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ON NONLOCAL PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

XiWang Dong, JinRong Wang, Yong Zhou

Abstract. In this paper, we study the existence and uniqueness of solutions to the nonlocal problems for the fractional differential equation in Banach spaces. New sufficient conditions for the existence and uniqueness of solutions are established by means of fractional calculus and fixed point method under some suitable conditions. Two examples are given to illustrate the results.

Keywords: nonlocal problems, fractional differential equations, existence, generalized singular Gronwall inequality, fixed point method.

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1. INTRODUCTION

During the past two decades, fractional differential equations have been proved to be valuable tools in the modelling of many phenomena in various fields of engineering, physics and economics. For more details, one can see the monographs of Kilbas *et al.* [6], Lakshmikantham *et al.* [7], Miller and Ross [8], Podlubny [12]. Very recently, fractional differential equations and optimal controls in Banach spaces are studied by Balachandran *et al.* [3, 4], N'Guérékata [9, 10], Mophou and N'Guérékata [11], Wang *et al.* [13–20], Zhou *et al.* [22–24] and etc.

Throughout this paper, $(X, \|\cdot\|)$ will be a Banach spaces, and $J = [0, T]$, $T > 0$. Let $C(J, X)$ be the Banach space of all continuous functions from J into X with the norm $\|u\|_C := \sup\{\|u(t)\| : t \in J\}$ for $u \in C(J, X)$.

We consider the following nonlocal problems of fractional differential equation

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), \ t \in J, \\ u(0) + g(u) = u_0, \end{cases} \tag{1.1}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $q \in (0,1)$, $f : J \times X \to X$ is strongly measurable with respect to t and is continuous with respect to u. The nonlocal term $g : C(J,X) \to X$ is a given function satisfying some assumptions that will be specified later. The nonlocal condition can be applied in physics with better effect than the classical initial value problem. Nonlocal conditions were initiated by Byszewski [1] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [2] and Deng [5], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

A pioneering work on the existence results of solutions for system (1.1) has been reported by N'Guérékata [9]. Also, N'Guérékata [10] reported that the results in [9] hold only in finite dimensional spaces. In the present paper, we revisit this interesting problem and establish some new existence principles of solutions to the system (1.1) by virtue of fractional calculus and fixed point theorems under some suitable conditions, which extend the results in [9] to infinite dimensional spaces.

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preparation results. In Section 3, we give an important priori estimation of solutions and obtain two main results (Theorems 3.4–3.5), the first result based on Banach contraction principle, the second result based on Krasnoselskii's fixed point theorem. At last, two examples are given to demonstrate the application of our main results.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us recall the following known definitions. For more details see [6].

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \ t > 0, \ \gamma > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0,\infty) \to R$ can be written as

$$^LD^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \ t > 0, \ n-1 < \gamma < n.$$

Definition 2.3. The Caputo derivative of order γ for a function $f : [0, \infty) \to R$ can be written as

$$ {}^{c}D^{\gamma} f(t) = {}^{L}D^{\gamma}\left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \ t > 0, \ n-1 < \gamma < n. $$

Remark 2.4. (i) If $f(t) \in C^n[0, \infty)$, then

$$ {}^{c}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^{(n)}(t), \ t > 0, \ n-1 < \gamma < n. $$

(ii) The Caputo derivative of a constant is equal to zero.
(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Lemma 2.5 (Bochner theorem). *A measurable function $f \colon J \to X$ is Bochner integral if $\|f\|$ is Lebesbuge integrable.*

Lemma 2.6 (Mazur lemma). *If $\mathcal{K}$ is a compact subset of X, then its convex closure $\overline{conv}\mathcal{K}$ is compact.*

Lemma 2.7 (Ascoli-Arzela theorem). *Let $\mathcal{S} = \{s(t)\}$ is a function family of continuous mappings $s : J \to X$. If $\mathcal{S}$ is uniformly bounded and equicontinuous, and for any $t^* \in J$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(t)\} (n = 1, 2, \cdots, t \in J)$ in $\mathcal{S}$.*

