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# EXISTENCE OF SOLUTIONS FOR A FOUR-POINT BOUNDARY VALUE PROBLEM OF A NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

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**Abstract.** In this paper, we discuss a four-point boundary value problem for a nonlinear differential equation of fractional order. The differential operator is the Riemann-Liouville derivative and the inhomogeneous term depends on the fractional derivative of lower order. We obtain the existence of at least one solution for the problem by using the Schauder fixed-point theorem. Our analysis relies on the reduction of the problem considered to the equivalent Fredholm integral equation.

**Keywords:** four-point boundary value problem, Riemann-Liouville fractional derivative, Green's function, Schauder fixed-point theorem.

**Mathematics Subject Classification:** 26A33, 34B10.

## 1. INTRODUCTION

Fractional differential equations have been of great interest recently. It is due to the development of the theory of fractional calculus itself and by the application of such constructions in various fields of science and engineering such as control theory, physics, mechanics, electrochemistry, porous media, etc. There are many papers discussing the solvability of nonlinear fractional differential equations and the existence of positive solutions of nonlinear fractional differential equations, see the monographs of Kilbas *et al.* [1], Miller and Ross [2], and the papers [3, 4, 8–12] and the references therein.

In [8], Bai and Lü considered the boundary value problem of a fractional order differential equation

$$\begin{cases} D_{0+}^{\alpha} + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $D^{\alpha}_{0+}$ is the standard Rieman-Liouville fractional derivative of order $1 < \alpha \le 2$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

In [5] the authors investigated the existence of solutions for a coupled system of nonlinear fractional differential equations with three-point boundary conditions

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{p}v(t)), & t \in (0,1), \\ D^{\beta}v(t) = g(t, u(t), D^{q}u(t)), & t \in (0,1), \\ u(0) = 0, \quad u(1) = \gamma u(\eta), & v(0) = 0, \quad v(1) = \gamma v(\eta), \end{cases} \tag{1.2}$$

where $1 < \alpha, \beta < 2$, $p, q, \gamma > 0$, $0 < \eta < 1$, $\alpha - q \ge 1$, $\beta - q \ge 1$, $\gamma\eta^{\alpha-1} < 1$, $\gamma\eta^{\beta-1} < 1$, $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Multi-point boundary value problems for ordinary differential equations arise in a variety of areas of applied mathematics, physics and engineering. For instance, the vibrations of a guy wire of uniform cross-section and composed of $N$ parts of different densities can be set up as a multi-point BVP, as in [6]; also, many problems in the theory of elastic stability can be handled by multi-point problems in [13].

Due to the above reason, we will consider a multi-point boundary problem for nonlinear fractional differential equations. No contributions exist, as far as we know, concerning the four-point boundary value problem of the following system:

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), D^{\mu}u(t)), & t \in (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = au(\eta_1) + bu(\eta_2), \end{cases} \tag{1.3}$$

where $2 < \alpha < 3$, $\mu > 0$, $\alpha - \mu \ge 1$, $0 < a < 1$, $0 < b < 1$, $0 < \eta_1 \le \eta_2 < 1$, $a\eta_1^{\alpha-1} + b\eta_2^{\alpha-1} < 1$, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $D$ is the standard Riemann-Liouville differentiation.

In this paper, we firstly give the corresponding Green's function of system (1.3). Then, problem (1.3) is deduced to an equivalent Fredholm integral equation of the second kind. Finally, by means of the Schauder fixed-point theorem, we obtain the existence of solutions of the boundary value problem (1.3).

## 2. PRELIMINARIES

For convenience of the reader, in this section, we give some definitions and fundamental results of fractional calculus theory. Let $\alpha > 0$ and $n = [\alpha] + 1 = N + 1$, where $N$ is the smallest integer less than or equal to $\alpha$.

**Definition 2.1** ([7])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds,$$

where $\Gamma$ denotes the gamma function, provided that the right side integral exists.

