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INVESTIGATING THE NUMERICAL RANGE AND Q-NUMERICAL RANGE OF NON SQUARE MATRICES

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Abstract. A presentation of numerical ranges for rectangular matrices is undertaken in this paper, introducing two different definitions and elaborating basic properties. Further, we extend to the q-numerical range.

Keywords: numerical range, projectors, matrix norms, singular values.

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1. INTRODUCTION

Let $\mathcal{M}_{m,n}(\mathbb{C})$ be the set of matrices $A = [a_{ij}]_{i,j=1}^{m;n}$ with entries $a_{ij} \in \mathbb{C}$. For $m = n$, the set

$$F(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\|_2 = 1\} \tag{1.1}$$

is the well known *numerical range* or *field of values* of A, for which basic properties can be found in [5, 8] and [6, Chapter 22]. Equivalently, we say that $F(A) = f(\mathcal{S}_n)$, where $\mathcal{S}_n$ is the unit sphere of $\mathbb{C}^n$ and the function f on $\mathcal{S}_n$ is defined by the bilinear mapping $g : \mathcal{S}_n \times \mathcal{S}_n \to \mathbb{C}$, such that $f(x) = g(x, x) = \langle Ax, x\rangle$. $F(A)$ is a closed and convex set and contains the spectrum $\sigma(A)$ of A.

Recently in [4], it has been proposed a definition of the numerical range of a matrix $A \in \mathcal{M}_{m,n}$ *with respect to a matrix* $B \in \mathcal{M}_{m,n}$ the compact and convex set

$$w_{\|\cdot\|}(A, B) = \bigcap_{z_0 \in \mathbb{C}} \mathcal{D}(z_0, \|A - z_0 B\|), \tag{1.2}$$

where $\|B\| \geq 1$ and $\|\cdot\|$ denotes any matrix norm. In this paper any special type of matrix norm associated with the vector norm will be followed by the corresponding index [7]. The definition (1.2) is an extension of the definition of $F(A)$ for square matrices in [2, 3] and clearly the numerical range is based on the notion of a matrix norm. In [4] it is proved that $w_{\|\cdot\|}(A,B)$ coincides with the disc in $\mathbb{C}$

$$\mathcal{D}\left(\frac{\langle A,B\rangle}{\|B\|^2}, \left\|A-\frac{\langle A,B\rangle}{\|B\|^2}B\right\|\sqrt{1-\|B\|^{-2}}\right) \tag{1.3}$$

in the case when the matrix norm $\|\cdot\|$ is induced by the inner product $\langle\cdot,\cdot\rangle$.

Another proposal for the definition of the numerical range for rectangular matrices is via the projection onto the lower or the higher dimensional subspace. Let $m > n$ and the vectors $v_1, \dots, v_n$ of $\mathbb{C}^m$ form an orthonormal basis of $\mathbb{C}^n$. Clearly, the matrix $P = HH^*$, where $H = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \in \mathcal{M}_{m,n}$, is an orthogonal projector of $\mathbb{C}^m \longrightarrow \mathbb{C}^n$. In this case, we define the (lower) numerical range of $A \in \mathcal{M}_{m,n}$ *with respect to* H, to be the set:

$$w_l(A) = F(H^*A) = \{\langle Ax, Hx\rangle : x \in \mathbb{C}^n, \|Hx\|_2 = 1\}, \tag{1.4}$$

where obviously H^*A is an $n \times n$ matrix. Moreover, the vector $y = Hx \in \mathbb{C}^m$ is projected onto $\mathbb{C}^n$ along $\mathcal{K}$, where $\mathcal{K}$ is the orthogonal complement of $\mathbb{C}^n$, i.e. $\mathbb{C}^m = \mathbb{C}^n \oplus \mathcal{K}$. Also, in (1.4) the second set has been defined in [1] as a "bioperative" numerical range $W(A,H)$, without requiring H to be an isometry. Since $\|y\|_2 = \langle Hx, Hx\rangle^{1/2} = \|x\|_2$, we can also define in a similar way the (upper) numerical range $w_h(A)$ using the higher dimensional $m \times m$ matrix AH^*. Namely,

$$w_h(A) = F(AH^*). \tag{1.5}$$

Similarly, if $m < n$, then $x = Hy$ and consequently

$$w_l(A) = F(AH), \quad w_h(A) = F(HA). \tag{1.6}$$

It is obvious that, for $m = n$ and $H = I$, $w_l(A)$ and $w_h(A)$ are reduced to the classical numerical range $F(A)$ in (1.1). Some additional properties of these sets are exposed in section 2, including the notion of the sharp point and a relation of $w_l(A), w_h(A)$ and $w_{\|\cdot\|}(A,B)$ for $B = H$.