Theorem 2.8 (Krasnoselskii). *Let $\mathfrak{B}$ be a closed convex and nonempty subsets of X. Suppose that $\mathcal{L}$ and $\mathcal{N}$ are in general nonlinear operators which map $\mathfrak{B}$ into X such that:*

(1) *$\mathcal{L}x + \mathcal{N}y \in \mathfrak{B}$ whenever $x, y \in \mathfrak{B}$;*
(2) *$\mathcal{L}$ is a contraction mapping;*
(3) *$\mathcal{N}$ is compact and continuous.*

Then there exists $z \in \mathfrak{B}$ such that $z = \mathcal{L}z + \mathcal{N}z$.

To end this section, we collect an important singular type Gronwall inequality which is introduce by Ye *et al.* [21] and can be used in fractional differential equations.

Theorem 2.9 ([21, Theorem 1]). *Suppose $\beta > 0$, $\widetilde{a}(t)$ is a nonnegative function locally integrable on J and $\widetilde{g}(t)$ is a nonnegative, nondecreasing continuous function defined on $\widetilde{g}(t) \le M$, $t \in J$, and suppose $u(t)$ is nonnegative and locally integrable on J with*

$$ u(t) \le \widetilde{a}(t) + \widetilde{g}(t) \int_0^t (t-s)^{\beta-1} u(s) ds, \ t \in J. $$

Then

$$ u(t) \le \widetilde{a}(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\widetilde{g}(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} \widetilde{a}(s) \right] ds, \ t \in J. $$

Remark 2.10. Under the hypothesis of Theorem 2.9, let $\widetilde{a}(t)$ be a nondecreasing function on J. Then we have

$$u(t) \leq \widetilde{a}(t) E_\beta(\widetilde{g}(t)\Gamma(\beta)t^\beta),$$

where E_β is the Mittag-Leffler function defined by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}.$$

3. MAIN RESULTS

We make some following assumptions.

[H1] For any $u \in X$, $f(t,u)$ is strongly measurable with respect to t on J.
[H2] For any $t \in J$, $f(t,u)$ is continuous with respect to u on X.
[H3] For arbitrary $u \in X$, there exists a $a_f > 0$, such that

$$\|f(t,u)\| \leq a_f(1 + \|u\|),$$

and for arbitrary $u \in C(J,X)$, there exists a $a_g \in (0,1)$ such that

$$\|g(u)\| \leq a_g(1 + \|u\|_C).$$

[H4] For arbitrary $u, v \in X$ satisfying $\|u\|, \|v\| \leq \rho$, there exists a constant $L_f(\rho) > 0$, such that

$$\|f(t,u) - f(t,v)\| \leq L_f(\rho)\|u - v\|,$$

and for arbitrary $u, v \in C(J,X)$ there exists a constant $L_g \in (0,1)$, such that

$$\|g(u) - g(v)\| \leq L_g\|u - v\|_C.$$

Now, let us recall the definition of a solution of the system (1.1).

Definition 3.1. A function $u \in C^1(J,X)$ is said to be a solution of the system (1.1) if u satisfies the equation ${}^cD^q u(t) = f(t,u(t))$ a.e. on J, and the condition $u(0) + g(u) = u_0$.

By Definition 2.1–2.3, one can obtain the following lemma.

Lemma 3.2. *If the hypothesis* [H1]–[H3] *hold. A function $u \in C(J,X)$ is a solution of the fractional integral equation*

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))ds, \tag{3.1}$$

if and only if u is a solution of the system (1.1).

