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ON THE BOCHNER SUBORDINATION OF EXIT LAWS

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Abstract. Let $\mathbb{P} = (P_t)_{t\geq 0}$ be a sub-Markovian semigroup on $L^2(m)$, let $\beta = (\beta_t)_{t\geq 0}$ be a Bochner subordinator and let $\mathbb{P}^\beta = (P_t^\beta)_{t\geq 0}$ be the subordinated semigroup of $\mathbb{P}$ by means of β, i.e. $P_s^\beta := \int_0^\infty P_r\, \beta_s(dr)$. Let $\varphi := (\varphi_t)_{t>0}$ be a $\mathbb{P}$-exit law, i.e.

$$P_t\varphi_s = \varphi_{s+t}, \qquad s,t > 0$$

and let $\varphi_t^\beta := \int_0^\infty \varphi_s\, \beta_t(ds)$. Then $\varphi^\beta := (\varphi_t^\beta)_{t>0}$ is a $\mathbb{P}^\beta$-exit law whenever it lies in $L^2(m)$. This paper is devoted to the converse problem when β is without drift. We prove that a $\mathbb{P}^\beta$-exit law $\psi := (\psi_t)_{t>0}$ is subordinated to a (unique) $\mathbb{P}$-exit law φ (i.e. $\psi = \varphi^\beta$) if and only if $(P_t u)_{t>0} \subset D(A^\beta)$, where $u = \int_0^\infty e^{-s}\psi_s ds$ and A^β is the $L^2(m)$-generator of $\mathbb{P}^\beta$.

Keywords: sub-Markovian semigroup, exit law, subordinator, Bernstein function, Bochner subordination.

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1. INTRODUCTION

Let $\mathbb{P} = (\mathbb{P}_t)_{t\geq 0}$ be a sub-Markovian semigroup on $L^2(m)$. A $\mathbb{P}$-exit law is a family $\varphi = (\varphi_t)_{t>0}$ of $L^2_+(m)$ satisfying the functional equation

$$P_s\varphi_t = \varphi_{s+t}, \qquad s,t > 0. \tag{1.1}$$

This notion is first introduced by Dynkin (cf. [5] and the related references) in the framework of potential theory without Green function. Moreover, the representation of potentials in terms of exit laws, allows explicit formulas for the energy and for the capacity (cf. [5, 6] and the related references).

Now, let $\beta = (\beta_t)_{t\geq 0}$ be a Bochner subodinator, that is a vaguely continuous convolution semigroup of sub-probability measures on $[0, +\infty[$. Let $\mathbb{P}^\beta$ be the subordinated semigroup of $\mathbb{P}$ by means of β, i.e

$$P_t^\beta f := \int_0^\infty P_s f\, \beta_t(ds), \qquad f \in L^2(m), t > 0. \tag{1.2}$$

If $(\varphi_t)_{t>0}$ is a $\mathbb{P}$-exit law then the family $\varphi^\beta := (\varphi_t^\beta)_{t>0}$ defined by

$$\varphi_t^\beta := \int_0^\infty \varphi_s \, \beta_t(ds), \qquad t > 0 \tag{1.3}$$

is a $\mathbb{P}^\beta$-exit law whenever it belongs to $L^2(m)$.

Conversely, let ψ be a $\mathbb{P}^\beta$-exit law. Does there exist a $\mathbb{P}$-exit law φ such that $\psi = \varphi^\beta$? Following [1], there are non trivial examples and counterexamples for this problem. Moreover, the converse is proved in many papers, under some additional hypotheses on $\mathbb{P}$ (cf. [1, 7–9]) and for exit laws ψ such that $\int_0^\infty \psi_t \, dt \in L^2(m)$. However, this condition is not satisfied in many interesting situations and for an important class of potentials (cf. Remark 3.2 below).