In the second part of the paper, section 3, we refer to the q-numerical range for a rectangular matrix A and $q \in [0,1]$. First, we extend the notion of the numerical range in [2, 3] to the q-numerical range, considering the algebra of operators on $\mathbb{C}^n$, which is identified with the algebra $\mathcal{M}_n$ of square $n \times n$ complex matrices. Using the nonempty set of linear functionals

$$\mathcal{L}_q = \{f : \mathcal{M}_n \to \mathbb{C} \quad \text{such that} \quad \|f\| = 1, f(I) = q \in [0,1]\}, \tag{1.7}$$

we may well define the *q-numerical range of A* to be the set

$$F_q(A) = \{f(A) : f \in \mathcal{L}_q\}. \tag{1.8}$$

Since any linear functional on $\mathcal{M}_n(\mathbb{C})$ is induced by a unit vector $y \in \mathbb{C}^n$ via $A \mapsto \langle Ax, y\rangle$, such that $\|y\|_2 = 1$ and $\langle x, y\rangle = q \leq 1$ for all unit vectors $x \in \mathbb{C}^n$, the set (1.8) is identified with the set [9–11]

$$F_q(A) = \{\langle Ax, y\rangle : x, y \in \mathbb{C}^n, \|x\|_2 = \|y\|_2 = 1, \langle x, y\rangle = q\}. \tag{1.9}$$

In section 3, we prove that

$$F_q(A) = \bigcap_{z_0 \in \mathbb{C}} \mathcal{D}(qz_0, \|A - z_0 I_n\|_2). \tag{1.10}$$

The set $F_q(A)$ in relation (1.10), when $q \neq 0$, is identified with the well known numerical range $w_{\|\cdot\|_2}(A, \frac{1}{q}I_n)$, defined in [4], of A with respect to the matrix $\frac{1}{q}I_n$. Adapting the arguments in [4] to our purpose and using any matrix norm, the relation (1.10) is extended to the *q-numerical range of* $A \in \mathcal{M}_{m,n}$ *with respect to* $B \in \mathcal{M}_{m,n}$, defining the set

$$w_{\|\cdot\|}(A, B; q) = \bigcap_{z_0 \in \mathbb{C}} \left\{ z \in \mathbb{C} : |z - qz_0| \leq \|A - z_0 B\|\,,\ \|B\| \geq q \right\}, \tag{1.11}$$

where $q \in [0, 1]$. Similarly the q-numerical range of $A \in \mathcal{M}_{m,n}$ with respect to the matrix $B \in \mathcal{M}_{m,n}$ in (1.11) is identified, when $q \neq 0$, with the well known numerical range $w_{\|\cdot\|}(A, \frac{1}{q}B)$ of A with respect to the matrix $\frac{1}{q}B$. A discussion for the case $q = 0$ is considered separately.

The set in (1.11) is compact and convex and for $q = 1$ $w_{\|\cdot\|}(A, B; 1)$ is reduced to the numerical range of A with respect to B in (1.2). Also, in section 3, we outline some basic properties of $w_{\|\cdot\|}(A, B; q)$ and prove that it coincides with a circular disc when the matrix norm $\|\cdot\|$ is induced by the inner product $\langle \cdot, \cdot \rangle$.

2. PROPERTIES

In this section we will study basic properties and the relations between the various numerical ranges for rectangular matrices. We also show that, when the norm is induced by an inner product, the union of the numerical ranges $w_{\|\cdot\|}(A, B)$, as B varies, is the disc $\mathcal{D}(0, \|A\|)$.

Proposition 2.1. *Let* $A, B \in \mathcal{M}_{m,n}$ *such that* $\|B\| \geq 1$. *Then for any matrix norm* $\|\cdot\|$ *induced by an inner product* $\langle \cdot, \cdot \rangle$, *the following statements hold:*

1. $\bigcup_{\|B\| \geq 1} w_{\|\cdot\|}(A, B) = \mathcal{D}(0, \|A\|)$.
2. *If* $rankB = k$ *and* $\|\sigma\|_2 \geq \sqrt{k}$, *where the vector* $\sigma = (\sigma_1, \ldots, \sigma_k)$ *corresponds to the singular values of* B, *then the centers of the discs in* (1.3), $\frac{\langle A,B \rangle}{\|B\|^2} \in \mathcal{D}(0, \|A\|_2)$, *with respect to the Frobenius inner product* $\langle A, B \rangle = tr(B^* A)$.

Proof. 1. By definition (1.2) we have $w_{\|\cdot\|}(A, B) = \cap_{\lambda \in \mathbb{C}} \mathcal{D}(\lambda, \|A - \lambda B\|)$. From this it is immediate that $w_{\|\cdot\|}(A, B) \subseteq \mathcal{D}(0, \|A\|)$ for every $B \in \mathcal{M}_{m,n}$, $\|B\| \geq 1$ and hence $\bigcup_{\|B\| \geq 1} w_{\|\cdot\|}(A, B) \subseteq \mathcal{D}(0, \|A\|)$.

Conversely, let $z \in \mathcal{D}(0, \|A\|)$. Then:

— if $z \neq 0$ then $z \in w_{\|\cdot\|}(A, \frac{1}{z}A)$,

— if $z = 0$ then, using the relation (1.3), $0 \in w_{\|\cdot\|}(A, B)$, where B is taken such that $\langle A, B \rangle = 0$, $\|B\| \geq 1$.

Hence $\mathcal{D}(0, \|A\|) \subseteq \bigcup_{\|B\| \geq 1} w_{\|\cdot\|}(A, B)$.