**Definition 2.2** ([7]). Let $\alpha > 0$ and $n = [\alpha] + 1$, where $[\alpha]$ is the smallest integer greater than or equal to $\alpha$. Then the Riemann-Liouville fractional derivative of order $\alpha$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $\Gamma$ denotes the gamma function, provided that the right side is pointwise defined on $(0, \infty)$.

**Lemma 2.3** ([1]). *Let $\alpha, \beta > 0$, $f : (0, +\infty) \to \mathbb{R}$ is a continuous function, and assume that the Riemann-Liouville fractional integral and fractional derivative of $f$ exist, then we have*

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t), \quad D^\alpha I^\alpha f(t) = f(t).$$

**Lemma 2.4** ([7]). *Let $\alpha > 0$ and assume that $u \in C(0,1) \cap L^1(0,1)$, then the general solution of the fractional differential equation*

$$D^\alpha u(t) = 0$$

*is given by $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$, where $C_i \in \mathbb{R}$, $i = 1, 2, \cdots, N$, $N$ is the smallest integer greater than or equal to $\alpha$.*

**Lemma 2.5** ([7]). *Assume that $u \in C(0,1) \cap L^1(0,1)$ with fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I^\alpha D^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$$

*for some $C_i \in \mathbb{R}$, $i = 1, 2, \cdots, N$.*

In the following, we give the Green's function of the fractional deferential equation with a four-point boundary value problem. For convenience, we introduce the following notation:

$$\zeta := 1 - a\eta_1^{\alpha-1} - b\eta_2^{\alpha-1},$$

$$G_1(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)},$$

$$G_2(t,s) = \frac{t^{\alpha-1}[a(\eta_1 - s)^{\alpha-1} + b(\eta_2 - s)^{\alpha-1} - (1-s)^{\alpha-1}]}{\zeta\Gamma(\alpha)},$$

$$G_3(t,s) = \frac{t^{\alpha-1}[b(\eta_2 - s)^{\alpha-1} - (1-s)^{\alpha-1}]}{\zeta\Gamma(\alpha)},$$

$$G_4(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\zeta\Gamma(\alpha)}.$$

**Lemma 2.6** ([7])**.** *Given* $y \in C(0,1)$ *and* $2 < \alpha < 3$*. Then the unique solution of*

$$\begin{cases} D^\alpha u(t) = y(t), \quad t \in (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = au(\eta_1) + bu(\eta_2), \end{cases} \tag{2.1}$$

*is given by*

$$u(t) = \int_0^1 G(t,s)y(s)\,ds,$$

*where*

$$G(t,s) = \begin{cases} G_1(t,s) + G_2(t,s), & 0 \le s \le t \le \eta_1 \le \eta_2 \le 1, \\ G_2(t,s), & 0 \le t \le s \le \eta_1 \le \eta_2 \le 1, \\ G_3(t,s), & 0 \le t \le \eta_1 \le s \le \eta_2 \le 1, \\ -G_4(t,s), & 0 \le t \le \eta_1 \le \eta_2 \le s \le 1, \end{cases} \tag{2.2}$$

*or*

$$G(t,s) = \begin{cases} G_1(t,s) + G_2(t,s), & 0 \le s \le \eta_1 \le t \le \eta_2 \le 1, \\ G_1(t,s) + G_3(t,s), & 0 \le \eta_1 \le s \le t \le \eta_2 \le 1, \\ G_3(t,s), & 0 \le \eta_1 \le t \le s \le \eta_2 \le 1, \\ -G_4(t,s), & 0 \le \eta_1 \le t \le \eta_2 \le s \le 1, \end{cases} \tag{2.3}$$

*or*

$$G(t,s) = \begin{cases} G_1(t,s) + G_2(t,s), & 0 \le s \le \eta_1 \le \eta_2 \le t \le 1, \\ G_1(t,s) + G_3(t,s), & 0 \le \eta_1 \le s \le \eta_2 \le t \le 1, \\ G_1(t,s) + G_4(t,s), & 0 \le \eta_1 \le \eta_2 \le s \le t \le 1, \\ -G_4(t,s), & 0 \le \eta_1 \le \eta_2 \le t \le s \le 1. \end{cases} \tag{2.4}$$

