MATRICES RELATED
TO SOME FOCK SPACE OPERATORS

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Abstract. Matrices of operators with respect to frames are sometimes more natural and easier to compute than the ones related to bases. The present work investigates such operators on the Segal-Bargmann space, known also as the Fock space. We consider in particular some properties of matrices related to Toeplitz and Hankel operators. The underlying frame is provided by normalised reproducing kernel functions at some lattice points.

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1. INTRODUCTION

In applied analysis it is often more natural to use in the underlying Hilbert space overcomplete systems of vectors rather than bases. Such systems satisfying the frame condition enjoy better stability of the reconstruction algorithms and are easier to obtain using some natural constructions. For example, in reproducing kernel Hilbert spaces, sequences of normalised kernel functions at suitably chosen discrete sets of points can be shown to be frames, while they rarely constitute Riesz bases. Descriptions of such discrete sets among regular lattices or even quite general sequences of points appeared in the 80's in the works of Rochberg, Coifmann, Daubechies with Grossman, Lyubarskii, Seip and Wallsten, to mention the most important contributions. One of their guidelines can be traced to earlier success in atomic decompositions of Hardy spaces over the complex half-plane.

The reproducing kernels lead to the Berezin transform approach to certain classes of operators which proved especially fruitful in the cases related to analytic structure, like Toeplitz or Hankel operators. In this note we show some relations to matrices with respect to frames obtained from reproducing kernels.

2. FRAMES AND KERNELS

In [3, 4] and [2] matrices with respect to frames for bounded linear operators on a Hilbert space H were studied. Given a fixed sequence $\mathcal{G} = (g_j), j \in \mathbb{N}$ of vectors in H one defines the *analysis operator* $C = C_\mathcal{G}$ by

$$Cf := (\langle f, g_j \rangle)_{j \in \mathbb{N}},$$

the sequence of inner products with the members of $\mathcal{G}$. If C maps H boundedly into ℓ^2, then $\mathcal{G}$ is referred to as a *Bessel sequence*. If moreover, for some constants $K, \kappa > 0$ (called *frame bounds*) one has

$$\kappa \|f\|^2 \le \sum_{j=1}^{\infty} |\langle f, g_j \rangle|^2 \le K\|f\|^2, \tag{2.1}$$

$\mathcal{G}$ is called a *frame in H*. These frame bounds may be equal, in which case we speak of a *tight frame with bound* $\kappa = K$. *Parseval Frames* are defined as tight frames with bound 1. For such frames (2.1) is the (generalised) Parseval identity and its polarised form represents the inner product $\langle f, h \rangle$ as the sum $\sum_j K^{-1} \langle f, g_j \rangle \langle g_j, h \rangle$.

The adjoint operator $C^* = C_\mathcal{G}^* : \ell^2 \to H$ of C, called *the synthesis operator* maps square–summable sequences $(\alpha_j) \in \ell^2$ to the (unconditionally norm-convergent) sums $\sum \alpha_j g_j \in H$ -cf. [7]. Finally, their composition $S := C^*C$, called *the frame operator* is given by $Sf = \sum \langle f, g_j \rangle g_j$, which is self–adjoint, positive, with $\kappa I_H \le S \le K I_H$. The *the canonical dual frame for $\mathcal{G}$* is the sequence $\widetilde{\mathcal{G}} := (S^{-1} g_j)$ essential in the following

$$(\textit{Reconstruction Formula}) \quad f = \sum_{j=1}^{\infty} \langle f, g_j \rangle \widetilde{g_j}, \quad \text{where} \quad \widetilde{g_j} := S^{-1} g_j. \tag{2.2}$$

The dual frame is a constant multiple of $\mathcal{G}$ iff the frame is tight, with equality $\widetilde{\mathcal{G}} = \mathcal{G}$ taking place iff $\mathcal{G}$ is a Parseval frame (then $\widetilde{g_j} = g_j$ for any j, see [7, 10]).