2. Denoting by $\lambda(\cdot)$ and $\sigma(\cdot)$ the eigenvalues and singular values of matrices, respectively and making use of known inequalities [8, p. 176–177] it follows that

$$\begin{aligned}\frac{|\langle A,B\rangle|}{\|B\|^2} &= \frac{|tr(B^*A)|}{\|B\|^2} = \frac{|\sum\lambda(B^*A)|}{\|B\|^2} \leq \frac{\sum|\lambda(B^*A)|}{\|B\|^2} \leq \frac{\sum\sigma(B^*A)}{\|B\|^2} \leq \\ &\leq \frac{\sum\sigma(B^*)\sigma(A)}{\|B\|^2} \leq \sigma_{\max}(A)\frac{\sum\sigma(B)}{\sum\sigma^2(B)}.\end{aligned} \tag{2.1}$$

Since $\|\sigma\|_2 \geq \sqrt{k}$, then $\sum\sigma^2(B) = \|\sigma\|_2^2 \geq \sqrt{k}\,\|\sigma\|_2 \geq \langle\mathbf{1},\sigma\rangle = \sum\sigma(B)$ and consequently by (2.1),

$$\frac{|\langle A,B\rangle|}{\|B\|^2} \leq \sigma_{\max}(A) = \|A\|_2\,. \qquad \square$$

Example 2.2. If $A = \begin{bmatrix} 6+i & 0 & 1/2 \\ -4 & -3-6i & 0 \end{bmatrix}$ and $\|B\|_F = [tr(B^*B)]^{1/2}$ denotes the Frobenius norm of B, Proposition 2.1 is illustrated in Figure 1, where the drawing discs $w_{\|\cdot\|_F}(A,B)$ in (1.3), for six different matrices B with $\|B\|_F \geq 1$, approximate the disc $\mathcal{D}(0,\|A\|_F)$. The dashed circle is the boundary of the disc $\mathcal{D}(0,\|A\|_2)$.

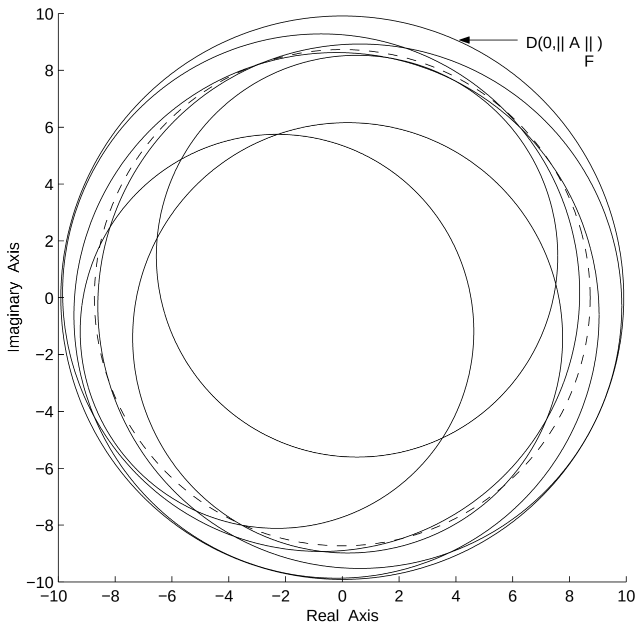

Fig. 1

In (1.4) and (1.5) we referred to the numerical ranges $w_l(A)$ and $w_h(A)$, respectively, for rectangular matrices with respect to an $m \times n$ isometry H $(m > n)$. For these numerical ranges we have the following.

Proposition 2.3. *Let $m > n$, $A \in \mathcal{M}_{m,n}$ and $H \in \mathcal{M}_{m,n}$ be an isometry, then:*

1. $w_l(A) \subseteq w_h(A)$,
2. $\bigcup_H w_l(A) = \bigcup_H w_h(A) = \mathcal{D}(0, \|A\|_2)$,
3. $\sigma(A, H) \subseteq w_l(A) \subseteq \mathcal{D}(0, \|A\|_2)$, *where* $\sigma(A,H) = \big\{\lambda \in \mathbb{C} : Ax = \lambda Hx,\, x \in \mathbb{C}^n \setminus \{0\}\big\}$ *denotes the set of the generalized eigenvalues of A.*

Proof. 1. Let the unitary matrix $U = \begin{bmatrix} H & R \end{bmatrix} \in \mathcal{M}_{m,m}$, where $H \in \mathcal{M}_{m,n}$. Then

$$w_h(A) = F(AH^*) = F(U^*AH^*U) = F\Big(\begin{bmatrix} H^*A & 0 \\ R^*A & 0 \end{bmatrix}\Big)$$

whereupon $w_l(A) = F(H^*A) \subseteq w_h(A)$.

2. Suppose $z \in \bigcup_H w_l(A) = \bigcup_H F(H^*A)$, then for an $m \times n$, isometry H

$$|z| \leq r(H^*A) \leq \|H^*A\|_2 \leq \|H^*\|_2 \, \|A\|_2 = \|A\|_2 \,,$$

where $r(\cdot)$ denotes the numerical radius of a matrix. Thereby, $\bigcup w_l(A) = \bigcup_H F(H^*A) \subseteq \mathcal{D}(0, \|A\|_2)$. On the other side, if $z = y^*Ax \in \mathcal{D}(0, \|A\|_2)$, then there exists an $m \times n$ isometry H such that $y = Hx$ and $z = x^*(H^*A)x \in F(H^*A)$. The assertion $\bigcup w_h(A) = \mathcal{D}(0, \|A\|_2)$ is established similarly.