*Proof.* We can apply Lemma 2.5 to reduce the first equation of (2.1) to an equivalent integral equation

$$u(t) = I^\alpha y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3}$$

for some $C_1, C_2, C_3 \in \mathbb{R}$. Hence, the general solution of Eq. (2.1) is

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}}\,ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3}.$$

It follows from $u(0) = u'(0) = 0$ that $C_2 = C_3 = 0$, and it follows from $u(1) = au(\eta_1) + bu(\eta_2)$ that

$$C_1 = \int_0^{\eta_1} \frac{a(\eta_1 - s)^{\alpha-1} y(s)}{\zeta\Gamma(\alpha)}\,ds + \int_0^{\eta_2} \frac{b(\eta_2 - s)^{\alpha-1} y(s)}{\zeta\Gamma(\alpha)}\,ds - \\ - \int_0^1 \frac{(1-s)^{\alpha-1} y(s)}{\zeta\Gamma(\alpha)}\,ds.$$

Therefore, the unique solution of (2.1) is given by

$$u(t) = I^{\alpha}y(t) + \int_0^{\eta_1} \frac{at^{\alpha-1}(\eta_1 - s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\, ds +$$

$$+ \int_0^{\eta_2} \frac{bt^{\alpha-1}(\eta_2 - s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\, ds - \int_0^{1} \frac{t^{\alpha-1}(1 - s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\, ds =$$

$$= \int_0^{t} \big[G_1(t,s) + G_2(t,s)\big]y(s)\, ds + \int_t^{\eta_1} G_2(t,s)y(s)\, ds +$$

$$+ \int_{\eta_1}^{\eta_2} G_3(t,s)y(s)\, ds - \int_{\eta_2}^{1} G_4(t,s)y(s)\, ds$$

for $0 \leq t \leq \eta_1 \leq \eta_2 \leq 1$ holds, where $G(t,s)$ is described by (2.2). Or

$$u(t) = \int_0^{t} \frac{(t-s)^{\alpha-1}y(s)}{\Gamma(\alpha)}\, ds + \int_0^{\eta_1} \frac{at^{\alpha-1}(\eta_1 - s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\, ds +$$

$$+ \int_0^{\eta_2} \frac{bt^{\alpha-1}(\eta_2 - s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\, ds - \int_0^{1} \frac{t^{\alpha-1}(1 - s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\, ds =$$

$$= \int_0^{\eta_1} \big[G_1(t,s) + G_2(t,s)\big]y(s)\, ds + \int_{\eta_1}^{t} \big[G_1(t,s) + G_3(t,s)\big]y(s)\, ds +$$

$$+ \int_t^{\eta_2} G_3(t,s)y(s)\, ds - \int_{\eta_2}^{1} G_4(t,s)y(s)\, ds =$$

$$= \int_0^{1} G(t,s)y(s)\, ds$$

for $0 \le \eta_1 \le t \le \eta_2 \le 1$ holds, where $G(t,s)$ is described by (2.3). Or

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}y(s)}{\Gamma(\alpha)}\,ds + \int_0^{\eta_1} \frac{at^{\alpha-1}(\eta_1-s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\,ds +$$

$$+ \int_0^{\eta_2} \frac{bt^{\alpha-1}(\eta_2-s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\,ds - \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\,ds =$$

$$= \int_0^{\eta_1} \big[G_1(t,s)+G_2(t,s)\big]y(s)\,ds + \int_{\eta_1}^{\eta_2} \big[G_1(t,s)+G_3(t,s)\big]y(s)\,ds +$$

$$+ \int_{\eta_2}^{t} \big[G_1(t,s)+G_4(t,s)\big]y(s)\,ds - \int_t^1 G_4(t,s)y(s)\,ds =$$

$$= \int_0^1 G(t,s)y(s)\,ds$$

for $0 \le \eta_1 \le \eta_2 \le t \le 1$ holds, where $G(t,s)$ is described by (2.4). Thus, we complete the proof. □

Next, we define the space $X = \{u(t) \in C[0,1] : D^\mu u(t) \in C[0,1]\}$ endowed with the norm $\|u\|_X = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |D^\mu u(t)|$.