The practical way of verifying the frame condition in some cases is to prove directly that the frame operator S itself is boundedly invertible by estimating its norm-distance from the identity by some number $\gamma < 1$ (cf. [9, 11]).

Recall, that a Hilbert space H of functions on some set Ω is a *reproducing kernel Hilbert space* (RKHS), if for any $w \in \Omega$ the linear functional $H \ni f \to f(w) \in \mathbb{C}$ of evaluation at w is continuous. Then for some *reproducing kernel* $K_w \in H$ one has

$$f(w) = \langle f, K_w \rangle, \quad f \in H, w \in \Omega. \tag{2.3}$$

The role of atoms is played by the normalised reproducing kernels,

$$k_w(z) := \frac{K_w(z)}{\|K_w\|} \tag{2.4}$$

and it is easy to see that $\|K_w\| = \sqrt{K_w(w)}$, $|K_w(z)|^2 \le K_w(w)K_z(z)$ and $K_z(w) = \overline{K_w(z)}$ for any $z, w \in \Omega$. Moreover, H is the (closed) linear span of the kernel functions

K_w as $w \in \Omega$. Actually, much smaller range of w's is enough here to span H, some discrete sequences in Ω suffice. The direct way of constructing K_w from an orthonormal basis (e_n) of H is the well-known formula

$$K_w(z) = \sum_n e_n(z)\overline{e_n(w)},$$

which in the case of a frame $\mathcal{G}$ and its dual $\tilde{\mathcal{G}}$ becomes

$$K_w(z) = \sum_n \widetilde{g}_n(z)\overline{g_n(w)}.$$

Let us recall the most important examples: The Hardy space case ($\Omega = \mathbb{D} := = \{z \in \mathbb{C} : |z| < 1\}, H = H^2, e_n(z) = z^n$) has the reproducing kernel

$$K_w(z) = (1 - z\bar{w})^{-1}.$$

For the Bergman space $L^2 \cap H(\mathbb{D}), e_n(z) = (n+1)^{\frac{1}{2}}z^n$, so the Bergman kernel is

$$K_w(z) = (1 - z\bar{w})^{-2}.$$

Here the L^2-norm is taken for the normalised area measure $\frac{1}{\pi}\,dx\,dy$ on $\mathbb{D}$ and

$$k_w(z) = \frac{1 - |w|^2}{(1 - z\bar{w})^2}.$$

If $dA(z) = dxdy$ is the Lebesgue (area) measure on the complex plane $\mathbb{C}$, let $d\mu(z) = \exp(-\pi|z|^2)dA(z)$ be the Gaussian measure. Then the subspace of $L^2(\mu)$ spanned by the orthonormal basis $\sqrt{\frac{\pi^n}{n!}}z^n$ is denoted by $F^2(\mathbb{C})$ and called the Fock space (or Segal-Bargmann space) over $\mathbb{C}$. It consists of all entire functions that are square μ-integrable. The reproducing kernels are $K_w(z) = \exp(\pi z\bar{w})$. Then

$$k_w(z) = \exp(\pi(z - \frac{w}{2})\bar{w}). \tag{2.5}$$

Some other normalisations are also common, e.g. replacing π either with $\frac{1}{2}$, or with 1 or with some arbitrary constant a. Passing from $\mathbb{C}$ to $\mathbb{C}^d$ (the "d degrees of freedom case") requires only the use of multiindices in place of $n \in \mathbb{N} \cup \{0\}$ (and the exponent d in the normalization constant), hence for the sake of simplicity we consider only $d = 1$ in what follows.