3. Since $\sigma(A, H) \subseteq \sigma(H^*A, I_n)$ for any $m \times n$ isometry H, we need merely to apply 2. □

Corollary 2.4. *For $A, B \in \mathcal{M}_{m,n}$ with $rankB = n$ and $B = QR$ is the QR-factorization of B, then*

$$W(A,B) = \{\langle Ax, Bx\rangle : x \in \mathbb{C}^n, \|Bx\|_2 = 1\} = F(Q^*AR^{-1}),$$

where $W(A,B)$ is the "bioperative" numerical range defined in [1].

Proof. Obviously,

$$\begin{aligned} W(A,B) &= \{\langle Q^*Ax, Rx\rangle : x \in \mathbb{C}^n, \|Rx\|_2 = 1\} = \\ &= \big\{\big\langle Q^*AR^{-1}\omega, \omega\big\rangle : \omega \in \mathbb{C}^n, \|\omega\|_2 = 1\big\} = F(Q^*AR^{-1}). \end{aligned}$$ □

By definitions (1.4), (1.5) or (1.6) the concept of the *sharp point* (i.e. the boundary point with nonunique tangents [8, p.50]) of $F(AH^*)$ or $F(H^*A)$ is transferred to the sharp point of $w_h(A)$ or $w_l(A)$, respectively. Especially, we note:

Proposition 2.5. *Let $A \in \mathcal{M}_{m,n}$, $m > n$ and $\lambda_0 (\neq 0)$ be a sharp point of $w_h(A)$ with respect to an isometry $H \in \mathcal{M}_{m,n}$. Then $\lambda_0 \in \sigma(H^*A)$ and is also a sharp point of $w_l(A)$ with respect to H.*

Proof. For the sharp point $\lambda_0 \in \partial w_h(A) = \partial F(AH^*)$ with $H^*H = I_n$ apparently, $\lambda_0 \in \sigma(AH^*) = \sigma(U^*AH^*U) = \sigma(H^*A) \cup \{0\}$, for the unitary matrix $U = \begin{bmatrix} H & R \end{bmatrix} \in \mathcal{M}_{m,m}$, i.e. $\lambda_0 \in \sigma(H^*A) \subseteq F(H^*A) = w_l(A)$.

Moreover, for λ_0, according to the definition of the sharp point, there exist $\theta_1, \theta_2 \in [0, 2\pi)$, $\theta_1 < \theta_2$ such that

$$\mathrm{Re}(e^{i\theta}\lambda_0) = \max\big\{\mathrm{Re}a : a \in e^{i\theta} w_h(A)\big\}$$

for all $\theta \in (\theta_1, \theta_2)$. Since $w_h(A) \supseteq w_l(A)$ we have

$$\mathrm{Re}(e^{i\theta}\lambda_0) = \max_{a \in e^{i\theta} w_h(A)} \mathrm{Re}a \geq \max_{b \in e^{i\theta} w_l(A)} \mathrm{Re}b$$

for all $\theta \in (\theta_1, \theta_2)$.

Furthermore, for every $\theta \in (\theta_1, \theta_2)$

$$\mathrm{Re}(e^{i\theta}\lambda_0) \in \mathrm{Re}(e^{i\theta}F(H^*A)) \leq \max\left\{\mathrm{Re}b : b \in e^{i\theta}F(H^*A)\right\}$$

and thus $\mathrm{Re}(e^{i\theta}\lambda_0) = \max\left\{\mathrm{Re}b : b \in e^{i\theta}F(H^*A)\right\}$ for all $\theta \in (\theta_1, \theta_2)$, concluding that $\lambda_0 (\neq 0)$ is a sharp point of $F(H^*A) = w_l(A)$. □

It is noticed here that the converse of Proposition 2.5 does not hold as is illustrated in the next example.

Example 2.6. If $A = \begin{bmatrix} 1+i & -7 & 0 \\ 5i & 0.02 & 0 \\ 0 & 0 & 6-i \\ 0 & 0 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 0 \\ I_3 \end{bmatrix}$, $\lambda_0 = 5i$ is a sharp point of $w_l(A)$ but not of $w_h(A)$. Note that in Figure 2 '$*$' denote the eigenvalues 0 and $5i$ of AH^*.

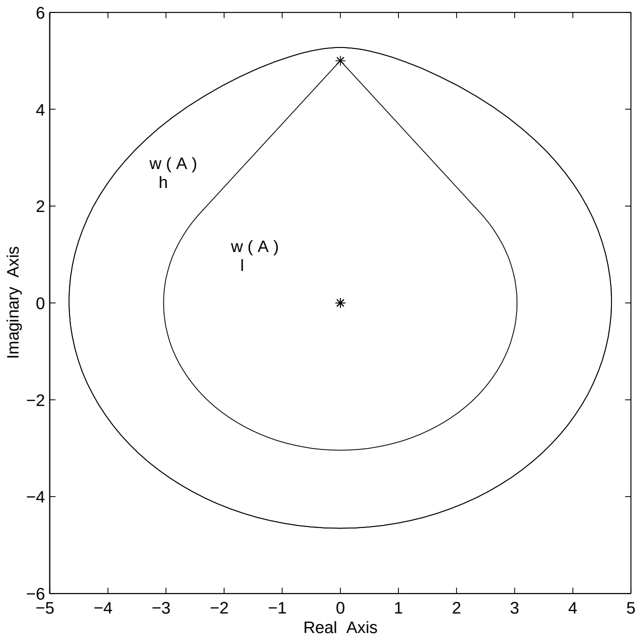

Fig. 2

Finally, we present the next inclusion relation.