**Lemma 2.7** ([9]). *$(X, \|\cdot\|_X)$ is a Banach space.*

In what follows, the Green's function's form of system (2.1) is described by (2.2), (2.3) and (2.4) can be considered similarly.

Considering the following integral equation

$$u(t) = \int_0^1 G(t,s)f(s,u(s),D^\mu u(s))\,ds. \tag{2.5}$$

**Lemma 2.8.** *Suppose that $f : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous, then $u \in X$ is a solution of* (1.3) *if and only if $u \in X$ is a solution of system* (2.5).

*Proof.* Let $u \in X$ be a solution of (1.3). Applying the method used in Lemma 2.6, we can obtain that $u$ is a solution of system (2.5). And the proof of the inverse condition is immediate from Lemma 2.6, so we omit it. □

Let us define an operator $T : X \to X$ as

$$Tu(t) = \int_0^1 G(t,s) f(s, u(s), D^\mu u(s))\, ds. \tag{2.6}$$

Then by Lemma 2.6, we know that the fixed point of operator $T$ coincides with the solution of system (1.3).

Let us set the following notation for convenience:

$$p =: \max_{t\in[0,1]} \int_0^1 |G(t,s)m(s)|\, ds + \frac{4}{\zeta\Gamma(\alpha-\mu)} \int_0^1 (1-s)^{\alpha-\mu-1} m(s)\, ds,$$

$$q =: \frac{a\eta_1^\alpha + b\eta_2^\alpha + 2}{\zeta\Gamma(\alpha+1)} + \frac{4}{\zeta\Gamma(\alpha-\mu)}.$$

## 3. MAIN RESULT

Our main result of this paper is as follows:

**Theorem 3.1.** *Let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that there exists nonnegative functions $a_1(t), a_2(t) \in C[0,1]$, $m(t) \in L[0,1]$ such that*

$$|f(t, z_1, z_2)| \le m(t) + a_1(t)|z_1|^{\lambda_1} + a_2(t)|z_2|^{\lambda_2},$$

*where $0 < \lambda_i < 1$, for $i = 1, 2$, then the system of (1.3) has a solution.*

*Proof.* In order to use the Schauder fixed-point theorem to prove our main result, we define

$$U = \{u(t) \in X : \|u\|_X \le R, t \in [0,1]\},$$

where $R \ge \max\{3p, (3k_1q)^{\frac{1}{1-\lambda_1}}, (3k_2q)^{\frac{1}{1-\lambda_2}}\}$, $k_1 = \max_{t\in[0,1]} a_1(t)$ and $k_2 = \max_{t\in[0,1]} a_2(t)$. Obviously, $U \subset X$ is a bounded and closed convex set.

As a first step, we prove that $T : U \to U$. Using Lemma 2.3 together with the result $D^\mu t^{\alpha-1} = \frac{\Gamma(\alpha)t^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}$, we obtain

$$|Tu(t)| = \left| \int_0^1 G(t,s) f(s,u(s),D^\mu u(s))\,ds \right| \le$$

$$\le \int_0^1 |G(t,s)m(s)|\,ds +$$

$$+ (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \int_0^1 |G(t,s)|\,ds \le$$

$$\le \int_0^1 |G(t,s)m(s)|\,ds +$$

$$+ (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \Bigg\{ \int_0^t [G_1(t,s) + G_2(t,s)]\,ds +$$

$$+ \int_t^{\eta_1} G_2(t,s)\,ds + \int_{\eta_1}^{\eta_2} G_3(t,s)\,ds + \int_{\eta_2}^1 G_4(t,s)\,ds \Bigg\} =$$

