintner [28] proved that (1.2) is oscillatory if

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_o}^{t} (t-s)\, q(s)\, ds = \infty. \tag{1.5}$$

Hartman [15] improved this result by proving that condition (1.5) can be replaced by the following weaker condition

$$-\infty < \liminf_{t\to\infty} \frac{1}{t}\int_{t_o}^{t} (t-s)\, q(s)\, ds < \limsup_{t\to\infty} \frac{1}{t}\int_{t_o}^{t} (t-s)\, q(s)\, ds \le \infty,$$

implies that every solution of (1.2) oscillates.

Kamenev [16] improved Wintner's result by proving that the condition

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}}\int_{t_o}^{t} (t-s)^{n-1} q(s)\, ds = \infty \text{ for some integer n} \ge 3,$$

is sufficient for the oscillation of (1.2).

For the oscillation of (1.3), the following Wong lemma [29], which is modified by Graef and Spikes [12], is a quite useful element in the following theorem.

Wong's lemma: let

$$\liminf_{t\to\infty} \int_{T}^{t} q(s)\, ds \ge 0 \text{ for all large } T, \tag{1.6}$$

then every nonoscillatory solution $x(t)$ of (1.3) which is not eventually constant must satisfy $x(t)\,\dot{x}(t) > 0$ for all large t.

Fu-Hsiang Wong and Cheh-Chih Yeh [30] proved that (1.3) is oscillatory if (1.6) holds and there exists a positive concave function ρ on $\mathbb{R}_+$ such that

$$\limsup_{t\to\infty} \frac{1}{t^{\beta}}\int_{0}^{t} (t-s)^{\beta} \rho(s)\, q(s)\, ds = \infty \text{ for some } \beta \ge 0. \tag{1.7}$$

Also, they [30] proved that the mixed type differential equation (1.3) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{t^{\beta}}\int_{0}^{t} (t-s)^{\beta} \rho(s)\, q(s)\, ds = \infty \text{ for some } \beta \ge 1. \tag{1.8}$$

For the oscillation of (1.4), Bihari [3] proved that if $q(t) > 0$ for all $t \ge t_0$ and

$$\lim_{t\to\infty} \int_{t_0}^{t} q(s)\, ds = \infty,$$

then every solution of (1.4) oscillates.

The following theorem extends the results of Bihari [3] to an equation in which q is of arbitrary sign.

Kartsatos [17] proved that (1.4) is oscillatory if

$$\lim_{t\to\infty}\int_{t_0}^{t} q(s)\,ds=\infty,$$

and there exists a constant $c_1 \in \mathbb{R}_+ = (0,\infty)$ such that

$$\Phi(m)=\int_0^m \frac{d\omega}{\varphi(1,\omega)} \ge -c_1 \text{ for every } m\in\mathbb{R}. \tag{1.9}$$

2. MAIN RESULTS

In this section, we will use the Riccati technique to establish sufficient conditions for (1.1) to be oscillatory. Comparisons between our results and the previously known results are presented and some examples illustrate the main results.

Theorem 2.1. *Assume that* (1.9) *holds. Furthermore, assume that*

$$\limsup_{t\to\infty}\frac{1}{t^\beta}\int_{t_0}^{t}(t-s)^\beta q(s)\,ds=\infty, \text{ for some } \beta\ge 0. \tag{2.1}$$

Then the differential equation (1.1) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of (1.1) such that $x(t)>0$ on $[T,\infty)$ for some $T\ge t_0$. Let $\omega(t)$ be defined by the Riccati transformation

$$\omega(t)=\frac{r(t)\,\psi(x(t))\,f(\dot{x}(t))}{g(x(t))},\ t\ge T.$$

This and (1.1) imply

$$\dot{\omega}(t)=-\varphi(1,\omega(t))\,q(t)-\frac{r(t)\,\psi(x(t))\,f(\dot{x}(t))\,g\prime(x(t))\,\dot{x}(t)}{(g(x(t)))^2},\ t\ge T.$$

Hence, for all $t\ge T$, we have

$$\dot{\omega}(t)\le -\varphi(1,\omega(t))\,q(t).$$

Or

$$\varphi(1,\omega(t))\,q(t)\le -\dot{\omega}(t),\ t\ge T.$$

Dividing this inequality by $\varphi(1,\omega(t))>0$, we obtain

$$q(t)\le -\frac{\dot{\omega}(t)}{\varphi(1,\omega(t))},\ t\ge T.$$

Integrating the above inequality multiplied by $(t-s)^{\beta}$ from T to $t(\geq T)$, we get

$$\int_T^t (t-s)^{\beta} q(s)\,ds \leq -\int_T^t (t-s)^{\beta} \frac{\dot{\omega}(s)}{\varphi(1,\omega(s))} ds. \qquad (2.2)$$

By the Bonnet's theorem [1], we see that for each $t \geq T$, there exists $\alpha_t \in [T,t]$ such that

$$-\int_T^t (t-s)^{\beta} \frac{\dot{\omega}(s)}{\varphi(1,w(s))} ds = -(t-T)^{\beta} \int_T^{\alpha_t} \frac{\dot{\omega}(s)}{\varphi(1,\omega(s))} ds = -(t-T)^{\beta} \int_{\omega(T)}^{\omega(\alpha_t)} \frac{dv}{\varphi(1,v)} =$$

$$= -(t-T)^{\beta} \left[\Phi(\omega(\alpha_t)) - \Phi(\omega(T))\right] \leq (c_1 + \Phi(\omega(T)))(t-T)^{\beta}. \qquad (2.3)$$

