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STABILITY OF THE POPOVICIU TYPE FUNCTIONAL EQUATIONS ON GROUPS

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Abstract. We consider the stability problem for a class of functional equations related to the Popoviciu equation.

Keywords: stability, Popoviciu equation, quadratic equation, additive function.

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1. INTRODUCTION

Dealing with some inequality for convex functions, T. Popoviciu [9] has introduced the functional equation

$$3f\Big(\frac{x+y+z}{3}\Big)+f(x)+f(y)+f(z)=2\Big[f\Big(\frac{x+y}{2}\Big)+f\Big(\frac{x+z}{2}\Big)+f\Big(\frac{y+z}{2}\Big)\Big]. \tag{1.1}$$

The solution and stability of (1.1) have been investigated by J. Brzdęk [2], W. Smajdor [12] and T. Trif [10]. The analogous problems for the following generalization of (1.1)

$$m^2f\Big(\frac{x+y+z}{m}\Big)+f(x)+f(y)+f(z)=n^2\Big[f\Big(\frac{x+y}{n}\Big)+f\Big(\frac{x+z}{n}\Big)+f\Big(\frac{y+z}{n}\Big)\Big], \tag{1.2}$$

where m and n are nonzero integers, have been studied in [7] (in the case where $m=3$ and $n=2$) and [8] (in the case where $m+1=2n$). Solutions of the particular case of (1.2), namely

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(x+z)+f(y+z), \tag{1.3}$$

have been considered by P. Kannappan [6]. Stability of (1.3) has been investigated by S.-M. Jung [5] and W. Fechner [4]. Solutions and stability of some further generalization of (1.1) have been investigated by T. Trif [11]. In [3] the general solution of the following functional equation

$$Mf\Big(\frac{x+y+z}{m}\Big)+f(x)+f(y)+f(z)=N\Big[f\Big(\frac{x+y}{n}\Big)+f\Big(\frac{x+z}{n}\Big)+f\Big(\frac{y+z}{n}\Big)\Big], \tag{1.4}$$

where m, n, M, N are arbitrary positive integers, has been determined in the case where the unknown function f maps a commutative group uniquely divisible by m and n into a commutative group uniquely divisible by 2. Let us recall that a group $(X, +)$ is said to be *uniquely divisible* by a given positive integer k provided, for every $x \in X$, there exists a unique $y \in X$ such that $x = ky$; such an element will be denoted in the sequel by $\frac{x}{k}$.

In the present paper we study the stability problem for (1.4) in a similar setting. Our work is inspired by the recent paper [1]. In the sequel we assume that m, n, M, N are positive integers, $(G, +)$ and $(H, +)$ are commutative groups, $(G, +)$ is uniquely divisible by m and n, $(H, +)$ is uniquely divisible by 2 and d is a metric on H such that:

(i) d is invariant with respect to $+$, that is

$$d(u+w, v+w) = d(u, v) \quad \text{for} \quad u, v, w \in H; \tag{1.5}$$

(ii) there exists a $\xi \in (0,1)$ such that

$$d\left(\frac{u}{2}, \frac{v}{2}\right) \leq \xi d(u, v) \quad \text{for} \quad u, v \in H; \tag{1.6}$$

(iii) (H, d) is a complete metric space.

2. RESULTS

We begin this section with the following simple observation.

Remark 2.1. Note that condition (1.5) implies the following two inequalities:

$$d(u+w, v+r) \leq d(u, v) + d(w, r) \quad \text{for} \quad u, v, w, r \in H; \tag{2.1}$$

$$d(-u, -v) = d(u, v) \quad \text{for} \quad u, v \in H. \tag{2.2}$$

In fact, for every $u, v, w, r \in H$, we have

$$d(u+w, v+r) = d(u-v, r-w) \leq d(u-v, 0) + d(0, r-w) = d(u, v) + d(w, r)$$

and

$$d(-u, -v) = d(-u + (u+v), -v + (u+v)) = d(v, u) = d(u, v).$$

Note also that, in view of (2.1), by induction we get

$$d(ku, kv) \leq kd(u, v) \quad \text{for} \quad u, v \in H,\ k \in \mathbb{N}. \tag{2.3}$$

In order to prove the stability result for (1.4) we need to recall that a function $Q : G \to H$ is said to be *quadratic* provided