$$= \int_0^1 |G(t,s)m(s)|\,ds + (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \left[ \frac{t^\alpha}{\alpha\Gamma(\alpha)} + \frac{t^{\alpha-1}(a\frac{\eta_1^\alpha}{\alpha} + b\frac{\eta_2^\alpha}{\alpha} + \frac{1}{\alpha})}{\zeta\Gamma(\alpha)} \right] =$$

$$= \int_0^1 |G(t,s)m(s)|\,ds + (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \left[ \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha-1}(a\eta_1^\alpha + b\eta_2^\alpha + 1)}{\zeta\Gamma(\alpha+1)} \right] \le$$

$$\le \int_0^1 |G(t,s)m(s)|\,ds + (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \frac{t^\alpha + t^{\alpha-1}(a\eta_1^\alpha + b\eta_2^\alpha + 1)}{\zeta\Gamma(\alpha+1)} \le$$

$$\le \int_0^1 |G(t,s)m(s)|\,ds + (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \frac{a\eta_1^\alpha + b\eta_2^\alpha + 2}{\zeta\Gamma(\alpha+1)}$$

and

$$
\begin{aligned}
|D^\mu(Tu(t))| &= \Bigg| D^\mu \Bigg[ I^\alpha f(t,u(t),D^\mu u(t)) + \int_0^{\eta_1} \frac{at^{\alpha-1}(\eta_1-s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\,ds + \\
&\quad + \int_0^{\eta_2} \frac{bt^{\alpha-1}(\eta_2-s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\,ds - \int_0^{1} \frac{t^{\alpha-1}(1-s)^{\alpha-1}y(s)}{\zeta\Gamma(\alpha)}\,ds \Bigg]\Bigg| = \\
&= |D^\mu[I^\alpha f(t,u(t),D^\mu u(t)) + \zeta^{-1}at^{\alpha-1}I^\alpha f(\eta_1) + \zeta^{-1}bt^{\alpha-1}I^\alpha f(\eta_2) - \zeta^{-1}t^{\alpha-1}I^\alpha f(1)]| = \\
&= |D^\mu I^\mu I^{\alpha-\mu} f(t,u(t),D^\mu u(t)) + I^\alpha f(\eta_1) D^\mu \zeta^{-1} a t^{\alpha-1} \\
&\quad + I^\alpha f(\eta_2) D^\mu \zeta^{-1} b t^{\alpha-1} - I^\alpha f(1) D^\mu \zeta^{-1} t^{\alpha-1}| = \\
&= |I^{\alpha-\mu} f(t,u(t),D^\mu u(t)) + \frac{a\Gamma(\alpha)t^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} I^\alpha f(\eta_1) + \\
&\quad + \frac{b\Gamma(\alpha)t^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} I^\alpha f(\eta_2) - \frac{\Gamma(\alpha)t^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} I^\alpha f(1)| = \\
&= \Bigg| \frac{1}{\Gamma(\alpha-\mu)} \int_0^t \frac{f(s,u(s),D^\mu u(s))}{(t-s)^{1-\alpha+\mu}}\,ds + \frac{at^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} \int_0^{\eta_1} \frac{f(s,u(s),D^\mu u(s))}{(\eta_1-s)^{1-\alpha}}\,ds + \\
&\quad + \frac{bt^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} \int_0^{\eta_2} \frac{f(s,u(s),D^\mu u(s))}{(\eta_2-s)^{1-\alpha}}\,ds - \frac{t^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} \int_0^{1} \frac{f(s,u(s),D^\mu u(s))}{(1-s)^{1-\alpha}}\,ds \Bigg| \le \\
&\le \frac{1}{\zeta\Gamma(\alpha-\mu)} \Bigg\{ \Bigg[ \int_0^t (t-s)^{\alpha-\mu-1} m(s)\,ds + (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \int_0^t (t-s)^{\alpha-\mu-1}\,ds \Bigg] + \\
&\quad + a\Bigg[ \int_0^{\eta_1} (\eta_1-s)^{\alpha-1} m(s)\,ds + (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \int_0^{\eta_1} (\eta_1-s)^{\alpha-1}\,ds \Bigg] + \\
&\quad + b\Bigg[ \int_0^{\eta_2} (\eta_2-s)^{\alpha-1} m(s)\,ds + (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \int_0^{\eta_2} (\eta_2-s)^{\alpha-1}\,ds \Bigg] + \\
&\quad + \Bigg[ \int_0^{1} (1-s)^{\alpha-1} m(s)\,ds + (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \int_0^{1} (1-s)^{\alpha-1}\,ds \Bigg] \Bigg\} \le \\
&\le \frac{1}{\zeta\Gamma(\alpha-\mu)} \Bigg[ (1+a+b+1) \int_0^1 (1-s)^{\alpha-\mu-1} m(s)\,ds + \\
&\quad + (1+a+b+1)(k_1R^{\lambda_1} + k_2R^{\lambda_2}) \int_0^1 (1-s)^{\alpha-\mu-1}\,ds \Bigg] \le \\
&\le \frac{4}{\zeta\Gamma(\alpha-\mu)} \Bigg[ \int_0^1 (1-s)^{\alpha-\mu-1} m(s)\,ds + (k_1R^{\lambda_1} + k_2R^{\lambda_2}) \int_0^1 (1-s)^{\alpha-\mu-1}\,ds \Bigg] \le \\
&\le \frac{4}{\zeta\Gamma(\alpha-\mu)} \int_0^1 (1-s)^{\alpha-\mu-1} m(s)\,ds + \frac{4(k_1R^{\lambda_1} + k_2R^{\lambda_2})}{\zeta\Gamma(\alpha-\mu)}.
\end{aligned}
$$