Proof. For any $r > 0$ and $u \in B_r = \{u \in C(J, X) : \|u\|_C \leq r\}$, according to [H1]–[H2], $f(t, u(t))$ is measurable function on J. For $t \in J$, we obtain that

$$\int_0^t (t-s)^{q-1}\|f(s,u(s))\|ds \leq \int_0^t (t-s)^{q-1} a_f (1 + \|u(s)\|)ds \leq$$

$$\leq a_f \int\limits_0^t (t-s)^{q-1}ds + a_f \int\limits_0^t (t-s)^{q-1} r ds \leq$$

$$\leq \frac{(1+r)a_f T^q}{q}.$$

Thus $\|(t-s)^{q-1}f(s,u(s))\|$ is Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$ and $u \in B_r$. Then from Bochner's theorem (Lemma 2.5) it follows that $(t-s)^{q-1}f(s,u(s))$ is Bochner integrable with respect to $s \in [0,t]$ for all $t \in J$.

Let $G(\tau,s) = (t-\tau)^{-q}|\tau - s|^{q-1}$. Since $G(\tau,s)$ is a nonnegative, measurable function on $D = [0,t] \times [0,t]$ for $t \in J$, we have

$$\int\limits_0^t \int\limits_0^t G(\tau,s)dsd\tau = \int\limits_D G(\tau,s)dsd\tau = \int\limits_0^t \int\limits_0^t G(\tau,s)d\tau ds$$

and

$$\int\limits_D G(\tau,s)dsd\tau = \int\limits_0^t \int\limits_0^t G(\tau,s)dsd\tau =$$

$$= \int\limits_0^t (t-\tau)^{-q} \int\limits_0^t |\tau - s|^{q-1}dsd\tau =$$

$$= \int\limits_0^t (t-\tau)^{-q} \left(\int\limits_0^\tau (\tau - s)^{q-1}ds \right) d\tau +$$

$$+ \int\limits_0^t (t-\tau)^{-q} \left(\int\limits_\tau^t (s-\tau)^{q-1}ds \right) d\tau \leq$$

$$\leq \frac{2T}{q(1-q)}.$$

Let $G_1(\tau,s) = (t-\tau)^{-q}(\tau-s)^{q-1}$. Note that $\|f(s,u(s))\| \le a_f(1+r)$, therefore, $G_1(\tau,s)f(s,u(s))$ is a Lebesbuge integrable function on D, then we have

$$\int_0^t \int_0^\tau G_1(\tau,s)f(s,u(s))dsd\tau = \int_0^t \int_s^t G_1(\tau,s)f(s,u(s))d\tau ds.$$

We now prove that

$${}^LD^q(I^q f(t,u(t))) = f(t,u(t)), \text{ for } t \in (0,T].$$

Indeed, we have

$${}^LD^q(I^q f(t,u(t))) = \frac{1}{\Gamma(1-q)\Gamma(q)}\frac{d}{dt}\int_0^t (t-\tau)^{-q}\int_0^\tau (\tau-s)^{q-1}f(s,u(s))dsd\tau =$$

$$= \frac{1}{\Gamma(1-q)\Gamma(q)}\frac{d}{dt}\int_0^t\int_0^\tau G_1(\tau,s)f(s,u(s))dsd\tau =$$

$$= \frac{1}{\Gamma(1-q)\Gamma(q)}\frac{d}{dt}\int_0^t\int_s^t G_1(\tau,s)f(s,u(s))d\tau ds =$$

$$= \frac{1}{\Gamma(1-q)\Gamma(q)}\frac{d}{dt}\int_0^t f(s,u(s))ds\int_s^t G_1(\tau,s)d\tau =$$

$$= \frac{d}{dt}\int_0^t f(s,u(s))ds =$$

$$= f(t,u(t)).$$

We claim that $u(t)$ is absolutely continuous on J. For that, for any disjoint family of open intervals $\{(a_i,b_i)\}_{i=1}^n$ on J with $\sum_{i=1}^n (b_i-a_i)\to 0$, we have