In the present paper, we solve the converse problem for general contraction semi-groups $\mathbb{P}$ and under a natural condition on ψ, namely

$$\int_s^\infty \psi_t \, dt \in L^2(m), \qquad s > 0. \tag{1.4}$$

More precisely, we suppose that β is without drift and we prove the following result: A $\mathbb{P}^\beta$-exit law ψ is subordinated to a (unique) $\mathbb{P}$-exit law φ if and only if

$$P_t u \in D(A^\beta), \qquad t > 0, \tag{1.5}$$

where $u := \int_0^\infty e^{-s} \, \psi_s ds$ and A^β is the L^2-generator of $\mathbb{P}^\beta$.
We give also many applications of this result.

2. PRELIMINARIES

Let $(E, \mathcal{E})$ be a standard measurable space and let m be a σ-finite positive measure on $(E, \mathcal{E})$. We denote by $L^2(m)$ the Hilbert space of square integrable (classes of) functions defined on E, by $\|.\|_2$ the associated norm and by $L^2_+(m)$ the positive elements of $L^2(m)$. In the sequel, equality and inequality holds always m-a.e. (i.e. almost everywhere with respect to m). In this paper we denote by ε_x the Dirac measure at point $x \in E$.

In this section we summarize some known results (cf. [4–6] and [13, 15, 16]).

2.1. EXIT EQUATION

A bounded operator $N : L^2(m) \to L^2(m)$ is said to be *sub-Markovian* if

$$(f \in L^2(m); \ 0 \le f \le 1) \quad \Rightarrow \quad (0 \le Nf \le 1).$$

In this case, N can be extended to a pseudo-kernel on $(E, \mathcal{E})$ with respect to the class of m-negligible sets. According to a regularization theorem ([4, Ch. XIII, §43]), we can assume that N is a sub-Markovian kernel (i.e. $N1 \leq 1$) on $(E, \mathcal{E})$. Therefore, we can apply the potential theory defined by kernels (cf. [4] for example), for such operators.

A *sub-Markovian semigroup* on E is a family $\mathbb{P} := (P_t)_{t\geq 0}$ of sub-Markovian bounded operators on $L^2(m)$ such that $P_0 = I$ the identity on E,

1. $P_s P_t = P_{s+t}$ for $s, t \geq 0$ (*semigroup property*),
2. $\|P_t f\|_2 \leq \|f\|_2$ for $f \in L^2(m)$ and $t \geq 0$ (*contraction property*),
3. $\lim_{t\to 0} \|P_t f - f\|_2 = 0$ for $f \in L^2(m)$ (*strong continuity*).

Let $\mathbb{P}$ be a sub-Markovian semigroup on E. The associated $L^2(m)$-*generator* A is defined by

$$Af := \lim_{t\to 0} \frac{1}{t}(P_t f - f)$$

on its domain $D(A)$ which is the set of all functions $f \in L^2(m)$ for which this limit exists in $L^2(m)$. It is known that:

1. A is closed and $D(A)$ is dense in $L^2(m)$.
2. If $u \in D(A)$ then $P_t u \in D(A)$ and $A(P_t u) = P_t A u$, for each $t > 0$.

Let $\mathbb{P}$ be a sub-Markovian semigroup on E. A $\mathbb{P}$-*exit law* is a family $\varphi := (\varphi_t)_{t>0}$ of positive elements in $L^2(m)$ such that the following *exit equation* holds

$$P_s \varphi_t = \varphi_{s+t}, \qquad s, t > 0. \tag{2.1}$$

For each $f \in L^2_+(m)$, the family $(P_t f)_{t>0}$ is a $\mathbb{P}$-exit law which is said to be *closed*. However, exit laws are not necessarily closed in general.

Example 2.1. Let $E = \mathbb{R}^d$ and $m(dx) = dx$ be the Lebesgue measure on $\mathbb{R}^d$, $d \geq 1$. A convolution semigroup on $\mathbb{R}^d$ is a family $\mu = (\mu_t)_{t\geq 0}$ of subprobability measures on $\mathbb{R}^d$ such that $\mu_0 = \varepsilon_0$ the Dirac measure at 0,

1. $\mu_s * \mu_t = \mu_{s+t}$ for $s, t > 0$,
2. $\lim_{t\to 0} \mu_t = \mu_0$ vaguely.