It follows from (2.2) and (2.3) that

$$\int_T^t (t-s)^{\beta} q(s)\,ds \leq (c_1 + \Phi(\omega(T)))(t-T)^{\beta}, \ t \geq T.$$

Dividing this inequality by t^{β} and taking the limit superior on both sides, we obtain

$$\limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_{t_0}^t (t-s)^{\beta} q(s)\,ds = \limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_{t_0}^T (t-s)^{\beta} q(s)\,ds +$$

$$+ \limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_T^t (t-s)^{\beta} q(s)\,ds < \infty,$$

which contradicts (2.1). Hence, the proof is complete. □

Example 2.2. Consider the differential equation

$$\left[t^3(\dot{x}(t))^3\right]^{\cdot} + (\frac{1}{t} + 2\sin t)x^9(t)\exp(-t^3(\dot{x}(t))^3/x^9(t)) = 0, \ t \geq 1. \qquad (2.4)$$

Here, $r(t)=t^3$, $q(t)=(\frac{1}{t}+2\sin t)$, $\psi(x)=1$, $f(x)=x^3$, $g(x)=x^9$ and $\varphi(u,v) = ue^{-\frac{v}{u}}$. Note that (1.9) is satisfied. By choosing $\beta = 2$, we have

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_1^t (t-s)^2 q(s)\,ds = \infty.$$

Thus, Theorem 2.1 ensures that every solution of (2.4) oscillates.
Note that, the results of Bihari [3] and Kartsatos [17] cannot be applied to (2.4).

Theorem 2.3. *Assume that* (1.9)*. Furthermore, assume that*

$$\limsup_{t\to\infty} \frac{1}{R^{\beta}(t)} \int_{t_0}^{t} (R(t) - R(s))^{\beta} q(s)\, ds = \infty \text{ for some } \beta \geq 0, \tag{2.5}$$

where $R(t) = \int_{t_0}^{t} \frac{ds}{r(s)}$, $t \geq t_0 > 0$*. Then the differential equation* (1.1) *is oscillatory.*

Proof. Assume that $x(t)$is a non-oscillatory solution of (1.1). Then there exists $T \geq t_0$ such that $x(t) \neq 0$ for $t \geq T$. Define

$$\omega(t) = \frac{r(t)\psi(x(t)) f(\dot{x}(t))}{g(x(t))}, \ t \geq T.$$

This and (1.1) imply

$$\dot{\omega}(t) = -q(t)\varphi(1, \omega(t)) - \frac{r(t)\psi(x(t))\dot{x}(t) f(\dot{x}(t)) g'(x(t))}{(g(x(t)))^2}, \ t \geq T.$$

Hence, for all $t \geq T$, we have

$$\dot{\omega}(t) \leq -\varphi(1, \omega(t)) q(t).$$

Or

$$\varphi(1, \omega(t)) q(t) \leq -\dot{\omega}(t), t \geq T.$$

Dividing this inequality by $\varphi(1, \omega(t))$, we obtain

$$q(t) \leq -\frac{\dot{\omega}(t)}{\varphi(1, \omega(t))}, \ t \geq T.$$

Integrating the above inequality multiplied by $(R(t) - R(s))^{\beta}$ from T to $t(\geq T)$, we get

$$\int_{T}^{t} (R(t) - R(s))^{\beta} q(s)\, ds \leq -\int_{T}^{t} (R(t) - R(s))^{\beta} \frac{\dot{\omega}(s)}{\varphi(1, \omega(s))} ds. \tag{2.6}$$

By the Bonnet's theorem, we see that, for each $t \geq T$, there exists $\alpha_t \in [T, t]$ such that

$$\begin{aligned} -\int_{T}^{t} (R(t) - R(s))^{\beta} \frac{\dot{\omega}(s)}{\varphi(1, \omega(s))} ds &= -(R(t) - R(T))^{\beta} \int_{T}^{\alpha_t} \frac{\dot{\omega}(s)}{\varphi(1, \omega(s))} ds = \\ &= -(R(t) - R(T))^{\beta} \int_{\omega(T)}^{\omega(\alpha_t)} \frac{du}{\varphi(1, u)} = \\ &= -(R(t) - R(T))^{\beta} [\Phi(\omega(\alpha_t)) - \Phi(\omega(T))] \leq \\ &\leq (c_1 + \Phi(\omega(T)))(R(t) - R(T))^{\beta}. \end{aligned}$$

It follows from (2.6) and (2.7) that

$$\int_T^t (R(t) - R(s))^\beta q(s)\, ds \le (c_1 + \Phi(\omega(T)))\, (R(t) - R(T))^\beta .$$

Dividing this inequality by $R^\beta(t)$ and taking the limit superior on both sides, we obtain

$$\limsup_{t\to\infty} \frac{1}{R^\beta(t)} \int_{t_0}^t (R(t) - R(s))^\beta q(s)\, ds \le \limsup_{t\to\infty} \Big([c_1 + \Phi(\omega(T))] \Big(1 - \frac{R(T)}{R(t)}\Big)^\beta \Big).$$