Hence,

$$\|Tu\|_X = \max_{t\in[0,1]} |Tu(t)| + \max_{t\in[0,1]} |D^\mu Tu(t)| \leq$$

$$\leq \int_0^1 |G(t,s)m(s)|\,ds + (k_1R^{\lambda_1} + k_2R^{\lambda_2})\frac{a\eta_1^\alpha + b\eta_2^\alpha + 2}{\zeta\Gamma(\alpha+1)} +$$

$$+ \frac{4}{\Gamma(\alpha-\mu)\zeta}\int_0^1 (1-s)^{\alpha-\mu-1}m(s)\,ds +$$

$$+ \frac{4(k_1R^{\lambda_1} + k_2R^{\lambda_2})}{\Gamma(\alpha-\mu)\zeta} \leq$$

$$\leq \int_0^1 |G(t,s)m(s)|\,ds +$$

$$+ \frac{4}{\Gamma(\alpha-\mu)\zeta}\int_0^1 (1-s)^{\alpha-\mu-1}m(s)\,ds +$$

$$+ (k_1R^{\lambda_1} + k_2R^{\lambda_2})\left[\frac{a\eta_1^\alpha + b\eta_2^\alpha + 2}{\zeta\Gamma(\alpha+1)} + \frac{4}{\Gamma(\alpha-\mu)\zeta}\right] \leq$$

$$\leq p + (k_1R^{\lambda_1} + k_2R^{\lambda_2})q \leq$$

$$\leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R.$$

So, we conclude that $\|Tu\|_X \leq R$. Since $Tu$, $D^\mu Tu$ are continuous on [0,1], therefore $T : U \to U$.

Now, we show that $T$ is a completely continuous operator. For this purpose we fix

$$M = \max_{t\in[0,1]} |f(t, u(t), D^\mu u(t))|.$$

For $u \in U$, $t, \tau \in [0,1] (t < \tau)$, we have

$$|Tu(t) - Tu(\tau)| =$$

$$= \Big| \int_0^1 G(t,s) f(s, u(s), D^\mu u(s))\, ds - \int_0^1 G(\tau, s) f(s, u(s), D^\mu u(s))\, ds \Big| \le$$

$$\le M \Big| \int_0^1 (G(t,s) - G(\tau,s))\, ds \Big| \le$$

$$\le M \Big\{ \int_0^t [G_1(\tau,s) - G_1(t,s) + G_2(\tau,s) - G_2(t,s)]\, ds +$$