It is known that each convolution semigroup μ induces a sub-Markovian semigroup $\mathbb{P}$ on $\mathbb{R}^d$ by the formula $P_t f := \mu_t * f$ for $t \geq 0$ and $f \in L^2(m)$ (cf. [2, Proposition 12.7] for example).

A particular case is the *Brownian* semigroup. It is induced by the convolution semigroup $\mu_t := \phi_t \cdot m$, where

$$\phi_t(x) = (4\pi t)^{-d/2} \exp\Big(-\frac{\|x\|^2}{4t}\Big), \quad t > 0, x \in \mathbb{R}^d$$

is the *Gaussian* function. By the classical Chapmann-Kolmogorov equation, we have $\phi_s * \phi_t = \phi_{s+t}$ for $s, t > 0$. Hence $(\phi_t)_{t>0}$ is a $\mathbb{P}$-exit law. Following [10, Corollary 4.5], $(\phi_t)_{t>0}$ is not closed.

2.2. BOCHNER SUBORDINATOR

For the following standard notions, we will refer to [2, 3] and [13, 14]. For each bounded measure τ on $[0,\infty[$, $\mathcal{L}$ denotes its Laplace transform, i.e. $\mathcal{L}(\tau)(r) := \int_0^\infty \exp(-rs)\,\tau(ds)$ for $r > 0$.

A *Bochner subordinator* is a convolution semigroup $\beta = (\beta_t)_{t\geq 0}$ on $\mathbb{R}$ such that, for each $t > 0$, $\beta_t \neq \varepsilon_0$ and β_t is supported by $[0,\infty[$.
The associated *Bernstein function* ℓ is defined by the Laplace transform $\mathcal{L}(\beta_t)(r) = \exp(-t\ell(r))$ for $r, t > 0$. Following [2, Theorem 9.8], ℓ admits the representation

$$l(r) = a + br + \int\limits_0^\infty (1 - \exp(-rs))\,\nu(ds), \qquad r > 0,$$

where $a, b \geq 0$ and ν is a σ-finite measure on $]0,\infty[$ verifying $\int_0^\infty \frac{s}{s+1}\,\nu(ds) < \infty$. The constants a, b are the *diffusion* and the *drift* coefficients of β, ν is the *Levy* measure of β.

In this paper, we will be interested in subordinators without drift, i.e.

$$l(t) = a + \int\limits_0^\infty (1 - e^{-ts})\,\nu(ds), \qquad t > 0. \tag{2.2}$$

Classical examples are: *Compound Poisson subordinators, One sided stable subordinators, Gamma subordinators,...*. We will refer to [2,3] and [14,15] for more details.

3. SUBORDINATION

The following notion of subordination is introduced by Bochner (cf. [2,3,13–15] and the related references).

3.1. SUBORDINATED EXIT LAW

Let $\mathbb{P}$ be a sub-Markovian semigroup on E and let β be a Bochner subordinator. For every $t \geq 0$ and for every $u \in L^2(m)$, we may define

$$P_t^\beta u := \int\limits_0^\infty P_s u\, \beta_t(ds). \tag{3.1}$$

Then $\mathbb{P}^\beta := (P_t^\beta)_{t\geq 0}$ is a sub-Markovian semigroup on E. It is said to be *subordinated* to $\mathbb{P}$ in the sense of Bochner by means of β. We denote by A^β be the generator of $\mathbb{P}^\beta$. The proof of the following classical result is given in [13, p. 269].

Lemma 3.1. *$D(A)$ is a subset of $D(A^\beta)$ and if β is without drift, then*

$$A^\beta u = -au + \int\limits_0^\infty (P_t u - u)\,\nu(dt), \qquad u \in D(A), \tag{3.2}$$

where a and ν are given in (2.2).

For each $\mathbb{P}$-exit law $\varphi := (\varphi_t)_{t>0}$, we define the family $\varphi^\beta := (\varphi_t^\beta)_{t>0}$ by

$$\varphi_t^\beta := \int_0^\infty \varphi_s \, \beta_t(ds), \qquad t > 0. \tag{3.3}$$

If $(\varphi_t^\beta)_{t>0} \subset L^2(m)$, then $\varphi^\beta := (\varphi_t^\beta)_{t>0}$ defined by (3.3) is a $\mathbb{P}^\beta$-exit law. φ^β is said to be subordinated to φ by means of β.