Proposition 2.7. *Let $m > n$ and $A, H \in \mathcal{M}_{m,n}$ with H an isometry. Then*

$$w_l(A) \subseteq w_{\|\cdot\|_2}(A, H) \subseteq w_h(A).$$

Proof. Using (1.4) the definition of the numerical range presented in [2] and equation (1.2) we have

$$\begin{aligned} w_l(A) = F(H^*A) &= \bigcap_{z_0\in\mathbb{C}} \{z : |z - z_0| \le \|H^*A - z_0 I_n\|_2\} = \\ &= \bigcap_{z_0\in\mathbb{C}} \{z : |z - z_0| \le \|H^*A - z_0 H^*H\|_2\} \subseteq \\ &\subseteq \bigcap_{z_0\in\mathbb{C}} \{z : |z - z_0| \le \|H^*\|_2 \, \|A - z_0 H\|_2\} = \\ &= \bigcap_{z_0\in\mathbb{C}} \{z : |z - z_0| \le \|A - z_0 H\|_2\} = w_{\|\cdot\|_2}(A, H). \end{aligned}$$

For the second inclusion we have

$$\begin{aligned} w_{\|\cdot\|_2}(A, H) &= \bigcap_{z_0\in\mathbb{C}} \{z : |z - z_0| \le \|A - z_0 H\|_2\} = \\ &= \bigcap_{z_0\in\mathbb{C}} \{z : |z - z_0| \le \|AH^*H - z_0 H\|_2\} \subseteq \\ &\subseteq \bigcap_{z_0\in\mathbb{C}} \{z : |z - z_0| \le \|AH^* - z_0 I_m\|_2 \, \|H\|_2\} = \\ &= \bigcap_{z_0\in\mathbb{C}} \{z : |z - z_0| \le \|AH^* - z_0 I_m\|_2\} = F(AH^*) = w_h(A). \qquad \square \end{aligned}$$

Combining Propositions 2.3(2) and 2.7, we obtain the following immediate corollary.

Corollary 2.8. *Let $m > n$ and $A, H \in \mathcal{M}_{m,n}$ with H an isometry. Then*

$$\bigcup_H w_{\|\cdot\|_2}(A, H) = \mathcal{D}(0, \|A\|_2).$$

Example 2.9. For $A = \begin{bmatrix} 5i & -1 & i \\ 2-7i & 0 & 4 \\ -1 & 2 & 0.3 \\ 0.5 & 5 & 1 \end{bmatrix}$, Proposition 2.7 is illustrated in Figure 3. The unshaded area approximates the set $w_{\|\cdot\|_2}(A, H)$, whereas the thin and the bold curve inside and outside this area, depict the boundaries of $w_l(A)$ and $w_h(A)$, respectively, with respect to the isometry

$$H = \begin{bmatrix} -0.1856 - 0.1899i & -0.2828 - 0.2242i & 0.8783 - 0.1527i \\ -0.4251 + 0.0565i & 0.6297 + 0.1258i & 0.1929 - 0.0260i \\ 0.2363 + 0.0733i & 0.1885 + 0.6418i & 0.2993 + 0.0222i \\ 0.8152 - 0.1407i & 0.0806 - 0.0589i & 0.1362 - 0.2426i \end{bmatrix}.$$

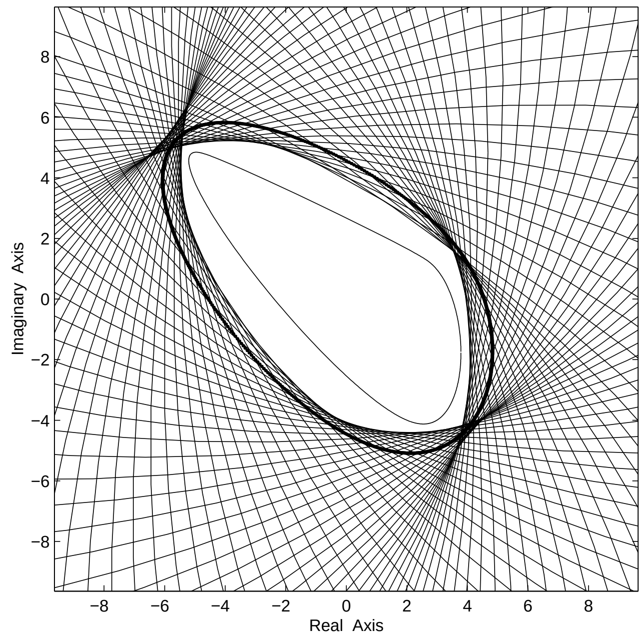

Fig. 3

3. THE Q-NUMERICAL RANGE

First, we present the q-numerical range of square $n \times n$ matrices A in (1.8) as an intersection of circular discs, yielding its convexity. Particularly, we obtain a generalization of Lemma 6.22.1 in [3] for the algebra of matrices $\mathcal{M}_n$ endowed with the matrix norm. This result also applies to the infinite dimensional case of operators.