$$\begin{aligned}
&\sum_{i=1}^n \frac{1}{\Gamma(q)}\left\| \int_0^{b_i} (b_i-s)^{q-1} f(s,u(s))ds - \int_0^{a_i} (a_i-s)^{q-1} f(s,u(s))ds \right\| \le \\
&\le \sum_{i=1}^n \frac{1}{\Gamma(q)}\left\| \int_{a_i}^{b_i} (b_i-s)^{q-1} f(s,u(s))ds \right\| + \\
&\quad + \sum_{i=1}^n \frac{1}{\Gamma(q)}\left\| \int_0^{a_i} \left((b_i-s)^{q-1}-(a_i-s)^{q-1}\right) f(s,u(s))ds \right\| \le \\
&\le \frac{a_f(1+r)}{\Gamma(q)} \sum_{i=1}^n \int_{a_i}^{b_i} (b_i-s)^{q-1} ds + \\
&\quad + \frac{a_f(1+r)}{\Gamma(q)} \sum_{i=1}^n \int_0^{a_i} \left((a_i-s)^{q-1}-(b_i-s)^{q-1}\right) ds \le \\
&\le \frac{a_f(1+r)}{\Gamma(1+q)} \sum_{i=1}^n (b_i-a_i)^q + \\
&\quad + \frac{a_f(1+r)}{\Gamma(1+q)} \sum_{i=1}^n \left((a_i)^q+(b_i-a_i)^{q-1}-(b_i)^q\right) \le \\
&\le \frac{2a_f(1+r)}{\Gamma(1+q)} \sum_{i=1}^n (b_i-a_i)^q \longrightarrow 0.
\end{aligned}$$

Thus, $u(t)$ is differentiable for almost all $t\in J$. According to the Remark 2.4, we have

$$\begin{aligned}
{}^cD^q u(t) &= {}^c D^q \left[u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))ds \right] = \\
&= {}^c D^q \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))ds \right] = \\
&= {}^c D^q (I^q f(t,u(t))) = \\
&= {}^L D^q (I^q f(t,u(t))) - [I^q f(t,u(t))]_{t=0} \frac{t^{-q}}{\Gamma(1-q)}.
\end{aligned}$$

Since $(t-s)^{q-1} f(s,u(s))$ is Lebesgue integrable with respect to $s\in[0,t]$ for all $t\in J$, we known that $[I^q f(t,u(t))]_{t=0}=0$ which implies that

$${}^cD^q u(t) = f(t,u(t)),\ \text{a.e. } t\in J.$$

Moreover, $u(0)+g(u)=u_0$. Thus, $u\in C(J,X)$ is a solution of system (1.1). On the other hand, if $u\in C(J,X)$ is a solution of system (1.1), then u satisfies the integral equation (3.1). □

In order to derive the existence results, we need important a priori estimation.

Lemma 3.3. *Suppose system* (1.1) *has a solution u on the time interval J. If the hypothesis* [H3] *holds, then there exists a constant $\rho>0$ such that*

$$\|u(t)\|\le\rho \text{ for all } t\in J.$$

Proof. By Lemma 3.2, the solution of system (1.1) is equivalent to the solution of integral equation

$$u(t)=u_0-g(u)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,u(s))ds.$$

According to the hypothesis [H3],

$$\begin{aligned}\|u(t)\|&\le\|u_0-g(u)\|+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\|f(s,u(s))\|ds\le\\ &\le\|u_0-g(u)\|+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}a_f(1+\|u(s)\|)ds\le\\ &\le\|u_0\|+a_g+a_g\|u\|_C+\frac{a_f}{\Gamma(q)}\int_0^t(t-s)^{q-1}ds+\frac{a_f}{\Gamma(q)}\int_0^t(t-s)^{q-1}\|u(s)\|ds,\end{aligned}$$

which implies that

$$(1-a_g)\|u\|_C\le\|u_0\|+a_g+\frac{a_f}{\Gamma(q)}\int_0^t(t-s)^{q-1}ds+\frac{a_f}{\Gamma(q)}\int_0^t(t-s)^{q-1}\|u(s)\|ds.$$

Thus,

$$\|u(t)\|\le\frac{\Gamma(q+1)(\|u_0\|+a_g)+a_fT^q}{(1-a_g)\Gamma(q+1)}+\frac{a_f}{(1-a_g)\Gamma(q)}\int_0^t(t-s)^{q-1}\|u(s)\|ds.$$