Conversely, let ψ be a $\mathbb{P}^\beta$-exit law. Does there exist a $\mathbb{P}$-exit law φ such that $\psi = \varphi^\beta$?

Remark 3.2. This problem is first studied in [1] for sub-Markovian semigroups defined by convolution semigroups on $\mathbb{R}^d$. In particular, non trivial examples and counterexamples are given in [1]. Moreover, for others particular semigroups, the converse is proved in [7] for the symmetric case, in [9] for the sector condition case and in [8] for the absolutely continuous case. However, in all these papers, it is supposed that the exit law ψ satisfy the condition $\int_0^\infty \psi_t \, dt \in L^2(m)$.

Consider, the Gaussian exit law ϕ given in Example 2.1. It is known that $\int_0^\infty \phi_t(x) \, dt = C/\|x\|^{d-2}$ which is not in $L^2(m)$. Hence, this standard example proves that the last condition is restrictive. But for the same example, we have $\int_s^\infty \phi_t(x) \, dt \in L^2(m)$ for each $s > 0$. Therefore the condition (1.4) seems to be natural and convenient.

The proofs of the following useful results are straightforward.

Lemma 3.3. *Let $\delta = (\delta_t)_{t\geq 0}$ be the subordinator defined by $\delta_t := e^{-t}\,\beta_t$. Then $D(A^\delta) = D(A^\beta)$ and $A^\delta u = A^\beta u - u$ whenever $u \in D(A^\beta)$.*

Lemma 3.4. *Let $\varphi = (\varphi_t)_{t>0} \subset L^2(m)$ and satisfying* (2.1) *and let $u = \int_0^\infty \varphi_t^\delta \, dt$. Then $(P_t u)_{t>0} \subset D(A^\delta)$ and*

$$A^\delta P_t u = -\varphi_t, \qquad t > 0. \tag{3.4}$$

3.2. THE BOUNDED CASE

In this paragraph, we suppose that the Bernstein function ℓ of β is bounded. Therefore, β is necessary without drift since $b = \lim_{t\to\infty} \ell(t)/t = 0$ (cf. [2, p. 66]).

Theorem 3.5. *Let $\mathbb{P}$ be a sub-Markovian semigroup and let β be a Bochner subordinator with bounded Bernstein function. Let ψ be a $\mathbb{P}^\beta$-exit law and let $u := \int_0^\infty e^{-t}\psi_t \, dt$. If $(P_t u)_{t>0} \subset L^2(m)$, then ψ is subordinated to a unique $\mathbb{P}$-exit law φ. Moreover φ is given explicitly by*

$$\varphi_t = (a+1) P_t u + \int_0^\infty (P_t u - P_{s+t} u) \, \nu(ds), \qquad t > 0. \tag{3.5}$$

Proof. Let ℓ be the Bernstein function of β and let κ be the subprobability measure defined on $[0,\infty[$ by $\kappa(F) = \int_0^\infty e^{-s}\,\beta_s(F)\,ds$ for any Borel subset of $[0,\infty[$. Then

$$0 \le \ell(r) = a + \int_0^\infty (1-\exp(-rs))\nu(ds) =$$

$$= a + \nu(]0,\infty[) - \int_0^\infty \exp(-rs)\nu(ds).$$

Hence, there exists a signed measure $\rho := (a+1+\nu(]0,\infty[))\varepsilon_0 - \nu$ on $]0,\infty[$ such that $\ell+1 = \mathcal{L}(\rho)$. Furthermore

$$\mathcal{L}(\rho * \kappa) = \mathcal{L}(\rho)\ \mathcal{L}(\kappa) = \frac{\ell+1}{\ell+1} = 1 = \mathcal{L}(\varepsilon_0).$$

By the injectivity of the Laplace transform, we deduce that $\rho * \kappa = \varepsilon_0$.