Proposition 3.1. *Let $A \in \mathcal{M}_n$ and $F_q(A)$ as in* (1.8)*, then*

$$F_q(A) = \bigcap_{z_0 \in \mathbb{C}} \mathcal{D}(qz_0, \|A - z_0 I_n\|),$$

where $\|\cdot\|$ is any matrix norm.

Proof. We denote $\Omega = \bigcap_{z_0 \in \mathbb{C}} \{z \in \mathbb{C} : |z - qz_0| \leq \|A - z_0 I_n\|\}$. If $\gamma \in F_q(A)$ in (1.8), then there exists a linear functional $f \in \mathcal{L}_q$ such that $\gamma = f(A)$. Thus

$$|\gamma - qz_0| = |f(A - z_0 I_n)| \leq \|f\| \, \|A - z_0 I_n\| = \|A - z_0 I_n\| \, ,$$

for every $z_0 \in \mathbb{C}$. Hence $\gamma \in \Omega$.

For the opposite inclusion $\supseteq$, let $\gamma \in \Omega$. If A is a scalar matrix, i.e. $A = cI_n$, $c \in \mathbb{C}$, then $|\gamma - qc| \leq \|A - cI_n\| = 0 \Rightarrow \gamma = qc$. Thus, for any linear functional $f \in \mathcal{L}_q$, we have $\gamma = cf(I_n) = f(cI_n) = f(A) \in F_q(A)$.

On the other hand, selecting $B \in \mathcal{M}_n$ such that $\{I_n, A, B\}$ are linearly independent and $\|B\| \leq 1$, we consider the subspace $\mathcal{X} = span\,\{I_n, A, B\}$. We then define the linear functional $\tilde{f} : \mathcal{X} \to \mathbb{C}$ such that

$$\tilde{f}(c_1 I_n + c_2 A + c_3 B) = c_1 q + c_2 \gamma + c_3, \quad c_1, c_2, c_3 \in \mathbb{C}.$$

Hence,

$$\begin{aligned} |\gamma - qz_0| \leq \|A - z_0 I_n\| &\quad \forall\, z_0 \in \mathbb{C} \Rightarrow \\ |\tilde{f}(A) - z_0 \tilde{f}(I_n)| \leq \|A - z_0 I_n\| &\quad \forall\, z_0 \in \mathbb{C} \Rightarrow \\ |\tilde{f}(A - z_0 I_n)| \leq \|A - z_0 I_n\| &\quad \forall\, z_0 \in \mathbb{C} \Rightarrow \quad \|\tilde{f}\| \leq 1. \end{aligned}$$

Since, $(c_1, c_2, c_3) = (0, 0, 1) \Rightarrow 1 = |\tilde{f}(B)| \leq \|\tilde{f}\|\,\|B\| \leq \|\tilde{f}\|$, clearly we have $\|\tilde{f}\| = 1$, whereas for $(c_1, c_2, c_3) = (1, 0, 0)$ and $(c_1, c_2, c_3) = (0, 1, 0)$ we obtain $\tilde{f}(I_n) = q$ and $\tilde{f}(A) = \gamma$, respectively. Due to the Hahn-Banach theorem, $\tilde{f}$ can be extended to a linear functional $f : \mathcal{M}_n \to \mathbb{C}$ such that $\|f\| = \|\tilde{f}\| = 1$ and $f_{|\mathcal{X}} = \tilde{f}$. Consequently, $f \in \mathcal{L}_q$ such that $f(A) = \gamma \in F_q(A)$. □

Remark 3.2. In case of the spectral norm $\|\cdot\|_2 = \langle \cdot, \cdot \rangle^{1/2}$ on $\mathbb{C}^n$, the q-numerical range defined by (1.8) is expressed with an inner product, due to the Riesz representation theorem. In fact, as we have noticed in the definition (1.8), the set (1.8) is identified with (1.9). This discussion along with the preceding proposition imply a new description of the q-numerical range (1.9) for *square* matrices, namely,

$$F_q(A) = \bigcap_{z_0 \in \mathbb{C}} \mathcal{D}(qz_0, \|A - z_0 I_n\|_2) = \{\langle Ax, y \rangle : \|x\|_2 = \|y\|_2 = 1, \langle x, y \rangle = q\}.$$

Following the analogous idea developed in [4], we extend the notion of the q-numerical range to rectangular matrices. That is,

$$w_{\|\cdot\|}(A, B; q) = \bigcap_{z_0 \in \mathbb{C}} \mathcal{D}(qz_0, \|A - z_0 B\|),$$

with respect to any matrix norm $\|\cdot\|$ and any matrix B such that $\|B\| \geq q$, where $q \in [0, 1]$. In the next proposition, we outline some basic properties of $w_{\|\cdot\|}(A, B; q)$.

Proposition 3.3. *For $A \in \mathcal{M}_{m,n}$, the following conditions hold:*

1. $w_{\|\cdot\|}(A, B; q) = w_{\|\cdot\|}(A, \frac{B}{q})$ *for* $q \neq 0$,
2. $w_{\|\cdot\|}(c_1 A + c_2 B, B; q) = c_1 w_{\|\cdot\|}(A, B; q) + c_2 q$ *for every* $c_1, c_2 \in \mathbb{C}$,
3. $q_1 w_{\|\cdot\|}(A, B; q_2) \subseteq q_2 w_{\|\cdot\|}(A, B; q_1)$ *for* $0 < q_1 \leq q_2 \leq 1$.