It is clear that $\lim_{t\to\infty} \frac{1}{R(t)} = L \in [0, \infty)$. Thus,

$$\limsup_{t\to\infty} \frac{1}{R^\beta(t)} \int_T^t (R(t) - R(s))^\beta q(s)\, ds \le \limsup_{t\to\infty} ([c_1 + \Phi(\omega(T))]\, (1 - LR(T))^\beta) < \infty,$$

which contradicts to (2.5). Hence, the proof is completed. □

Example 2.4. Consider the differential equation

$$\Big[\frac{\dot{x}^3}{x^2 + 4}\Big]^{\cdot} + (\frac{1}{t} + 2\cos t) x^3 (2 + \exp(-\dot{x}^3 / x^3 (4 + x^2)) + \exp(\dot{x}^3 / x^3 (4 + x^2)) = 0. \quad (2.7)$$

Here, $r(t) = 1$, $q(t) = \frac{1}{t} + 2\cos t, \psi(x) = \frac{1}{4+x^2}, g(x) = f(x) = x^3$ and $\varphi(u, v) = u(2 + e^{-\frac{v}{u}} + e^{\frac{v}{u}})$.
Note that (1.9) is satisfied. By choosing $\beta = 2$, we have

$$\limsup_{t\to\infty} \frac{1}{R^2(t)} \int_T^t (R(t) - R(s))^2 q(s)\, ds = \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_1^t (t - s)^2 q(s)\, ds = \infty.$$

Thus, Theorem 2.3 ensures that, every solution of (2.7) oscillates. Note that, Theorem 2.1 can be applied to (2.7), but the results of Bihari [3] and Kartsatos [17] can not be applied to (2.7).

Theorem 2.5. *Assume that $f(x) \ge bx$ for all $x \in \mathbb{R}$ and for some constant $b > 0$. Furthermore, assume that there exists a constant $c_2 \in \mathbb{R}_+ = (0, \infty)$ such that*

$$\Phi(m) = \int_0^m \frac{d\omega}{\varphi(1, \omega)} \ge c_2 m \text{ for every } m \in \mathbb{R}, \quad (2.8)$$

$$0 < \int_0^{\pm\varepsilon} \frac{\psi(u)}{g(u)} du < \infty \text{ for all } \varepsilon > 0, \quad (2.9)$$

$$\limsup_{t\to\infty} R(t) = A < \infty, \tag{2.10}$$

where $R(t) = \int_{t_0}^{t} \frac{ds}{r(s)}$, *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)} \int_{t_0}^{s} q(u)duds = \infty. \tag{2.11}$$

Then the differential equation (1.1) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of (1.1) such that $x(t) > 0$ on $[T, \infty)$ for some $T \geq t_0$. Let $\omega(t)$ be defined by the Riccati transformation

$$\omega(t) = \frac{r(t)\,\psi(x(t))\,f(\dot{x}(t))}{g(x(t))},\ t \geq T.$$

This and (1.1) imply

$$\dot{\omega}(t) \leq -\varphi(1, \omega(t))\,q(t),\ t \geq T.$$

Dividing this inequality by $\varphi(1, \omega(t)) > 0$, we obtain

$$q(t) \leq -\frac{\dot{\omega}(t)}{\varphi(1, \omega(t))},\ t \geq T.$$

Integrating the above inequality from T to t $(\geq T)$, we get

$$\int_{T}^{t} q(s)\,ds \leq -\int_{T}^{t} \frac{\dot{\omega}(s)\,ds}{\varphi(1, \omega(s))} = -\int_{\omega(T)}^{\omega(t)} \frac{dv}{\varphi(1, v)} = -\left[\int_{0}^{\omega(t)} \frac{dv}{\varphi(1, v)} - \int_{0}^{\omega(T)} \frac{dv}{\varphi(1, v)}\right] \leq$$
$$\leq -\Phi(\omega(t)) + \Phi(\omega(T)). \tag{2.12}$$

By (2.8), we get

$$\int_{T}^{t} q(s)\,ds \leq -c_2\omega(t) + \Phi(\omega(T)).$$

Integrating the above inequality multiplied by $\frac{1}{r(t)}$ from T to $t(\geq T)$, we get

$$b_1 \int_{T}^{t} \frac{\psi(x(s))\,\dot{x}(s)}{g(x(s))} ds \leq c_2 \int_{T}^{t} \frac{\psi(x(s))\,f(\dot{x}(s))}{g(x(s))} ds \leq$$
$$\leq \Phi(\omega(T))\,R(t) - \int_{T}^{t} \frac{1}{r(s)} \int_{T}^{s} q(u)\,duds,\ b_1 = bc_2.$$

From (2.10) and (2.11), we have that

$$I(t) = \int_T^t \frac{\psi(x(s))\,\dot{x}(s)}{g(x(s))}ds \to -\infty \text{ as } t \to \infty.$$

Now, if $x(t) \geq x(T)$for large t,then $I(t) \geq 0$, which is a contradiction.
Hence, for large t, $x(t) \leq x(T)$, so

$$I(t) = -\int_{x(t)}^{x(T)} \frac{\psi(u)}{g(u)}du > -\int_0^{x(T)} \frac{\psi(u)}{g(u)}du > -\infty,$$

which is again a contradiction. This completes the proof of Theorem 2.5. □

Theorem 2.6. *Assume that $f(x) \geq bx$ for all $x \in \mathbb{R}$ and for some constant $b > 0$. And the conditions* (2.8)*,* (2.9) *and* (2.10) *hold. Furthermore, suppose that, there exists a function $\rho : [t_0, \infty) \to (0, \infty)$ such that*

$$\dot{\rho}(t) \geq 0 \textit{ for all } t \geq t_0,$$

and

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{1}{\rho(s)\,r(s)} \left(\int_{t_0}^s \rho(u)\,q(u)\,du \right) ds = \infty. \tag{2.13}$$