Let ψ be a $\mathbb{P}^\beta$-exit law and let $t>0$. Since $P_t u \in L^2(m)$ then

$$P_t u = \int_0^\infty P_s(P_t u)\,\varepsilon_0(ds) = \int_0^\infty P_{s+t}u\,(\rho*\kappa)(ds) =$$

$$= \int_0^\infty P_{s+t+r}u\,\rho(dr)\,\kappa(ds) = \int_0^\infty P_t\varphi_s\,\kappa(ds),$$

where

$$\varphi_t := \int_0^\infty P_{r+t}u\rho(dr) = \int_0^\infty P_{r+t}u((a+1+\nu(]0,\infty[))\varepsilon_0 - \nu)(dr) =$$

$$= (a+1)P_t u + \int_0^\infty P_t u\nu(dr) - \int_0^\infty P_{r+t}u\ \nu(dr) =$$

$$= (a+1)P_t u + \int_0^\infty (P_t u - P_{r+t}u)\nu(dr).$$

Hence (3.5) holds. Since ℓ is bounded then $\varphi_t \in L^2(m)$ by the contraction property of $\mathbb{P}$. From (3.5) and the semigroup property of $\mathbb{P}$, we deduce that φ satisfies the exit equation (2.1).

Now, by the definition of κ, (2.1) and (3.1), we have also

$$P_t u = \int_0^\infty P_t\varphi_s\,\kappa(ds) = \int_0^\infty\int_0^\infty e^{-r}\,P_t\varphi_s\,\beta_r(ds)dr = \int_0^\infty P_r^\delta\varphi_t\,dr.$$

From Lemma 3.4 we deduce that $\varphi_t = -A^\delta P_t u$ for each $t > 0$. On the other hand, since $P_s^\delta P_t = P_t P_s^\delta$ and $P_s^\delta u = \int_s^\infty e^{-r}\,\psi_r\,dr$, we get

$$P_s^\delta(P_t u) - P_t u = -\int_0^s e^{-r}\,P_t\psi_r\,dr.$$

Therefore,

$$\varphi_t = \lim_{s\to 0}\frac{1}{s}\int_0^s e^{-r}\,P_t\psi_r\,dr, \qquad t > 0.$$

Hence φ_t is a positive function as limit of positive functions. Consequently $(\varphi_t)_{t>0}$ is a $\mathbb{P}$-exit law. Finally $P_t^\delta u = \int_t^\infty e^{-s}\,\psi_s\,ds = \int_t^\infty e^{-s}\,\varphi_s^\beta\,ds$ for all $t > 0$ and by derivation we get $\psi_s = \varphi_s^\beta$. □

3.3. THE GENERAL CASE

The following two useful results are proved in [13, p. 308–309].

Lemma 3.6. *Let ℓ be a Bernstein function and let $g_n(x) = \frac{nx}{n+x}$, $x > 0, n \in \mathbb{N}^*$. Then there exists a sequence $(\gamma_n)_n$ of sub-probability measures on $[0,\infty[$ such that*

$$\mathcal{L}(\gamma_n) = \frac{g_n \circ \ell}{\ell}, \qquad n \in \mathbb{N}^* \tag{3.6}$$

and $\gamma_n \to \varepsilon_0$ weakly as $n \to \infty$.

Lemma 3.7. *Let $\mathbb{P}$ be a sub-Markovian semigroup and let $(\gamma_n)_{n\in\mathbf{N}}$ be the sequence of measures given by (3.6). For each $u \in L^2(m)$, the sequence of functions defined by*

$$u_n := \int_0^\infty P_s u\,\gamma_n(ds), \qquad n \in \mathbb{N}^* \tag{3.7}$$

converges strongly to u.

Proposition 3.8. *Let $\mathbb{P}$ be a sub-Markovian semigroup and let β be a Bochner subordinator without drift. Let ψ be a $\mathbb{P}^\beta$-exit law and let $u := \int_0^\infty \psi_t\,dt$. Suppose that*

$$P_t^\beta u \in L^2(m) \qquad (t > 0) \tag{3.8}$$

and

$$P_t u \in D(A^\beta) \qquad (t > 0). \tag{3.9}$$

Then ψ is subordinated to a unique $\mathbb{P}$-exit law.