Applying the singular type Gronwall inequality (Lemma 2.9),

$$\|u(t)\|\le\frac{\Gamma(q+1)(\|u_0\|+a_g)+a_fT^q}{(1-a_g)\Gamma(q+1)}\sum_{n=0}^{\infty}\frac{(a_fT^q)^n}{\Gamma(nq+1)(1-a_g)^n},$$

where $\sum_{n=0}^{\infty} \frac{(a_f T^q)^n}{\Gamma(nq+1)(1-a_g)^n}$ is just the well known Mittag-Leffler function. Thus, there exists a constant $\rho > 0$ such that

$$\|u(t)\| \leq \rho, \text{ for } t \in J.$$

□

Our first result is based on Banach contraction principle.

Theorem 3.4. *Assume that* [H1]–[H4] *hold. If the following two conditions:*

$$a_g + \frac{a_f T^q}{\Gamma(q+1)} < 1 \tag{3.2}$$

and

$$\Upsilon_{T,q,\rho} = L_g + \frac{L_f(\rho) T^q}{\Gamma(q+1)} < 1 \tag{3.3}$$

hold, then system (1.1) *has an unique solution.*

Proof. Let

$$\rho \geq \frac{\|u_0\| + a_g + \frac{a_f T^q}{\Gamma(q+1)}}{1 - a_g - \frac{a_f T^q}{\Gamma(q+1)}},$$

and define

$$C_\rho = \{x \in C(J, X) : \|u(t)\| \leq \rho, \ t \in J\}. \tag{3.4}$$

Define a operator F: $C_\rho \to C_\rho$ as follows

$$(Fu)(t) = u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds. \tag{3.5}$$

By Lemma 3.2, it is obvious that F is well defined on C_ρ in the sense of Bochner integrable.

We divide our proof into two steps.

Step 1. $Fu \in C_\rho$ for every $u \in C_\rho$.

For every $u \in C_\rho$,

$$\begin{aligned}
\|(Fu)(t+\delta)-(Fu)(t)\| &\le \frac{1}{\Gamma(q)}\int_0^t ((t-s)^{q-1}-(t+\delta-s)^{q-1})\|f(s,u(s))\|ds+\\
&+\frac{1}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}\|f(s,u(s))\|ds\le\\
&\le \frac{a_f}{\Gamma(q)}\int_0^t ((t-s)^{q-1}-(t+\delta-s)^{q-1})(1+\|u(s)\|)ds+\\
&+\frac{a_f}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}(1+\|u(s)\|)ds\le\\
&\le \frac{a_f(1+\rho)}{\Gamma(q)}\left(\frac{t^q}{q}-\frac{(t+\delta)^q}{q}+\frac{\delta^q}{q}\right)+\frac{a_f(1+\rho)}{\Gamma(q)}\frac{\delta^q}{q}\le\\
&\le \frac{2a_f(1+\rho)}{\Gamma(q+1)}\delta^q.
\end{aligned}$$

It is easy to see that the right-hand side of the above inequality tends to zero as $\delta \to 0$. Therefore $Fu \in C(J,X)$.

Moreover, for all $t \in J$, $u \in C_\rho$, due to the condition (3.2),

$$\begin{aligned}
\|Fu(t)\| &\le \|u_0\|+\|g(u)\|+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,u(s))\|ds\le\\
&\le \|u_0\|+a_g(1+\|u\|_C)+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}a_f(1+\|u(s)\|)ds\le\\
&\le \|u_0\|+a_g(1+\rho)+\frac{a_f(1+\rho)}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds\le\\
&\le \|u_0\|+a_g(1+\rho)+\frac{a_f(1+\rho)T^q}{\Gamma(q+1)}\le\rho,
\end{aligned}$$

which implies that $Fu \in C_\rho$.