Then the differential equation (1.1) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of (1.1) such that $x(t) > 0$ on $[T, \infty)$ for some $T \geq t_0$. Let $\omega(t)$ be defined by the Riccati transformation

$$\omega(t) = \rho(t) \frac{r(t)\,\psi(x(t))\,f(\dot{x}(t))}{g(x(t))}, \text{ for all} t \geq T.$$

This and (1.1) imply

$$\dot{\omega}(t) \leq -\varphi\Big(1, \frac{\omega(t)}{\rho(t)}\Big)\rho(t)\,q(t) + \dot{\rho}(t)\,\frac{\omega(t)}{\rho(t)}.$$

Hence, for all $t \geq T$,we have

$$\int_T^t \rho(s)\,q(s)\,ds \leq -\int_T^t \rho(s)\,\frac{\frac{d}{ds}(\frac{\omega(s)}{\rho(s)})ds}{\varphi(1, \frac{\omega(s)}{\rho(s)})}. \tag{2.14}$$

By the Bonnet's Theorem that for each $t \geq T$, there exists a $T_0 \in [T,t]$ such that

$$-\int_T^t \rho(s)\frac{\frac{d}{ds}(\frac{\omega(s)}{\rho(s)})ds}{\varphi(1,\frac{\omega(s)}{\rho(s)})} = -\rho(t)\int_{T_0}^t \frac{\frac{d}{ds}(\frac{\omega(s)}{\rho(s)})ds}{\varphi(1,\frac{\omega(s)}{\rho(s)})} = -\rho(t)\int_{\frac{\omega(T_0)}{\rho(T_0)}}^{\frac{\omega(t)}{\rho(t)}} \frac{dv}{\varphi(1,v)} = \tag{2.15}$$
$$= -\rho(t)\,\Phi\Big(\frac{\omega(t)}{\rho(t)}\Big) + m_1\rho(t), \text{ where } m_1 = \Phi\Big(\frac{\omega(T_0)}{\rho(T_0)}\Big).$$

By (2.8), (2.14) and (2.15), we get

$$\int_T^t \rho(s)\,q(s)\,ds \leq -c_2\omega(t) + m_1\rho(t). \tag{2.16}$$

Integrating the above inequality multiplied by $\frac{1}{\rho(t)r(t)}$ from T to $t(\geq T)$, we get

$$b_1\int_T^t \frac{\psi(x(s))\,\dot{x}(s)}{g(x(s))}ds \leq c_2\int_T^t \frac{\psi(x(s))\,f(\dot{x}(t))}{g(x(s))}ds \leq$$
$$\leq m_1R(t) - \int_T^t \frac{1}{\rho(s)\,r(s)}\int_T^s \rho(u)\,q(u)\,du\,ds.$$

From (2.10) and (2.13), we have

$$I(t) = \int_T^t \frac{\psi(x(s))\,\dot{x}(s)}{g(x(s))}ds \to -\infty \text{ as } t \to \infty.$$

Now, if $x(t) \geq x(T)$ for large t, then $I(t) \geq 0$, which is a contradiction.
Hence, for large t, $x(t) \leq x(T)$, so

$$I(t) = -\int_{x(t)}^{x(T)} \frac{\psi(u)}{g(u)}du > -\int_0^{x(T)} \frac{\psi(u)}{g(u)}du > -\infty,$$

which is again a contradiction. This completes the proof of Theorem 2.6. □

Example 2.7. Consider the differential equation

$$\Big[e^t\left(x^2(t)+1\right)\dot{x}(t)\Big]^{\cdot} + \frac{e^{2t}e^{|x(t)|}\operatorname{sgn}x(t)\left[x^2(t)+1\right]}{1+\exp(e^t\dot{x}(t)/e^{|x(t)|}\operatorname{sgn}x(t))} = 0,\ t \geq 0. \tag{2.17}$$

Here, $r(t) = e^t$, $q(t) = e^{2t}$, $\psi(x) = x^2+1, g(x) = e^{|x|}\left(x^2+1\right)\operatorname{sgn}x$ and $\varphi(u,v) = u/(1+\exp\frac{v}{u})$. Note that (1.9) is not satisfied, but (2.8), (2.9) and (2.10) are satisfied.

Thus, Theorem 2.1 and the results of Bihari [3] and Kartsatos [17] cannot be applied to (2.17). Now, by choosing $\rho(t) = 1$, we have

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)\rho(s)} \int_{t_0}^{s} \rho(u)\, q(u)\, du ds = \limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{e^s} \int_{t_0}^{s} e^{2u} du ds = \infty,$$

then, Theorem 2.6 ensures that every solution of (2.17) oscillates.

We need the following lemma which is an extension to the Lemma of Y.S.W. Wang [29] and Greaf and Spikes [12].