Proof. Let ψ be a $\mathbb{P}^\beta$-exit law, since $P_t^\beta u \in L^2(m)$ then $P_t^\beta u = \int_t^\infty \psi_s\, ds$ and therefore $\psi_t = -A^\beta P_t^\beta u$. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by (3.7), then by Fubini's Theorem

$$P_t^\beta u_n = \int_0^\infty P_r(P_t^\beta u)\,\gamma_n(dr).$$

Since $P_t^\beta u \in L^2(m)$, it follows from [13, Theorem 4.3.17 and Corollary 4.3.18] that $P_t^\beta u_n \in D(A^\beta)$. Let $\psi_t^n = -A^\beta P_t^\beta u_n$, since $(P_s u)_{s>0} \subset D(A^\beta)$, we get

$$\begin{aligned}\psi_t^n &= -\int_0^\infty A^\beta P_s u_n\, \beta_t(ds) = \\ &= -\int_0^\infty A^\beta \int_0^\infty P_r(P_s u)\,\gamma_n(dr)\,\beta_t(ds) = -\int_0^\infty\int_0^\infty P_r(A^\beta P_s u)\,\gamma_n(dr)\,\beta_t(ds).\end{aligned}$$

Hence

$$\psi_t^n = \int_0^\infty P_r \varphi_t^\beta\, \gamma_n(dr),$$

where $\varphi_t := -A^\beta P_t u$. On the other hand, since $P_s^\beta P_t = P_t P_s^\beta$ and $P_s^\beta u = \int_s^\infty \psi_r\, dr$, we get

$$P_s^\beta(P_t u) - P_t u = -\int_0^s P_t \psi_r\, dr.$$

Therefore,

$$\varphi_t = \lim_{s\to 0} \frac{1}{s} \int_0^s P_t\psi_r\, dr, \qquad t > 0.$$

Hence φ_t is a positive function as limit of positive functions. Consequently $\varphi = (\varphi_t)_{t>0}$ is a $\mathbb{P}$-exit law. By the Fubini's Theorem we get

$$P_s\psi_t^n = \int_0^\infty P_r(P_s\varphi_t^\beta)\,\gamma_n(dr).$$

From (2.1), we have for all $s, t > 0$

$$P_s\varphi_t^\beta = \int_0^\infty P_s\varphi_r\, \beta_t(dr) = \int_0^\infty P_r\varphi_s\, \beta_t(dr) = P_t^\beta \varphi_s \in L^2(m).$$

Consequently, by Lemma 3.7, we get

$$\lim_{n\to\infty} P_s\psi_t^n = P_s\varphi_t^\beta, \qquad t > 0.$$

But $P_s\psi_t^n = A^\beta P_s P_t^\beta u_n$ for all $s,t>0$. Since $P_t^\beta u_n \to P_t^\beta u$ as $n\to\infty$ by Lemma 3.7, then the closeness of A^β implies that

$$\lim_{n\to\infty} P_s\psi_t^n = A^\beta P_s P_t^\beta u = P_s\psi_t.$$

The uniqueness of the limit yields $P_s\psi_t = P_s\varphi_t^\beta$ and by integration with respect to β, we obtain $P_s^\beta\psi_t = P_s^\beta\varphi_t^\beta$ for all $s,t>0$. Finally, from (2.1) we have

$$\psi_{s+t} = \varphi_{s+t}^\beta, \qquad s,t>0$$

and consequently the result holds. □

Remark 3.9. If $u=\int_0^\infty \psi_t\,dt \in L^2(m)$ then (3.8) is fulfilled. However, for the most important examples, $u\notin L^2(m)$ (cf. Remark 3.2).