Step 2. F is a contraction mapping on C_ρ.In fact, for any $u, v \in C_\rho$, we get

$$\begin{aligned}\|(Fu)(t)-(Fv)(t)\| &\leq \|g(u)-g(v)\| + \int\limits_0^t (t-s)^{q-1}\|f(s,u(s))-f(s,v(s))\|ds \leq \\ &\leq L_g\|u-v\|_C + \frac{L_f(\rho)}{\Gamma(q)}\int\limits_0^t (t-s)^{q-1}\|u(s)-v(s)\|ds \leq \\ &\leq \left[L_g + \frac{L_f(\rho)T^q}{\Gamma(q+1)}\right]\|u-v\|_C,\end{aligned}$$

which implies that

$$\|Fu-Fv\|_C \leq \Upsilon_{T,q,\rho}\|u-v\|_C.$$

Thus, F is a contraction mapping on C_ρ due to our condition (3.3). By applying Banach's contraction mapping principle we know that the operator F has a unique fixed point on C_ρ. Therefore, system (1.1) has an unique solution. ☐

Our second result uses the well known Krasnoselskii's fixed point theorem. For that, we make the following assumption.

[H5]: For every $t \in J$, the set $K = \{(t-s)^{q-1}f(s,u(s)) : u \in C(J,X), s \in [0,t]\}$ is relatively compact.

Theorem 3.5. *Assume that* [H1]–[H3] *and* [H5] *hold. If the condition* (3.2) *holds, then system* (1.1) *has at least one solution.*

Proof. We subdivide the operator F defined by (3.5) into two operators P and Q on C_ρ as follows

$$\begin{aligned}(Pu)(t) &= \frac{1}{\Gamma(q)}\int\limits_0^t (t-s)^{q-1}f(s,u(s))ds,\ t\in J,\\ (Qv)(t) &= u_0 - g(v),\ t \in J,\end{aligned}$$

where C_ρ is given by (3.4).

Therefore, the existence of a solution of system (1.1) is equivalent to that the operator $P+Q$ has a fixed point on C_ρ.

The proof is divided into several steps.

Step 1. $Pu + Qv \in C_\rho$ for every pair $u, v \in C_\rho$.

In fact, for every pair $u, v \in C_\rho$,

$$\begin{aligned}\|(Pu)(t) + (Qv)(t)\| &\le \|u_0\| + \|g(v)\| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,u(s))\|ds \le \\ &\le \|u_0\| + a_g(1+\|v\|_C) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}a_f(1+\|u(s)\|)ds \le \\ &\le \|u_0\| + a_g(1+\rho) + \frac{a_f(1+\rho)}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \le \\ &\le \|u_0\| + a_g(1+\rho) + \frac{a_f(1+\rho)T^q}{\Gamma(q+1)} \le \rho,\end{aligned}$$

which implies that $Pu + Qv \in C_\rho$.

Step 2. Q is a contraction mapping on C_ρ.

In fact, for every $v_1, v_2 \in C_\rho$,

$$\|Qv_1 - Qv_2\|_C = \|g(v_1) - g(v_2)\| \le L_g\|v_1 - v_2\|_C.$$

Thus Q is a contraction mapping due to $L_g \in (0,1)$.

Step 3. P is a continuous operator.

For that, let $\{u_n\}$ be a sequence of C_ρ such that $u_n \to u$ in C_ρ. Then, $f(s,u_n(s)) \to f(s,u(s))$ as $n \to \infty$ due to the hypotheses [H2].

Now, for all $t \in J$, we have

$$\|(Pu_n)(t) - (Pu)(t)\| \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,u_n(s)) - f(s,u(s))\|ds.$$

On the one other hand using [H3], we get for each $t \in J$,

$$\|f(s,u_n(s)) - f(s,u(s))\| \le L_f(\rho)\|u_n(s) - u(s)\| \le 2\rho L_f(\rho).$$

On the other hand, using the fact that the functions $s \to 2\rho L_f(\rho)(t-s)^{q-1}$ is integrable on J, by means of the Lebesgue Dominated Convergence Theorem yields

$$\int_0^t (t-s)^{q-1}\|f(s,u_n(s)) - f(s,u(s))\|ds \to 0.$$

Thus, $Pu_n \to Pu$ as $n \to \infty$ which implies that P is continuous.