Proof. 1. By definition, for $q \neq 0$

$$\begin{aligned} w_{\|\cdot\|}(A, B; q) &= \bigcap_{\lambda \in \mathbb{C}} \{z : |z - q\lambda| \le \|A - \lambda B\|\,,\ \|B\| \ge q\} = \\ &= \bigcap_{\lambda \in \mathbb{C}} \left\{z : |z - q\lambda| \le \left\|A - q\lambda \frac{B}{q}\right\|,\ \frac{\|B\|}{q} \ge 1\right\} = \\ &= \bigcap_{\mu \in \mathbb{C}} \left\{z : |z - \mu| \le \left\|A - \mu \frac{B}{q}\right\|,\ \frac{\|B\|}{q} \ge 1\right\} = w_{\|\cdot\|}\Big(A, \frac{B}{q}\Big). \end{aligned}$$

2. By statement 1 and Proposition 8 in [4], for $q \neq 0$ and any $c_1, c_2 \in \mathbb{C}$ we have

$$\begin{aligned} w_{\|\cdot\|}(c_1 A + c_2 B, B; q) &= w_{\|\cdot\|}\Big(c_1 A + c_2 B, \frac{B}{q}\Big) = w_{\|\cdot\|}\Big(c_1 A + c_2 q \frac{B}{q}, \frac{B}{q}\Big) = \\ &= c_1 w_{\|\cdot\|}\Big(A, \frac{B}{q}\Big) + c_2 q = c_1 w_{\|\cdot\|}(A, B; q) + c_2 q. \end{aligned}$$

If $q = 0$ and $c_1 \neq 0$, then for any c_2:

$$\begin{aligned} w_{\|\cdot\|}(c_1 A + c_2 B, B; 0) &= \bigcap_{z_0 \in \mathbb{C}} \{z : |z| \le \|c_1 A + c_2 B - z_0 B\|\} = \\ &= \bigcap_{z_0 \in \mathbb{C}} \left\{z : \left|\frac{z}{c_1}\right| \le \left\|A - \frac{z_0 - c_2}{c_1} B\right\|\right\} = \\ &= \bigcap_{\mu \in \mathbb{C}} \{c_1 u : |u| \le \|A - \mu B\|\} = c_1 w_{\|\cdot\|}(A, B; 0). \end{aligned}$$

If $c_1 = q = 0$ then it is immediate (take $z_0 = c_2$) that $w_{\|\cdot\|}(c_2 B, B; 0) = \{0\}$.

3. It is proved in [4] that $w_{\|\cdot\|}(A, B) \subseteq b^{-1} w_{\|\cdot\|}(A, b^{-1} B)$, when $|b| < 1$. According to this and 1, since $0 < \frac{q_1}{q_2} < 1$, it is

$$\frac{q_1}{q_2} w_{\|\cdot\|}(A, B; q_2) = \frac{q_1}{q_2} w_{\|\cdot\|}\Big(A, \frac{B}{q_2}\Big) \subseteq w_{\|\cdot\|}\Big(A, \frac{q_2}{q_1}\frac{B}{q_2}\Big) = w_{\|\cdot\|}\Big(A, \frac{B}{q1}\Big) = w_{\|\cdot\|}(A, B; q_1).$$

This statement is an analogue for the q-numerical range in [9]. □

Proposition 3.4. *Let $A, B \in \mathcal{M}_{m,n}$ with $\|B\| \ge q$, $q \in [0, 1]$ and the matrix norm is induced by the inner product $\langle \cdot, \cdot \rangle$. Then*

$$w_{\|\cdot\|}(A, B; q) = \mathcal{D}\left(q \frac{\langle A, B\rangle}{\|B\|^2}, \left\|A - \frac{\langle A, B\rangle}{\|B\|^2} B\right\| \frac{\sqrt{\|B\|^2 - q^2}}{\|B\|}\right)$$

and even $w_{\|\cdot\|}(A, 0; 0) = \mathcal{D}(0, \|A\|)$.

Proof. For $q \in (0, 1]$ the relation is verified readily by combining Proposition 3.3(1) and Proposition 13 in [4]. For the case $q = 0$, we have

$$\begin{aligned} w_{\|\cdot\|}(A, B; 0) &= \bigcap_{z_0 \in \mathbb{C}} \{z : |z| \le \|A - z_0 B\|, \ \ \|B\| \ge 0\} = \\ &= \left\{ z : |z| \le \min_{z_0 \in \mathbb{C}} \|A - z_0 B\|, \ \ \|B\| \ge 0 \right\}. \end{aligned} \tag{3.1}$$

Using the translation property of the q-numerical range (Proposition 3.3(2)) for the matrix $C = A - \frac{\langle A,B \rangle}{\|B\|^2} B$ and Lemma 11 in [4], we obtain

$$w_{\|\cdot\|}(A, B; 0) = w_{\|\cdot\|}(C, B; 0)$$

with $\langle C, B \rangle = 0$. Hence, by (3.1) and non zero B,

$$\begin{aligned} w_{\|\cdot\|}(A, B; 0) &= \left\{ z : |z| \le \min_{z_0 \in \mathbb{C}} \|C - z_0 B\| \right\} = \\ &= \left\{ z : |z| \le \min_{\rho \in \mathbb{R}_+} \sqrt{\|C\|^2 + \rho^2 \|B\|^2} \right\} = \\ &= \{z : |z| \le \|C\|\} = \mathcal{D}\Big(0, \|A - \frac{\langle A, B \rangle}{\|B\|^2} B\|\Big). \end{aligned}$$

Since $\|B\| = 0 \Leftrightarrow B = 0$, by (3.1) we have $w_{\|\cdot\|}(A, 0; 0) = \mathcal{D}(0, \|A\|)$. □

In the next example we present the set $w_{\|\cdot\|}(A, B; q)$ in (1.11) with respect to two different types of matrix norm.