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \quad \text{for} \quad x, y \in G;$$

and a function $A : G \to H$ is said to be *additive* provided

$$A(x+y) = A(x) + A(y) \quad \text{for} \quad x, y \in G.$$

Theorem 2.2. *Assume that a function $f : G \to H$ satisfies inequality*

$$
\begin{aligned}
&d\Big(Mf\Big(\frac{x+y+z}{m}\Big) + f(x) + f(y) + f(z), \Big[f\Big(\frac{x+y}{n}\Big) + f\Big(\frac{x+z}{n}\Big) + f\Big(\frac{y+z}{n}\Big)\Big]\Big) \leq \\
&\leq \delta(x,y,z) \quad \textit{for} \quad x,y,z \in G,
\end{aligned} \tag{2.4}
$$

where $\delta : G^3 \to [0,\infty)$ is an arbitrary function such that, for some $0 < \eta < \frac{1}{\xi}$, it holds

$$\delta(2x,2y,2z) \leq \eta\delta(x,y,z) \quad \textit{for} \quad x,y,z \in G. \tag{2.5}$$

Then there exists a uniquely determined quadratic function $Q : G \to H$ and an additive function $A : G \to H$ such that

$$(N - n^2)Q(x) = (M - m^2)Q(x) = 0 \quad \textit{for} \quad x \in G, \tag{2.6}$$

$$(Mn + mn - 2mN)A(x) = 0 \quad \textit{for} \quad x \in G \tag{2.7}$$

and

$$
\begin{aligned}
&d(f(x), Q(x) + A(x) + f(0)) \leq \\
&\leq \frac{\xi^4}{1-\xi\eta}[\delta(x,x,0) + \delta(-x,-x,0) + \delta(2x,0,0) + \delta(-2x,0,0) + \\
&\quad + \delta(x,x,-2x) + \delta(-x,-x,2x)] + \frac{\xi^3}{1-\xi^2\eta}[\delta(x,x,-x) + \delta(-x,-x,x) + \\
&\quad + \delta(x,0,-x) + \delta(-x,0,x) + \delta(0,x,0) + \delta(0,-x,0)] + \\
&\quad + \frac{\xi^4}{1-\xi^2\eta}[\delta(2x,0,-2x) + \delta(-2x,0,2x)] + \frac{\xi^3}{1-\xi^2\eta}(3+4\xi)\delta(0,0,0)
\end{aligned} \tag{2.8}
$$

for $x \in G$.

Proof. Let $f_e : G \to H$ and $f_o : G \to H$ be given by

$$f_e(x) := \frac{f(x) + f(-x)}{2} - f(0) \quad \text{for} \quad x \in G$$

and

$$f_o(x) := \frac{f(x) - f(-x)}{2} \quad \text{for} \quad x \in G,$$

respectively. Then f_e is even, f_o is odd, $f_o(0) = f_e(0) = 0$ and

$$f(x) = f_e(x) + f_o(x) + f(0) \quad \text{for} \quad x \in G. \tag{2.9}$$

Applying (2.4) with $x = y = z = 0$, we obtain

$$d((M+3)f(0), 3Nf(0)) \leq \delta(0,0,0). \tag{2.10}$$

Furthermore, by (2.4), we get

$$\begin{aligned}&d\Big(Mf\Big(\frac{-(x+y+z)}{m}\Big)+f(-x)+f(-y)+f(-z),\\&\quad N\Big[f\Big(\frac{-(x+y)}{n}\Big)+f\Big(\frac{-(x+z)}{n}\Big)+f\Big(\frac{-(y+z)}{n}\Big)\Big]\Big)\le\delta(-x,-y,-z)\end{aligned}\tag{2.11}$$

for $x, y, z \in G$. Therefore, taking into account (1.6)–(2.1) and (2.10), from (2.4) and (2.11) we derive that