$$+ \int_t^\tau [G_1(\tau,s) + G_2(\tau,s) - G_2(t,s)]\, ds + \int_\tau^{\eta_1} [G_2(\tau,s) - G_2(t,s)]\, ds +$$

$$+ \int_{\eta_1}^{\eta_2} [G_3(\tau,s) - G_3(t,s)]\, ds + \int_{\eta_2}^1 [G_4(\tau,s) - G_4(t,s)]\, ds \Big\} =$$

$$= M \Big[ \int_0^1 \frac{(\tau^{\alpha-1} - t^{\alpha-1})(1-s)^{\alpha-1}}{\zeta \Gamma(\alpha)}\, ds + \int_0^{\eta_1} \frac{a(\tau^{\alpha-1} - t^{\alpha-1})(\eta_1 - s)^{\alpha-1}}{\zeta \Gamma(\alpha)}\, ds +$$

$$+ \int_0^{\eta_2} \frac{b(\tau^{\alpha-1} - t^{\alpha-1})(\eta_2 - s)^{\alpha-1}}{\zeta \Gamma(\alpha)}\, ds + \int_0^\tau \frac{(\tau - s)^{\alpha-1}}{\Gamma(\alpha)}\, ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds \Big] =$$

$$= M \Big[ \frac{\tau^{\alpha-1} - t^{\alpha-1}}{\zeta \Gamma(\alpha)} \cdot \frac{1}{\alpha} + \frac{\tau^{\alpha-1} - t^{\alpha-1}}{\zeta \Gamma(\alpha)} \cdot \frac{a \eta_1^\alpha}{\alpha} +$$

$$+ \frac{\tau^{\alpha-1} - t^{\alpha-1}}{\zeta \Gamma(\alpha)} \cdot \frac{b \eta_2^\alpha}{\alpha} + \frac{1}{\Gamma(\alpha)} \frac{\tau^\alpha}{\alpha} - \frac{1}{\Gamma(\alpha)} \cdot \frac{t^\alpha}{\alpha} \Big] =$$

$$= M \Big[ \frac{\tau^{\alpha-1} - t^{\alpha-1}}{\zeta \Gamma(\alpha+1)} (1 + a\eta_1^\alpha + b\eta_2^\alpha) + \frac{\tau^\alpha - t^\alpha}{\Gamma(\alpha+1)} \Big]$$