Lemma 2.8. *Let* $f(\dot{x}) = (\dot{x})^\gamma$, *where* $\gamma > 0$ *is the ratio of odd positive integers, and assume that* (1.6) *and* (2.8) *hold. Furthermore, suppose that,*

$$\frac{\partial}{\partial v}\varphi(u,v) \le 0 \quad \forall\, u \ne 0 \quad and \quad v \in \mathbb{R}, \tag{2.18}$$

$$0 < \int_{0}^{\pm\varepsilon} \Big(\frac{\psi(u)}{g(u)}\Big)^{\frac{1}{\gamma}} du < \infty, \tag{2.19}$$

and

$$\int_{t_0}^{\infty} \Big(\frac{1}{r(t)}\Big)^{\frac{1}{\gamma}} dt = \infty. \tag{2.20}$$

Then every nonoscillatory solution of (1.1) *which is not eventually a constant must satisfy* $x(t)\,\dot{x}(t) > 0$ *for all large* t.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) and without loss of generality, we assume that $x(t) > 0$ for $t \ge T_1$, $T_1 \ge t_0$. If the Lemma is not true, then either $\dot{x}(t) < 0$ for all large t or $\dot{x}(t)$ oscillates. In the former case, we may suppose that T_1 is sufficiently large that

$$\int_{T_1}^{t} q(s)\, ds \ge 0 \text{ and } \dot{x}(t) < 0 \text{ for } t \ge T_1.$$

From equation (1.1), we get

$$\frac{(r(t)\,\psi(x(t))\,(\dot{x}(t))^\gamma)^{\cdot}}{g(x(t))} + q(t)\,\varphi(1,\omega(t)) = 0, \tag{2.21}$$

where $\omega(t) = \frac{r(t)\psi(x(t))(\dot{x}(t))^\gamma}{g(x(t))}$. Then, for all $t \ge T_1$, we obtain

$$\frac{r(t)\,\psi(x(t))\,(\dot{x}(t))^\gamma}{g(x(t))} - \frac{r(T)\,\psi(x(T))\,(\dot{x}(T))^\gamma}{g(x(T))} + \\ + \int_{T_1}^{t} \frac{r(s)\,\psi(x(s))\,(\dot{x}(s))^{\gamma+1}}{(g(x(s)))^2} g'(x(s)) ds + \int_{T_1}^{t} F(\omega(s))\, q(s)\, ds = 0,$$

where $F(\omega) = \varphi(1, \omega)$. Since $g'(x) > 0$ for all $x \neq 0$, we have

$$\frac{r(t)\,\psi(x(t))\,(\dot{x}(t))^{\gamma}}{g(x(t))} + \int_{T_1}^{t} F(\omega(s))\,q(s)\,ds < 0.$$

Integrating the integral by parts, we get

$$\omega(t) + F(\omega(t)) \int_{T_1}^{t} q(s)\,ds - \int_{T_1}^{t} \Big(F'(\omega(s))\,\dot{\omega}(s) \int_{T_1}^{s} q(u)\,du \Big) ds < 0.$$

Then,

$$\omega(t) - \int_{T_1}^{t} \Big(F'(\omega(s))\,\dot{\omega}(s) \int_{T_1}^{s} q(u)\,du \Big) ds < 0, \ \ t \geq T_1. \tag{2.22}$$

We define

$$H(t) = \omega(t) - \int_{T_1}^{t} \Big(F'(\omega(s))\,\dot{\omega}(s) \int_{T_1}^{s} q(u)\,du \Big) ds < 0, \ \ t \geq T_1. \tag{2.23}$$

Then, for all $t \geq T_1$, we get

$$\dot{H}(t) = \dot{\omega}(t) - F'(\omega(t))\,\dot{\omega}(t) \int_{T_1}^{t} q(s)\,ds.$$

Thus,

$$\dot{H}(t) = \dot{\omega}(t) \Big(1 - F'(\omega(t)) \int_{T_1}^{t} q(s)\,ds \Big). \tag{2.24}$$

There are three cases to consider

Case 1. Assume that $\dot{H}(t) \leq 0$ on $[T_2, t]$, $T_2 \geq T_1$.

Since $(1 - F'(\omega(t)) \int_{T_1}^{t} q(u)\,du) > 0$ for all $t \geq T_2$,

$$\dot{\omega}(t) \leq 0 \text{ for all } t \geq T_2.$$

Hence,

$$\omega(t) \leq \omega(T_2) < 0.$$

Thus,

$$r(t)\,\frac{\psi(x(t))(\dot{x}(t))^{\gamma}}{g(x(t))} \leq \omega(T_2), \ \ t \geq T_2.$$

Then,

$$\int_{T_2}^{t} \Big(\frac{\psi\,(x\,(s))}{g\,(x\,(s))}\Big)^{\frac{1}{\gamma}} \dot{x}\,(t)\,ds \le (\omega(T_2))^{\frac{1}{\gamma}} \int_{T_2}^{t} \Big(\frac{1}{r\,(s)}\Big)^{\frac{1}{\gamma}} ds.$$

Since $\dot{x}\,(t) < 0$ on $[T_2, t]$,

$$-\int_{x(t)}^{x(T_2)} \Big(\frac{\psi\,(u)}{g\,(u)}\Big)^{\frac{1}{\gamma}} du \le (\omega(T_2))^{\frac{1}{\gamma}} \int_{T_2}^{t} \Big(\frac{1}{r\,(s)}\Big)^{\frac{1}{\gamma}} ds.$$

From (2.20), we have

$$-\int_{x(t)}^{x(T_2)} \Big(\frac{\psi(u)}{g\;(u)}\Big)^{\frac{1}{\gamma}} du \to -\infty \;\text{ as } t \to \infty.$$

But

$$-\int_{x(t)}^{x(T_2)} \Big(\frac{\psi(u)}{g(u)}\Big)^{\frac{1}{\gamma}} d\;u \ge -\int_{0}^{x(T_2)} \Big(\frac{\psi(u)}{g(u)}\Big)^{\frac{1}{\gamma}} du > -\infty \;\text{ for large } t,$$

which is a contradiction.
Case 2. Assume that $\dot{H}\,(t) \ge 0$ on $[T_3, t]$, $T_3 \ge T_1$.
Since $(1 - F'\,(\omega\,(t)) \int_{T_1}^{t} q\,(s)\,ds) > 0$ for all $t \ge T_3$,

$$\dot{\omega}\,(t) \ge 0 \text{ for all } t \ge T_3.$$

Hence,

$$-F\,(\omega\,(t))\,q\,(t) - \frac{r\,(t)\,\psi\,(x\,(t))\,(\dot{x}\,(t))^{\gamma+1}}{(g\,(x\,(t)))^2} g'\,(x\,(t)) \ge 0.$$

Thus, $q(t) \le 0,\; t \ge T_3$. Then, for all $t \ge T_3$, we obtain

$$\int_{T_3}^{t} q\,(s)\,ds \le 0,$$

which contradicts (1.6).