$$\begin{aligned}&d\Big(Mf_o\Big(\frac{x+y+z}{m}\Big)+f_o(x)+f_o(y)+f_o(z),\\&\quad N\Big[f_o\Big(\frac{x+y}{n}\Big)+f_o\Big(\frac{x+z}{n}\Big)+f_o\Big(\frac{y+z}{n}\Big)\Big]\Big)\le\xi\Delta(x,y,z)\quad\text{for}\quad x,y,z\in G\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}&d\Big(Mf_e\Big(\frac{x+y+z}{m}\Big)+f_e(x)+f_e(y)+f_e(z),\\&\quad N\Big[f_e\Big(\frac{x+y}{n}\Big)+f_e\Big(\frac{x+z}{n}\Big)+f_e\Big(\frac{y+z}{n}\Big)\Big]\Big)\le\xi\big(\Delta(x,y,z)+\frac{1}{2}\Delta(0,0,0)\big)\end{aligned}\tag{2.13}$$

for $x, y, z \in G$, where $\Delta : G^3 \to [0,\infty)$ is given by

$$\Delta(x,y,z)=\delta(x,y,z)+\delta(-x,-y,-z)\quad\text{for}\quad x,y,z\in G.\tag{2.14}$$

Obviously, by (2.5) and (2.14), we get

$$\Delta(2x,2y,2z)\le\eta\Delta(x,y,z)\quad\text{for}\quad x,y,z\in G.\tag{2.15}$$

Moreover, since $f_o(0) = 0$, taking in (2.12) $z = 0$ and then $y = z = 0$, we obtain

$$\begin{aligned}&d\Big(Mf_o(\frac{x+y}{m})+f_0(x)+f_0(y),\\&\quad N\Big[f_0\Big(\frac{x+y}{n}\Big)+f_0\Big(\frac{x}{n}\Big)+f_0\Big(\frac{y}{n}\Big)\Big]\Big)\le\xi\Delta(x,y,0)\quad\text{for}\quad x,y\in G\end{aligned}\tag{2.16}$$

and

$$d\left(Mf_o\left(\frac{x}{m}\right)+f_0(x),2Nf_0\left(\frac{x}{n}\right)\right)\le\xi\Delta(x,0,0)\quad\text{for}\quad x\in G,\tag{2.17}$$

respectively. Making use of (2.2), from (2.17) we derive that

$$d\Big(-Mf_o\Big(\frac{x+y}{m}\Big)-f_0(x+y),-2Nf_0\Big(\frac{x+y}{n}\Big)\Big)\le\xi\Delta(x+y,0,0)\quad\text{for}\ \ x,y\in G.$$

Hence, taking into account (2.16), by (2.1), we get

$$\begin{aligned}&d\left(f_o(x)+f_o(y)-f_0(x+y),N\left[f_0\left(\frac{x}{n}\right)+f_0\left(\frac{y}{n}\right)-f_0\left(\frac{x+y}{n}\right)\right]\right)\le\\&\le\xi(\Delta(x,y,0)+\Delta(x+y,0,0))\quad\text{for}\quad x,y\in G.\end{aligned}$$

On the other hand, as f_0 is odd, from (2.12) we derive that

$$\begin{aligned}
&d\left(f_o(x)+f_o(y)-f_o(x+y), N\left[f_o\left(\frac{x+y}{n}\right)-f_o\left(\frac{x}{n}\right)-f_o\left(\frac{y}{n}\right)\right]\right)=\\
&=d\left(Mf_o\left(\frac{x+y-(x+y)}{m}\right)+f_o(x)+f_o(y)+f_o(-(x+y)),\right.\\
&\qquad\left.N\left[f_o\left(\frac{x+y}{n}\right)+f_o\left(\frac{y-(x+y)}{n}\right)+f_o\left(\frac{x-(x+y)}{n}\right)\right]\right)\leq\\
&\leq \xi\Delta(x,y,-(x+y)) \quad \text{for} \quad x,y\in G.
\end{aligned}$$

Therefore, using (2.1), we get

$$d(2(f_o(x)+f_o(y)-f_0(x+y)),0)\leq \xi[\Delta(x,y,0)+\Delta(x+y,0,0)+\Delta(x,y,-(x+y))]$$

for $x,y\in G$, whence by (1.5) and (1.6), we obtain

$$d(f_0(x+y),f_o(x)+f_o(y))\leq \xi^2[\Delta(x,y,0)+\Delta(x+y,0,0)+\Delta(x,y,-(x+y))]$$

for $x,y\in G$. Note also that in view of (2.15), the function $\chi_0: G^2\to[0,\infty)$ given by

$$\chi_0(x,y):=\xi^2[\Delta(x,y,0)+\Delta(x+y,0,0)+\Delta(x,y,-(x+y))] \quad \text{for} \quad x,y\in G \tag{2.18}$$

satisfies

$$\chi_0(2x,2y)\leq \eta\chi_0(x,y) \quad \text{for} \quad x,y\in G. \tag{2.19}$$