and

$$
\begin{aligned}
&|D^\mu(Tu(t)) - D^\mu(Tu(\tau))| = \\
&= \Big| I^{\alpha-\mu} f(t, u(t), D^\mu u(t) + \frac{at^{\alpha-\mu-1}\Gamma(\alpha)}{\zeta\Gamma(\alpha-\mu)} I^\alpha f(\eta_1) + \\
&\quad + \frac{bt^{\alpha-\mu-1}\Gamma(\alpha)}{\zeta\Gamma(\alpha-\mu)} I^\alpha f(\eta_2) - \frac{t^{\alpha-\mu-1}\Gamma(\alpha)}{\zeta\Gamma(\alpha-\mu)} I^\alpha f(1) + \\
&\quad - I^{\alpha-\mu} f(\tau, u(\tau), D^\mu u(\tau) - \frac{a\tau^{\alpha-\mu-1}\Gamma(\alpha)}{\zeta\Gamma(\alpha-\mu)} I^\alpha f(\eta_1) + \\
&\quad - \frac{b\tau^{\alpha-\mu-1}\Gamma(\alpha)}{\zeta\Gamma(\alpha-\mu)} I^\alpha f(\eta_2) + \frac{\tau^{\alpha-\mu-1}\Gamma(\alpha)}{\zeta\Gamma(\alpha-\mu)} I^\alpha f(1) \Big| = \\
&= \Big| \frac{1}{\Gamma(\alpha-\mu)} \int_0^t \frac{f(s, u(s), D^\mu u(s))}{(t-s)^{1-\alpha+\mu}}\, ds + \frac{at^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} \int_0^{\eta_1} \frac{f(s, u(s), D^\mu u(s))}{(\eta_1-s)^{1-\alpha}}\, ds + \\
&\quad + \frac{bt^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} \int_0^{\eta_2} \frac{f(s, u(s), D^\mu u(s))}{(\eta_2-s)^{1-\alpha}}\, ds - \frac{t^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} \int_0^1 \frac{f(s, u(s), D^\mu u(s))}{(1-s)^{1-\alpha}}\, ds + \\
&\quad - \frac{1}{\Gamma(\alpha-\mu)} \int_0^\tau \frac{f(s, u(s), D^\mu u(s))}{(\tau-s)^{1-\alpha+\mu}}\, ds - \frac{a\tau^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} \int_0^{\eta_1} \frac{f(s, u(s), D^\mu u(s))}{(\eta_1-s)^{1-\alpha}}\, ds + \\
&\quad - \frac{b\tau^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} \int_0^{\eta_2} \frac{f(s, u(s), D^\mu u(s))}{(\eta_2-s)^{1-\alpha}}\, ds + \frac{\tau^{\alpha-\mu-1}}{\zeta\Gamma(\alpha-\mu)} \int_0^1 \frac{f(s, u(s), D^\mu u(s))}{(1-s)^{1-\alpha}}\, ds \Big| \le \\
&\le \frac{M}{\Gamma(\alpha-\mu)} \Big| \int_0^t (t-s)^{\alpha-\mu-1}\, ds - \int_0^\tau (\tau-s)^{\alpha-\mu-1}\, ds \Big| + \\
&\quad + \frac{aM}{\zeta\Gamma(\alpha-\mu)} \Big| (t^{\alpha-\mu-1} - \tau^{\alpha-\mu-1}) \int_0^{\eta_1} (\eta_1-s)^{\alpha-1}\, ds \Big| + \\
&\quad + \frac{bM}{\zeta\Gamma(\alpha-\mu)} \Big| (t^{\alpha-\mu-1} - \tau^{\alpha-\mu-1}) \int_0^{\eta_2} (\eta_2-s)^{\alpha-1}\, ds \Big| + \\
&\quad + \frac{M}{\zeta\Gamma(\alpha-\mu)} \Big| (t^{\alpha-\mu-1} - \tau^{\alpha-\mu-1}) \int_0^1 (1-s)^{\alpha-1}\, ds \Big| = \\
&= \frac{M(\tau^{\alpha-\mu} - t^{\alpha-\mu})}{\Gamma(\alpha-\mu+1)} + \frac{M(\tau^{\alpha-\mu+1} - t^{\alpha-\mu+1})}{\zeta\Gamma(\alpha-\mu)\alpha} (a\eta_1^\alpha + b\eta_2^\alpha + 1).
\end{aligned}
$$

Since the functions $t^{\alpha-\mu}$, $t^{\alpha-\mu-1}$, $t^{\alpha-1}$, $t^\alpha$ are uniformly continuous on $[0, 1]$, therefore it follows from the above estimates that $TU$ is an equicontinuous set. Also, it is uniformly bounded as $TU \subset U$. Thus, we conclude that $T$ is a completely continuous operator. Hence, by the Schauder fixed point theorem, there exists a solution of (1.3). This completes the proof. □

**Example 3.2.** Consider the following four-point boundary value problem

$$\begin{cases} D^{\frac{15}{7}}u(t) = a + (t - \frac{1}{3})^4[(u(t))^{\rho_1} + (D^{\frac{2}{5}}u(t))^{\rho_2}], & t \in [0,1], \\ u(0) = u'(0) = 0, \quad u(1) = \frac{4}{5}u(\frac{1}{3}) + \frac{4}{5}u(\frac{2}{3}), \end{cases} \tag{3.1}$$

where $0 < \rho_1, \rho_2 < 1$ and $a$ is an constant different from 0. Obviously, it follows by Theorem 3.1 that there exists a solution of (3.1).

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