Case 3. Assume that $\dot{H}\,(t)$ oscillates on $[T_4, t]$, $T_4 \ge T_1$, then $\dot{\omega}\,(t)$ oscillates on $[T_4, t]$, $T_4 \ge T_1$. As in the proof of Theorem 2.5, we can obtain the inequality,

$$\int_{T_4}^{t} q\,(s)\,ds \le -\Phi\,(\omega\,(t)) + \Phi\,(\omega\,(T_4))\,. \tag{2.25}$$

Since, $\Phi'(\omega) = \frac{1}{\varphi(\omega)} > 0$ for all $\omega \in \mathbb{R}$ and $\omega(T_4) = \frac{r(T_4)\psi(x(T_4))(\dot{x}(T_4))^\gamma}{g(x(T_4))} < 0$, then

$$\Phi(\omega(T_4)) < 0. \tag{2.26}$$

Thus, from (2.8), (2.25) and (2.26), we get

$$\int_{T_4}^{t} q(s)\, ds \le -c_2 \omega(t).$$

Choose, $T_5 \ge T_4$ such that $\int_{T_5}^{t} q(s)\, ds \ge c_3 > 0$, then

$$\int_{T_5}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} ds \le -(\frac{c_2}{c_3})^{\frac{1}{\gamma}} \int_{x(T_5)}^{x(t)} \left(\frac{\psi(u)}{g(u)}\right)^{\frac{1}{\gamma}} du = \left(\frac{c_2}{c_3}\right)^{\frac{1}{\gamma}} \int_{x(t)}^{x(T_5)} \left(\frac{\psi(u)}{g(u)}\right)^{\frac{1}{\gamma}} du.$$

Thus, from (2.19) we get

$$\int_{T_5}^{\infty} \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} dt < \infty,$$

which contradicts (2.20). If $\dot{x}(t)$ oscillates, Then there exists a sequence $\{\tau_n : n = 1, 2, ...\}$ such that $\dot{x}(\tau_n) = 0$ and $\tau_n \to \infty$. For $t \ge T_1$, we define

$$\omega(t) = \frac{r(t)\psi(x(t))(\dot{x}(t))^\gamma}{g(x(t))}.$$

This and (1.1) imply

$$\dot{\omega}(t) \le -\varphi(1, \omega(t))\, q(t),\ t \ge T_1.$$

Or

$$\varphi(1, \omega(t))\, q(t) \le -\dot{\omega}(t),\ t \ge T_1.$$

Dividing this inequality by $\varphi(1, \omega(t)) > 0$, we obtain

$$q(t) \le -\frac{\dot{\omega}(t)}{\varphi(1, \omega(t))},\ t \ge T_1.$$

Integrating the above inequality from τ_n to τ_{n+1}, we get

$$\int_{\tau_n}^{\tau_{n+1}} q(s)\, ds \le -\int_{\tau_n}^{\tau_{n+1}} \frac{\dot{\omega}(s)\, ds}{\varphi(1, \omega(s))} = -\int_{\omega(\tau_n)}^{\omega(\tau_{n+1})} \frac{du}{\varphi(1, u)} = -\int_{0}^{0} \frac{du}{\varphi(1, u)} = 0, \tag{2.27}$$

which contradicts (1.6). Hence $\dot{x}(t) > 0$. □

Theorem 2.9. *Let $f(\dot{x}) = (\dot{x})^{\gamma}$, where $\gamma > 0$ is the ratio of odd positive integers, and assume that* (1.6), (2.1), (2.8), (2.18), (2.19) *and* (2.20) *hold. Then the differential equation* (1.1) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of (1.1) such that $x(t) > 0$ on $[T, \infty)$ for some $T \geq t_0$. It follows from Lemma 2.8 that $\dot{x}(t) > 0$ on $[T_1, \infty)$ for some $t \geq T_1 \geq T$. Define

$$\omega(t) = \frac{r(t)\,\psi(x(t))\,(\dot{x}(t))^{\gamma}}{g(x(t))},\ t \geq T_1.$$

This and (1.1) imply

$$\dot{\omega}(t) = -\varphi(1, \omega(t))\,q(t) - \frac{r(t)\,\psi(x(t))\,(\dot{x})^{\gamma+1}(t)}{g^2(x(t))}g'(x(t)).$$

Hence, for all $t \geq T_1$, we have

$$\int_{T_1}^{t} (t-s)^{\beta}\, q(s)\, ds \leq -\int_{T_1}^{t} (t-s)^{\beta}\, \frac{\dot{\omega}(s)}{\varphi(1, \omega(s))} ds. \tag{2.28}$$