So, applying [1, Corollary 3.2], we conclude that there exists a unique additive function $A: G\to H$ such that

$$d(f_o(x),A(x))\leq \frac{\xi^2}{1-\xi\eta}\chi_0(x,x) \quad \text{for} \quad x\in G. \tag{2.20}$$

Moreover, taking into account (2.3), from (2.20) we deduce that

$$d(f_o(mnx),A(mnx))\leq \frac{\xi^2}{1-\xi\eta}\chi_0(mnx,mnx) \quad \text{for} \quad x\in G, \tag{2.21}$$

$$d(Mf_o(nx),MA(nx))\leq \frac{M\xi^2}{1-\xi\eta}\chi_0(nx,nx) \quad \text{for} \quad x\in G \tag{2.22}$$

and

$$d(2Nf_o(mx),2NA(mx))\leq \frac{2N\xi^2}{1-\xi\eta}\chi_0(mx,mx) \quad \text{for} \quad x\in G. \tag{2.23}$$

Making use of (2.1), from (2.21) and (2.22) we derive that

$$\begin{aligned}
&d(Mf_o(nx)+f_o(mnx),MA(nx)+A(mnx))\leq\\
&\leq \frac{\xi^2}{1-\xi\eta}(\chi_0(mnx,mnx)+M\chi_0(nx,nx)) \quad \text{for} \quad x\in G.
\end{aligned}$$

On the other hand, by (2.17), for every $x \in G$, we get

$$d(Mf_o(nx) + f_o(mnx), 2Nf_o(mx)) \leq \xi\Delta(mnx, 0, 0).$$

Therefore, in view of (2.22), for every $x \in G$, we obtain

$$\begin{aligned} & d((Mn + mn - 2Nm)A(x), 0) = d(MA(nx) + A(mnx), 2NA(mx)) \leq \\ & \leq \frac{\xi^2}{1-\xi\eta}[\chi_0(mnx, mnx) + M\chi_0(nx, nx) + 2N\chi_0(mx, mx)] + \xi\Delta(mnx, 0, 0). \end{aligned}$$

Thus, as A is additive, Δ satisfies (2.15) and χ_0 satisfies (2.19), using (1.5), we get

$$\begin{aligned} & d((Mn + mn - 2Nm)A(x), 0) = d(2^{-k}(Mn + mn - 2Nm)A(2^k x), 0) \leq \\ & \leq (\xi\eta)^k \left[\frac{\xi^2}{1-\xi\eta}(\chi_0(mnx, mnx) + M\chi_0(nx, nx) + 2N\chi_0(mx, mx)) \right. \\ & \quad \left. + \xi\Delta(mnx, 0, 0)\right] \quad \text{for} \quad x \in G,\ k \in \mathbb{N}. \end{aligned}$$

Since $\eta < \frac{1}{\xi}$, this yields (2.7).

Next consider inequality (2.13). Since f_e is even and $f_e(0) = 0$, taking into account (2.13) $z = -x$, we obtain

$$\begin{aligned} & d\Big(Mf_e\Big(\frac{y}{m}\Big) + 2f_e(x) + f_e(y), \\ & \quad N\Big[f_e\Big(\frac{x+y}{n}\Big) + f_e\Big(\frac{x-y}{n}\Big)\Big]\Big) \leq \xi\big(\Delta(x, y, -x) + \frac{1}{2}\Delta(0, 0, 0)\big) \end{aligned} \tag{2.24}$$

for $x, y \in G$. Applying (2.24), with $y = 0$ and next with $x = 0$, we get

$$d\left(2f_e(x), 2Nf_e\left(\frac{x}{n}\right)\right) \leq \xi\left(\Delta(x, 0, -x) + \frac{1}{2}\Delta(0, 0, 0)\right) \tag{2.25}$$

for $y \in G$ and

$$d\left(Mf_e\left(\frac{y}{m}\right) + f_e(y), 2Nf_e\left(\frac{y}{n}\right)\right) \leq \xi\left(\Delta(0, y, 0) + \frac{1}{2}\Delta(0, 0, 0)\right)$$