Step 4. P is a compact operator.

Let $\{u_n\}$ be a sequence on C_ρ, then

$$\|(Pu_n)(t)\| \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,u_n(s))\|ds \leq$$
$$\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}a_f(1+\|u_n(s)\|)ds \leq \frac{(1+\rho)T^q a_f}{\Gamma(q+1)}.$$

Thus, $\{u_n\}$ is uniform boundedness.

Now we prove that $\{Pu_n\}$ is is equicontinuous. For $0 \leq t_1 < t_2 \leq T$, we get

$$\|(Pu_n)(t_1)-(Pu_n)(t_2)\| \leq \frac{1}{\Gamma(q)}\int_0^{t_1}((t_1-s)^{q-1}-(t_2-s)^{q-1})\|f(s,u_n(s))\|ds+$$
$$+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\|f(s,u_n(s))\|ds \leq$$
$$\leq \frac{a_f}{\Gamma(q)}\int_0^{t_1}((t_1-s)^{q-1}-(t_2-s)^{q-1})(1+\|u_n(s)\|)ds+$$
$$+\frac{a_f}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}(1+\|u_n(s)\|)ds \leq$$
$$\leq \frac{a_f(1+\rho)}{\Gamma(q)}\left(\frac{t_1^q}{q}-\frac{t_2^q}{q}+\frac{(t_2-t_1)^q}{q}\right)+$$
$$+\frac{a_f(1+\rho)}{\Gamma(q)}\frac{(t_2-t_1)^q}{q} \leq$$
$$\leq \frac{2a_f(1+\rho)}{\Gamma(q+1)}(t_2-t_1)^q.$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. Therefore $\{Pu_n\}$ is equicontinuous.

In view of the condition [H5] and the Lemma 2.6, we know that $\overline{convK}$ is compact.

For any $t^* \in J$,

$$(Pu_n)(t^*) = \frac{1}{\Gamma(q)}\int_0^{t^*}(t^*-s)^{q-1}f(s,u_n(s))ds =$$
$$= \frac{1}{\Gamma(q)}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}\Big(t^*-\frac{it^*}{k}\Big)^{q-1}f\Big(\frac{it^*}{k},u_n\Big(\frac{it^*}{k}\Big)\Big) =$$
$$= \frac{t^*}{\Gamma(q)}\zeta_n,$$

where

$$\zeta_n = \lim_{k\to\infty} \sum_{i=1}^{k} \frac{1}{k}\Big(t^* - \frac{it^*}{k}\Big)^{q-1} f\Big(\frac{it^*}{k}, u_n\Big(\frac{it^*}{k}\Big)\Big).$$

Since $\overline{convK}$ is convex and compact, we know that $\zeta_n \in \overline{convK}$. Hence, for any $t^* \in J$, the set $\{Pu_n\}$ $(n = 1, 2, \cdots)$ is relatively compact. From Ascoli-Arzela theorem every $\{Pu_n(t)\}$ contains a uniformly convergent subsequence $\{Pu_{n_k}(t)\}$ $(k = 1, 2, \cdots)$ on J. Thus, the set $\{Pu : u \in C_\rho\}$ is relatively compact.

Therefore, the continuity of P and relatively compactness of the set $\{Pu : u \in C_\rho\}$ imply that P is a completely continuous operator. By Krasnoselskii's fixed point theorem, we get that $P + Q$ has a fixed point on C_ρ. Hence system (1.1) has a solution, and this completes the proof. □

4. EXAMPLES

In this section we give two examples to illustrate the usefulness of our main results.

Example 4.1. Let us consider the following nonlocal problem of fractional differential equation

$$\begin{cases} {}^cD^q u(t) = \frac{e^{-t}\rho|u(t)|}{(1+Le^t)(1+|u(t)|)},\ q \in (0,1),\ t \in J = [0,T], \\ u(0) + \sum_{j=1}^{m} \lambda_j u(t_j) = 0,\ 0 < t_1 < t_2 < \cdots < t_m < T, \end{cases} \tag{4.1}$$

where $\rho, L, \lambda_j > 0$, $j = 1, 2, \cdots, m$.