By Bonnet's theorem, we see that for each $t \geq T_1$, there exists $\alpha_t \in [T_1, t]$ such that

$$\begin{aligned} -\int_{T_1}^{t} (t-s)^{\beta}\, \frac{\dot{\omega}(s)}{\varphi(1, \omega(s))} ds &= -(t-T_1)^{\beta} \int_{T_1}^{\alpha_t} \frac{\dot{\omega}(s)}{\varphi(1, \omega(s))} ds = \\ &= -(t-T_1)^{\beta} \int_{\omega(T_1)}^{\omega(\alpha_t)} \frac{dv}{\varphi(1, v)} \end{aligned} \tag{2.29}$$

Thus, from (2.8), (2.28) and (2.29) we get,

$$\begin{aligned} \int_{T_1}^{t} (t-s)^{\beta}\, q(s)\, ds &\leq -(t-T_1)^{\beta}\,[\Phi(\omega(\alpha_t)) - \Phi(\omega(T_1))] \leq \\ &\leq -(t-T_1)^{\beta}\,(\Phi(\omega(\alpha_t)) - m_2),\ m_2 = \Phi(\omega(T_1)) \leq \\ &\leq -(t-T_1)^{\beta}\,(c_2\omega(\alpha_t) - m_2) \leq \\ &\leq m_2\,(t-T_1)^{\beta}. \end{aligned} \tag{2.30}$$

Dividing this inequality (2.30) by t^{β} and taking limit superior of both sides, we obtain

$$\limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_{T_1}^{t} (t-s)^{\beta}\, q(s)\, ds < \infty,$$

which contradicts (2.1). Hence, the proof is complete. □

Example 2.10. Consider the differential equation

$$\left[t\dot{x}(t)\right]^{\cdot}+\left(\frac{1}{t}+\sin t\right)x^{\frac{1}{3}}(t)(1+(2+\exp[t\dot{x}(t)/x^{\frac{1}{3}}(t)])^{-1})=0,\ t\geq 1. \tag{2.31}$$

Here, $r(t)=t$, $\psi(x)=1$, $f(x)=x$, $q(t)=\frac{1}{t}+\sin t$, $g(x)=x^{\frac{1}{3}}$ and $\varphi(u,v)=\frac{u}{2+e^{\frac{v}{u}}}$. By choosing $\beta=1$, we have

$$\liminf_{t\to\infty}\int_1^t q(s)\,ds=\liminf_{t\to\infty}(\ln t-\cos t+\cos 1)>0,\ \text{and}$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_1^t(t-s)\left(\frac{1}{s}+\sin s\right)ds=\limsup_{t\to\infty}\left(\ln t-\frac{(t-1)}{t}+\cos 1-\frac{\cos 1}{t}-\frac{\sin t}{t}+\frac{\sin 1}{t}\right)=\infty.$$

Thus, Theorem 2.9 ensures that every solution of (2.31) oscillates.

Theorem 2.11. *Let $f(\dot{x})=(\dot{x})^{\gamma}$, where $\gamma>0$ is the ratio of odd positive integers, and assume* (1.6), (2.8), (2.18), (2.19) *and* (2.20) *hold. Furthermore, suppose that, there exists a function $\rho:[t_0,\infty)\to(0,\infty)$ such that $\dot{\rho}(t)\geq 0$ for all $t\geq t_0$, and*

$$\limsup_{t\to\infty}\frac{1}{\rho(t)}\int_{t_0}^t\rho(s)\,q(s)\,ds=\infty. \tag{2.32}$$

Then the differential equation (1.1) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of (1.1) such that $x(t)>0$ on $[T,\infty)$ for some $T\geq t_0$. It follows from Lemma 2.8 that $\dot{x}(t)>0$ on $[T_1,\infty)$ for some $t\geq T_1\geq T$. Define

$$\omega(t)=\rho(t)\frac{r(t)\psi(x(t))(\dot{x}(t))^{\gamma}}{g(x(t))},\ t\geq T_1.$$

This and (1.1) imply

$$\dot{\omega}(t)\leq-\varphi\left(1,\frac{\omega(t)}{\rho(t)}\right)\rho(t)q(t)+\dot{\rho}(t)\frac{\omega(t)}{\rho(t)}.$$

Then, for all $t\geq T_1$, we obtain

$$\int_{T_1}^t\rho(s)q(s)ds\leq-\int_{T_1}^t\rho(s)\frac{\frac{d}{ds}\left(\frac{\omega(s)}{\rho(s)}\right)}{\varphi\left(1,\frac{\omega(s)}{\rho(s)}\right)}ds. \tag{2.33}$$

By Bonnet's Theorem, we see that, for each $t \geq T_1$, there exist $T_2 \in [T_1, t]$ such that

$$\begin{aligned} -\int_{T_1}^{t} \rho(s)\frac{d}{ds}\Big(\frac{w\,(s)}{\rho\,(s)}\Big)ds &= -\rho(t)\int_{T_2}^{t} \frac{\frac{d}{ds}\big(\frac{\omega(s)}{\rho(s)}\big)}{\varphi(1, \frac{\omega(s)}{\rho(s)})}ds = \\ &= -\rho(t)\Big[\Phi\Big(\frac{\omega(t)}{\rho(t)}\Big) - \Phi\Big(\frac{\omega(T_2)}{\rho(T_2)}\Big)\Big] \leq \\ &\leq -c_2\omega\,(t) + \rho(t)\Phi\Big(\frac{\omega(T_2)}{\rho(T_2)}\Big) \leq \\ &\leq \rho(t)\Phi\Big(\frac{\omega(T_2)}{\rho(T_2)}\Big). \end{aligned} \tag{2.34}$$

Hence, from (2.33) and (2.34), we have

$$\int_{T_1}^{t} \rho(s)q(s)ds \leq \rho(t)\Phi\Big(\frac{\omega(T_2)}{\rho(T_2)}\Big).$$

Dividing this inequality by $\rho(t)$ and taking the limit

$$\limsup_{t\to\infty} \frac{1}{\rho(t)}\int_{T_1}^{t} \rho(s)q(s)ds < \infty,$$

which contradicts to (2.32). Hence, the proof is complete. □

REFERENCES

[1] R.G. Bartle, *The elements of real analysis*, 7th ed., John Wiley and Sons, **233** (1976).