for $x \in G$, respectively. In view of (1.5), the last two inequalities imply that

$$d\left(f_e(y), Mf_e\left(\frac{y}{m}\right)\right) \leq \xi\left(\Delta(y, 0, -y) + \Delta(0, y, 0) + \Delta(0, 0, 0)\right) \tag{2.26}$$

for $y \in G$. Note also that, by (1.6), from (2.25) we deduce that

$$d\left(f_e(x), Nf_e\left(\frac{x}{n}\right)\right) \leq \xi^2\left(\Delta(x, 0, -x) + \frac{1}{2}\Delta(0, 0, 0)\right) \tag{2.27}$$

for $x \in G$. Therefore, using (1.5) and (2.1), from (2.24), (2.26) and (2.27), we obtain

$$\begin{aligned} & d\left(f_e(x+y) + f_e(x-y), 2f_e(x) + 2f_e(y)\right) \leq \\ & \leq \xi(\Delta(x, y, -x) + \Delta(y, 0, -y) + \Delta(0, y, 0)) + \\ & \quad + \xi^2(\Delta(x+y, 0, -(x+y)) + \Delta(x-y, 0, -(x-y))) + \\ & \quad + (\frac{3}{2}\xi + \xi^2)\Delta(0, 0, 0) \quad \text{for} \quad x, y \in G. \end{aligned}$$

Furthermore, in view of (2.15), the function $\chi_1 : G^2 \to [0, \infty)$ given by

$$\begin{aligned}\chi_1(x,y) := {}&\xi(\Delta(x,y,-x) + \Delta(y,0,-y) + \Delta(0,y,0)) + \\ &+ \xi^2(\Delta(x+y,0,-(x+y)) + \Delta(x-y,0,-(x-y))) + \\ &+ \Big(\frac{3}{2}\xi + \xi^2\Big)\Delta(0,0,0)\end{aligned} \tag{2.28}$$

for $x, y \in G$, satisfies

$$\chi_1(2x, 2y) \le \eta\chi_1(x,y) \quad \text{for} \quad x, y \in G. \tag{2.29}$$

Thus, as $f_e(0) = 0$, applying [1, Corollary 5.2], we obtain that there exists a unique quadratic function $Q : G \to H$ such that

$$d(f_e(x), Q(x)) \le \frac{\xi^2}{1-\xi^2\eta}\chi_1(x,x) \quad \text{for} \quad x \in G. \tag{2.30}$$

Moreover, taking into account (2.3), from (2.26) we derive that

$$\begin{aligned}d(Q(mx), MQ(x)) \le d(f_e(mx), Q(mx)) + d\left(f_e(mx), Mf_e\left(\frac{mx}{m}\right)\right) + \\ + d(Mf_e(x), MQ(x)) \le \frac{\xi^2}{1-\xi^2\eta}[\chi_1(mx,mx) + M\chi_1(x,x)] + \\ + \xi[\Delta(mx,0,-mx) + \Delta(0,mx,0) + \Delta(0,0,0)] \quad \text{for} \quad x \in G.\end{aligned}$$

Since Q is quadratic and the functions Δ and χ_1 satisfy (2.15) and (2.29), respectively, making use of (1.5) and (1.6), from the latter inequality we obtain that

$$\begin{aligned}d((M-m^2)Q(x), 0) = d(4^{-k}(M-m^2)Q(2^k x), 0) \le \\ \le (\xi^2\eta)^k\Big[\frac{\xi^2}{1-\xi^2\eta}(\chi_1(mx,mx) + M\chi_1(x,x)) + \\ + \xi[\Delta(mx,0,-mx) + \Delta(0,mx,0) + \Delta(0,0,0)]\Big]\end{aligned}$$

for every $x \in G$ and $k \in \mathbb{N}$. As $\eta < \frac{1}{\xi} < \frac{1}{\xi^2}$, this means that $(M - m^2)Q(x) = 0$ for $x \in G$. In a similar way, using (2.27), we obtain that $(N - n^2)Q(x) = 0$ for $x \in G$. So, (2.6) holds.