Set

$$f(t,u) = \frac{e^{-t}\rho u}{(1+Le^t)(1+u)},\ (t,u) \in J \times [0,\rho],$$

and

$$g(u) = \sum_{j=1}^{m} \lambda_j u(t_j).$$

Let $u_1, u_2 \in X$ and $t \in J$. Then we have

$$\begin{aligned} |f(t,u_1) - f(t,u_2)| &\le \frac{e^{-t}\rho}{1+Le^t}|u_1 - u_2| \le \\ &\le \frac{\rho}{1+L}|u_1 - u_2|, \end{aligned}$$

and

$$\begin{aligned} |g(u_1) - g(u_2)| &\le \sum_{j=1}^{m} \lambda_j |u_1(t_j) - u_2(t_j)| \le \\ &\le \sum_{j=1}^{m} \lambda_j \max_{t_j\in J} \{|u_1(t_j) - u_2(t_j)|\}. \end{aligned}$$

Obviously, for all $u \in X$ and each $t \in J$,

$$|f(t,u)| \leq \frac{\rho}{1+L}\|u\|,$$

and

$$\begin{aligned} |g(u)| &\leq \sum_{j=1}^{m} \lambda_j |u(t_j)| \leq \\ &\leq \sum_{j=1}^{m} \lambda_j \max_{t_j \in J}\{|u(t_j)|\}. \end{aligned}$$

It is obviously that our assumptions in Theorem 3.4 can be satisfied by choosing a sufficient large $L > 0$ and small enough T and λ_j such that $\sum_{j=1}^{m} \lambda_j + \frac{\rho T^q}{(1+L)\Gamma(q+1)} < 1$ for some $q \in (0,1)$. Therefore, the problem (4.1) has an unique solution.

Example 4.2. Let us consider another nonlocal problem of fractional differential equation

$$\begin{cases} {}^cD^q u(t) = \frac{e^{-vt}|u(t)|}{(1+9e^t)(1+|u(t)|)}, \ v > 0, \ q \in (0,1), \ t \in J = [0,T], \\ u(0) + \sum_{j=1}^{m} \lambda_j u(t_j) = 0, \ 0 < t_1 < t_2 < \cdots < t_m < T. \end{cases} \tag{4.2}$$

Set

$$f_1(t,u) = \frac{e^{-vt}u}{(1+9e^t)(1+u)}, \ (t,u) \in J \times [0,+\infty),$$

and

$$g(u) = \sum_{j=1}^{m} \lambda_j u(t_j), \ \text{where} \ \sum_{j=1}^{m} \lambda_j < 1.$$

Let $v = \frac{1}{t^2}$, $t \in (0,T]$, it is obvious that $\lim_{t\to 0^+} \frac{t^{q-1}}{e^{\frac{1}{t}}} = 0$. As a result, the set $K_1 = \left\{(t-s)^{q-1}\frac{e^{-vs}|u(s)|}{(1+9e^s)(1+|u(s)|)} : u \in C(J), s \in [0,t]\right\}$ is bounded and closed which implies that K_1 is compact. Thus, all the assumptions in Theorem 3.5 are satisfied by choosing a small enough T and λ_j such that $1 - \sum_{j=1}^{m} \lambda_j - \frac{T^q}{10\Gamma(q+1)} > 0$, our results can be applied to the problem (4.2).

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XiWang Dong
jzdxw@yahoo.com.cn

Guizhou University
Department of Mathematics
Guiyang, Guizhou 550025, P.R. China

JinRong Wang
wjr9668@126.com

Guizhou University
Department of Mathematics
Guiyang, Guizhou 550025, P.R. China

Yong Zhou
yzhou@xtu.edu.cn

Xiangtan University
Department of Mathematics
Xiangtan, Hunan 411105, P.R. China

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