Theorem 3.10. *Let $\mathbb{P}$ be a sub-Markovian semigroup and let β be a Bochner subordinator without drift. Let ψ be a $\mathbb{P}^\beta$-exit law and let $u=\int_0^\infty e^{-s}\psi_s ds$. Then ψ is subordinated to a (unique) $\mathbb{P}$-exit law if and only if*

$$P_t u \in D(A^\beta), \qquad t>0. \tag{3.10}$$

Proof. Let ψ be a $\mathbb{P}^\beta$-exit law and let $\xi_t := e^{-t}\psi_t$, then $\xi := (\xi_t)_{t>0}$ is a $\mathbb{P}^\delta$-exit law where $\delta=(\delta_t)_{t>0}$ is the Bochner subordinator defined by $\delta_t = e^{-t}\beta_t$. Now, for each $t>0$

$$P_t^\delta u = e^{-t}K^\beta\psi_t \in L^2(m).$$

Hence, if $(P_t u)_{t>0}\subset D(A^\beta)=D(A^\delta)$ by Lemma 3.3, then Proposition 3.8 may be applied for $\mathbb{P}^\delta$ and ξ: There exists a unique $\mathbb{P}$-exit law φ such that $\xi=\varphi^\delta$, then

$$e^{-t}\psi_t = \xi_t = \int_0^\infty \varphi_s\,\delta_t(ds) = \int_0^\infty \varphi_s\,e^{-t}\beta_t(ds)$$

and consequently $\psi_t = \varphi_t^\beta$ for each $t>0$.

Conversely, suppose that $\psi=\varphi^\beta$ for some $\mathbb{P}$-exit law φ. Using the preceding notations, we get $\xi=\varphi^\delta$. Then, Lemma 3.4 implies that $(P_t u)_{t>0}\subset D(A^\delta)=D(A^\beta)$. Therefore (3.10) holds. □

3.4. SOME APPLICATIONS

In the sequel, we denote by $\langle .,. \rangle$ the inner product of $L^2(m)$.

Corollary 3.11. *Let $\mathbb{P}$ be a sub-Markovian semigroup satisfying the sector condition, i.e. there exists a constant $M>0$ such that*

$$|\langle -Av,w\rangle| \le M\,\langle -Av,v\rangle^{1/2}\cdot\langle -Aw,w\rangle^{1/2}, \qquad v,w\in D(A)$$

and let β be a Bochner subordinator without drift. Let ψ be a $\mathbb{P}^\beta$-exit law and let $u=\int_0^\infty e^{-t}\psi_t dt$. If $(P_t u)_{t>0}\subset L^2(m)$, then ψ is subordinated to a unique $\mathbb{P}$-exit law φ.

Proof. Following [6, p. 292], the sector condition implies that $P_t(L^2(m)) \subset D(A)$ and therefore $P_t(L^2(m)) \subset D(A) \subset D(A^\beta)$ by Lemma 3.1. By the semigroup property $P_t u = P_{t/2}(P_{t/2}u)$ and therefore $(P_t u)_{t>0} \subset D(A^\beta)$. We conclude by Theorem 3.10 and Lemma 3.1 again. $\square$

Remark 3.12. If $\mathbb{P}$ is m-symmetric, i.e. $\langle P_t v, w\rangle = \langle v, P_t w\rangle$ for $t > 0$ and $v, w \in L^2(m)$, then the sector condition is trivially fulfilled (for $M = 1$).

Example 3.13. Let $\mathbb{P} = (P_t)_{t\geq 0}$ be the sub-Markovian semigroup induced by a convolution semigroup $\mu := (\mu_t)_{t\geq 0}$ on $\mathbb{R}^d$, i.e. $P_t f = \mu_t * f$ (cf. Example 2.1). Let $\Psi : \mathbb{R}^d \longrightarrow \mathbb{C}$ be the negative definite function associated to μ, i.e. Ψ is defined by the following Fourier transform

$$\hat{\mu}_t(x) = \exp(-t\Psi(x)), \qquad t > 0, x \in \mathbb{R}^d.$$

Using [13, Example 4.3.3], we get

$$Af = \int e^{i\langle \cdot, y\rangle}\Psi(y)\hat{f}(y)dy$$

on its domain $D(A)$, which is the set of all functions $f \in L^2(m)$ such that

$$\int (1 + |\Psi(x)|)^2 |\hat{f}(x)|^2 dx < \infty.$$

Let $f, g \in D(A)$. Then

$$\langle -Af, g\rangle = \int \bar{\hat{f}}(y)\hat{g}(y)\Re\Psi(y)\, dy.$$

Hence the sector condition is equivalent to the following one, given in terms of Ψ by

$$|\Psi(y)| \leq L(1 + \Re\Psi(y)) \qquad (y \in \mathbb{R}^d)$$

for some constant $L > 0$.