Example 3.5. $A = \begin{bmatrix} i & 2-i & -1 & 0.5 \\ -1 & 0 & -3 & 2 \\ 4 & 0.3i & 0 & 0.6 \end{bmatrix}$, $B = \begin{bmatrix} 0.65 & 0 & 0 & 0 \\ 0 & 0.5 - 0.35i & 0 & 0 \\ 0 & 0 & 0.1 & 0.75i \end{bmatrix}$, where $\|B\|_1 = 0.75$ and $\|B\|_2 = 0.7566$. The $w_{\|\cdot\|_1}(A, B; 0.5)$ and $w_{\|\cdot\|_2}(A, B; 0.5)$ are illustrated in Figure 4 and in Figure 5, respectively.

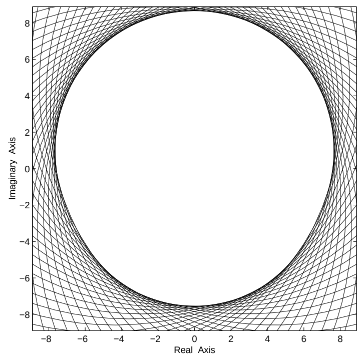

Fig. 4

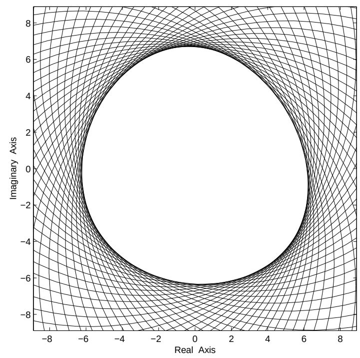

Fig. 5

By Proposition 3.3(1), we obtain an analogous statement of Proposition 2.1(1).

Proposition 3.6. *Let $A, B \in \mathcal{M}_{m,n}$ such that $\|B\| \geq q$, where the matrix norm is induced by an inner product and $0 \leq q \leq 1$. Then*

$$\bigcup_{\|B\| \geq q, 0 \leq q \leq 1} w_{\|\cdot\|}(A, B; q) = \mathcal{D}(0, \|A\|).$$

Proof. We have

$$\bigcup_{\|B\| \geq q, 0 < q \leq 1} w_{\|\cdot\|}(A, B; q) = \bigcup_{\frac{\|B\|}{q} \geq 1} w_{\|\cdot\|}\Big(A, \frac{B}{q}\Big) = \bigcup_{\|\Gamma\| \geq 1} w_{\|\cdot\|}(A, \Gamma) = \mathcal{D}(0, \|A\|),$$

where $\Gamma = \frac{B}{q}$, for all $B \in \mathcal{M}_{m,n}$ and all $q \in (0, 1]$ such that $\|\Gamma\| \geq 1$.

For $q = 0$, $B \neq 0$, using Proposition 3.4, we have

$$\bigcup_{B \neq 0} w_{\|\cdot\|}(A, B; 0) = \bigcup_{B \neq 0} \mathcal{D}\Big(0, \Big\|A - \frac{\langle A, B\rangle}{\|B\|^2} B\Big\|\Big) \subseteq \mathcal{D}(0, \|A\|).$$

Hence, since $w_{\|\cdot\|}(A, 0; 0) = \mathcal{D}(0, \|A\|)$, we get the result. □

Especially, for square matrices we have the following.

Proposition 3.7. *Let $A \in \mathcal{M}_n$ and $\mathcal{G} = \{B \in \mathcal{M}_n : rank B = 1, tr BB^* = 1, tr B = q, q \in [0, 1]\}$. If the matrix norm $\|\cdot\|$ is induced by the Frobenius inner product, then*

$$\bigcup_{B \in \mathcal{G}} w_{\|\cdot\|}(A, B) = F_q(A),$$

with $F_q(A)$ in (1.9).

Proof. Since $B \in \mathcal{G}$, we write $B = yx^*$ with $\|x\|_2 \|y\|_2 = 1$ and $\langle x, y\rangle = q$. Hence, by (1.3)

$$w_{\|\cdot\|}(A, B) = \mathcal{D}(\langle A, yx^*\rangle, 0) = \frac{y^* A x}{\|y\|_2 \|x\|_2}$$

and consequently for $\hat{x} = x / \|x\|_2$, $\hat{y} = y / \|y\|_2$ we obtain

$$\bigcup_{B \in \mathcal{G}} w_{\|\cdot\|}(A, B) = \{\hat{y}^* A \hat{x} : \|\hat{x}\|_2 = \|\hat{y}\|_2 = 1, \hat{y}^* \hat{x} = q\} = F_q(A).$$ □

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