Finally, (1.5), (2.1), (2.9), (2.20) and (2.30) imply that

$$d(f(x), Q(x) + A(x) + f(0)) \le \frac{\xi^2}{1-\xi\eta}\chi_0(x,x) + \frac{\xi^2}{1-\xi^2\eta}\chi_1(x,x)$$

for $x \in G$. Thus, taking into account (2.14), (2.18) and (2.28), after straightforward calculations, we get (2.8). □

From Theorem 2.2 and [3, Theorem 1] we obtain the following stability result for (1.4).

Corollary 2.3. *If $f : G \to H$ satisfies* (2.4) *with δ satisfying* (2.5)*, then there exists a unique solution $F : G \to H$ of* (1.4) *such that*

$$\begin{aligned}
&d(f(x), F(x)) \leq \\
&\leq \frac{\xi^4}{1-\xi\eta}[\delta(x,x,0) + \delta(-x,-x,0) + \delta(2x,0,0) + \delta(-2x,0,0)+ \\
&\quad + \delta(x,x,-2x) + \delta(-x,-x,2x)] + \frac{\xi^3}{1-\xi^2\eta}[\delta(x,x,-x) + \delta(-x,-x,x)+ \\
&\quad + \delta(x,0,-x) + \delta(-x,0,x) + \delta(0,x,0) + \delta(0,-x,0)]+ \\
&\quad + \frac{\xi^4}{1-\xi^2\eta}[\delta(2x,0,-2x) + \delta(-2x,0,2x)] + \frac{\xi^3}{1-\xi^2\eta}(3+4\xi)\delta(0,0,0)
\end{aligned}$$

for $x \in G$.

In the case where $(H, \|\cdot\|)$ is a Banach space and $\delta \in [0,\infty)$, conditions (1.5), (1.6) and (2.5) hold with $\xi = \frac{1}{2}$ and $\eta = 1$. Therefore, from Theorem 2.2 we deduce the following result.

Corollary 2.4. *Assume that $(H, \|\cdot\|)$ is a Banach space, $\delta \in [0,\infty)$ and the function $f : G \to H$ satisfies inequality*

$$\begin{aligned}
&\Big\| Mf\Big(\frac{x+y+z}{m}\Big) + f(x) + f(y) + f(z) - \\
&- N\Big[f\Big(\frac{x+y}{n}\Big) + f\Big(\frac{x+z}{n}\Big) + f\Big(\frac{y+z}{n}\Big)\Big]\Big\| \leq \delta \quad \text{for } x,y,z \in G.
\end{aligned}$$

Then there exists a uniquely determined quadratic function $Q : G \to H$ and an additive function $A : G \to H$ such that (2.6), (2.7) *hold and*

$$\|f(x) - Q(x) - A(x) - f(0)\| \leq \frac{11}{4}\delta \quad \text{for} \quad x \in G.$$

Finally note that if $(H, \|\cdot\|)$ is a Banach space and $\delta : G^3 \to [0,\infty)$ is given by

$$\delta(x,y,z) = \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \quad \text{for} \quad x,y,z \in G,$$

where $\varepsilon \in [0,\infty)$ and $p \in (0,1)$ are fixed, then conditions (1.5), (1.6) and (2.5) hold with $\xi = \frac{1}{2}$ and $\eta = 2^p$. Therefore, applying Theorem 2.2, we obtain the following result.

Corollary 2.5. *Assume that $(H, \|\cdot\|)$ is a Banach space, $\varepsilon \in [0,\infty)$ and $p \in (0,1)$ are fixed and a function $f : G \to H$ satisfies inequality*

$$\begin{aligned}
&\Big\| Mf\Big(\frac{x+y+z}{m}\Big) + f(x) + f(y) + f(z) - \\
&- N\Big[f\Big(\frac{x+y}{n}\Big) + \Big(\frac{x+z}{n}\Big) + f\Big(\frac{y+z}{n}\Big)\Big]\Big\| \leq \\
&\leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \quad \text{for} \quad x,y,z \in G.
\end{aligned}$$

Then there exists a uniquely determined quadratic function $Q : G \to H$ *and an additive function* $A : G \to H$ *such that* (2.6), (2.7) *hold and*

$$\|f(x) - Q(x) - A(x) - f(0)\| \leq \left(\frac{2 + 2^p}{4 - 2^{p+1}} + \frac{6 + 2^p}{4 - 2^p} \right) \varepsilon \|x\|^p \quad \textit{for} \quad x \in G.$$

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