Corollary 3.14. *Let $\mathbb{P}$ be a sub-Markovian semigroup and let β be a Bochner subordinator without drift. Let ψ be a $\mathbb{P}^\beta$-exit law and let $u = \int_0^\infty e^{-t}\psi_t\, dt$. If there exists a function $K :]0, \infty[\to]0, \infty[$ such that*

$$\|P_{s+t}u - P_t u\|_2 \leq K(t)\, s \quad \text{as } s \to 0, \qquad t > 0. \tag{3.11}$$

Then ψ is subordinated to a unique $\mathbb{P}$-exit.

Proof. Let

$$\varphi_t = aP_t u + \int_0^\infty (P_t u - P_{s+t}u)\,\nu(ds), \qquad t > 0.$$

where a and ν are given in (3.5). Then we get

$$\|\varphi_t\|_2 \le a\,\|P_t u\|_2 + \int_0^\infty \|P_t u - P_{s+t}u\|_2\,\nu(ds) \le$$

$$\le a\,\|P_t u\|_2 + K(t)\int_0^1 s\,\nu(ds) + 2\,\|P_t u\|_2\,\nu([1,\infty[) < \infty.$$

Hence $\varphi_t \in L^2(m)$ for each $t > 0$. Now, let $V_k := \int_0^\infty e^{-ks} P_s\,ds$ and let $\varphi_{t,k} := kV_k\varphi_t$. Following [2, p. 81–82], we have $\varphi_{t,k} \in L^2(m)$ and

$$\lim_{k\to\infty} \varphi_{t,k} = \varphi_t, \qquad t > 0.$$

On the other hand, by the contraction property of $\mathbb{P}$, we have for each $k > 0$

$$\int_0^\infty\int_0^\infty \|e^{-kr}P_r(P_t u - P_{s+t}u)\|_2\,dr\,\nu(ds) \le \frac{1}{k}\int_0^\infty \|P_t u - P_{s+t}u\|_2\,\nu(ds) < \infty.$$

Therefore, we obtain

$$\varphi_{t,k} = a\,kV_kP_t u + \int_0^\infty (Id - P_s)(kV_kP_t u)\,\nu(ds) \qquad k,t > 0.$$

But $V_kP_t u \in D(A)$ (cf. [2, Proposition 11.10]). Hence by (3.2) we get

$$\varphi_{t,k} = A^\beta(kV_kP_t u).$$

Since $kV_kP_t u \to P_t u$ as $k \to \infty$, it follows by the closeness of the operator A^β that $(P_t u)_{t>0} \subset D(A^\beta)$. We conclude by Theorem 3.10. □

FINAL REMARKS

1. Condition (3.11) is fulfilled whenever $(P_t u)_{t>0} \subset D(A)$, in this case we take $K(t) := \|AP_t u\|_2$. However, condition (3.11) seems to be more general and it is considered in [14].
2. Theorem 3.5 may be applied for bounded Bernstein functions (for example for component Poisson subordinators). Moreover, we have an explicit formula for φ if $\psi = \varphi^\beta$.
3. Theorem 3.10 (and Corollaries 3.11 and 3.14) may be applied for the general case (for example for the one sided stable subordinators and for the gamma subordinators). But we have no explicit formula for φ in this case.
4. Proposition 3.8 is a intermediary result in order to prove Theorem 3.5. But it can be applied (instead of Theorem 3.5) for some particular cases.

5. Theorem 3.10 solves the converse problem in the general case. In particular, this result generalizes our earlier papers in a significant manner.
6. The approximation used in the proof of Theorem 3.10, is also used in [12] but for a different problem.

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