[2] N.P. Bhatia, *Some oscillation theorems for second order differential equations*, J. Math. Appl. **15** (1966), 442–446.

[3] I. Bihari, *An oscillation theorem concerning the half linear differential equation of the second order*, Magyar Tud. Akad. Mat. Kutato Int. Kozl. **8** (1963), 275–280.

[4] R. Blasko, J.R. Greaf, M. Hacik, P. Spikes, *Oscillatory behavior of solutions of nonlinear differential equations of the second order*, J. Math. Anal. Appl. **151** (1990), 330–343.

[5] G.J. Butler, *Integral averages and the oscillation of second order ordinary differential equations*, SIAM J. Math. Anal. **11** (1980), 190–200.

[6] W.J. Coles, *An oscillation criterion for the second order differential equations*, Proc. Amer. Math. Soc. **19** (1968), 755–759.

[7] W.J. Coles, *Oscillation criteria for nonlinear second order equations*, Ann. Math. Pura Appl. **82** (1969), 132–134.

[8] E.M. Elabbasy, *On the oscillation of nonlinear second order differential equations*, Panam. Math. J. **4** (1996), 69–84.

[9] E.M. Elabbasy, T.S. Hassan, S.H. Saker, *Oscillation of second-order nonlinear differential equations with a damping term*, Electronic J. of Differential Equations **76** (2005), 1–13.

[10] W.B. Fite, *Concerning the zeros of the solutions of certian differential equations*, Trans. Amer. Math. Soc. **19** (1918), 341–352.

[11] S.R. Grace, *Oscillation theorems for nonlinear differential equations of second order*, J. Math. Anal. Appl. **171** (1992), 220–240.

[12] J.R. Graef, P.W. Spikes, *A note on a paper of Wong*, Bull. Inst. Math. Acad. Sinica **5** (1977), 375–377.

[13] J.R. Graef, P.W. Spikes, *On the oscillatory behavior of solutions of second order nonlinear differential equation*, Czech. Mth. J. **36** (1986), 275–284.

[14] J.R. Graef, P.W. Spikes, S.M. Rankin, P.W. Spikes, *Oscillation theorems for perturbed nonlinear differential equations*, J. Math. Anal. Appl. **65** (1978), 375–390.

[15] P. Hartman, *Non-oscillatory linear differential equations of second order*, Amer. J. Math. **74** (1952), 389–400.

[16] I.V. Kamenev, *Integral criterion for oscillation of linear differential equations of second order*, Math. Zametki **23** (1978), 249–251.

[17] A.G. Kartsatos, *On oscillation of nonlinear equations of second order*, J. Math. Anal. Appl. **24** (1968), 665–668.

[18] M. Kirane, Y.V. Rogovechenko, *On oscillation of nonlinear second order differential equation with damping term*, Appl. Math. and Comp. **117** (2–3) (2001), 177–192.

[19] Q. Kong, *Interval criteria for oscillation of second order linear ordinary differential equations*, J. Math. Anal. Appl. **229** (1999), 258–270.

[20] W. Leighton, *The detection of the oscillation of solutions of a second order linear differential equation*, Duke J. Math. **17** (1950), 57–62.

[21] M. Li, M. Wang, J. Yan, *On oscillation of nonlinear second order differential equations with damping term*, J. Appl. Math. and Comp., **13** (2003), 223–232.

[22] Ch.G. Philos, *Oscillation theorems for linear differential equations of second order*, Arch. Math. **53** (1989), 482–492.

[23] Y.V. Rogovechenko, *Note on oscillation criteria for second order linear differential equations*, J. Math. Anal. Appl. **202** (1996), 560–563.

[24] Yuri. V. Rogovechenko, *Oscillation theorems for second order equations with damping*, J. Math. Anal. **42** (2000), 1005–1028.

[25] S.P. Rogovechenco, Y.V. Rogovechenko, *Oscillation of second-order differential equations with damping*, Math. Anal. **10** (2003), 447–461.

[26] S.H. Saker, P.Y.H. Pang, Ravi P. Agarwal, *Oscillation theorems for second order nonlinear functional differential equations with damping*, Dynamic System and Applications **12** (2003), 307–322.

[27] Y.G. Sun, *New Kamenev-type oscillation criteria for second-order nonlinear differential equations with damping*, J. Math. Anal. **291** (2004), 341–351.

[28] A. Wintner, *A criterion of oscillatory stability*, Quart. Appl. Math. **7** (1949), 15–117.

[29] J.S.W. Wong, *Oscillation theorems for second order nonlinear differential equations*, Bull. Inst. Math. Acad. Simca **3** (1975), 283–309.

[30] F.H. Wong, C.C. Yeh, *Oscillation criteria for second order super-linear differential equations*, Math. Japonica **37** (1992), 573–584.

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