If $\{z_j : j \in \mathbb{N}\}$ is a discrete subset of $\mathbb{C}$ satisfying the separation condition:

$$\inf\{|z_j - z_k| : j \neq k\} = \delta > 0$$

and the density condition:

$$\sup_{w \in \mathbb{C}} \inf_j |z_j - w| = \epsilon_0$$

with ϵ_0 sufficiently small, then it is shown in Theorem 8.2 of [9] that the normalised reproducing kernels $g_j := k_{z_j}$ form a frame in $F^2(\mathbb{C})$. For regular lattices (of the

form $\epsilon_1 \mathbb{Z}^{2d}$, where $\mathbb{Z}^{2d} = \{(\lambda_1, \ldots, \lambda_d) : \lambda_j \in \mathbb{C}, \Re\lambda_j \in \mathbb{Z}, \Im\lambda_j \in \mathbb{Z}\}$) the result has been obtained by Daubechies and Grossman (1988) and refined by Lyubarskij and Seip in 1992. It should be mentioned that in [9] a general approach to various RKHS is developed, giving frames for some sequence of points $z_j \in \Omega$ and applying to analogous spaces with L^p-norms for $1 \le p \le +\infty$, where instead of frames decompositions one speaks then of *atomic decompositions*.

3. MATRICES REPRESENTING OPERATORS

If $\mathcal{G}_1 = (g_{j1})_{j=1}^\infty, \mathcal{G}_2 = (g_{j2})_{j=1}^\infty$ are frames in Hilbert spaces H_1, H_2, the coefficients of a bounded linear operator $T : H_1 \longrightarrow H_2$ are defined by

$$T_{nm} := \langle Tg_{m1}, g_{n2}\rangle.$$

The so obtained matrix (T_{nm}) will be denoted as $\mathrm{Matr}(T)$, or $\mathrm{Matr}^{(\mathcal{G}_2,\mathcal{G}_1)}(T)$.

Conversely, to a bounded (as an operator on ℓ^2) matrix $A = (A_{nk})_{n,k \in \mathbb{N}}$ one assigns the operator $\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(M) = \mathcal{O}(A)$ that maps $f \in H_1$ into the sum

$$\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(A)f := \sum_{k=1}^\infty \sum_{n=1}^\infty A_{kn}\langle f, g_{n1}\rangle g_{k2}.$$

The basic properties of the correspondence between matrices and bounded linear operators are collected below for the reader's convenience. (These are Theorem 3.1, Proposition 3.2 and Corollary 3.3 in [3], Theorem 3.2 and Corollary 4.1 in [2].)

Theorem 3.1. (i) *Treated as an operator on ℓ^2, the matrix $\mathrm{Matr}(T)$ is bounded and*

$$\mathrm{Matr}^{(\mathcal{G}_2,\mathcal{G}_1)}(T) = C_{\mathcal{G}_2} \circ T \circ C_{\mathcal{G}_1}^* \quad \text{with} \quad \|\mathrm{Matr}^{(\mathcal{G}_2,\mathcal{G}_1)}(T)\| \le \sqrt{K_1 K_2}\|T\|.$$

(ii) *The operator $\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(A) : H_1 \longrightarrow H_2$ is bounded and satisfies*

$$\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(A) = C_{\mathcal{G}_2}^* \circ A \circ C_{\mathcal{G}_1} \quad \text{with} \quad \|\mathcal{O}(A)\| \le \sqrt{K_1 K_2}\|A\|.$$

(iii) *For any frames $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and operators $T : H_1 \to H_2, L : H_2 \to H_3$ the Product Formula holds:*

$$\mathrm{Matr}^{(\mathcal{G}_3,\mathcal{G}_2)}(L) \cdot \mathrm{Matr}^{(\widetilde{\mathcal{G}_2},\mathcal{G}_1)}(T) = \mathrm{Matr}^{(\mathcal{G}_3,\mathcal{G}_1)}(LT),$$

which together with $\mathcal{G} = \widetilde{\widetilde{\mathcal{G}}}$ and (i) implies that $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}$ is a (non unital) continuous homomorphism of Banach algebra $\mathcal{B}(H)$ into $\mathcal{B}(\ell^2)$.

(iv) *We have the following Operator Reconstruction Formula:*

$$\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(\mathrm{Matr}^{(\widetilde{\mathcal{G}_2},\widetilde{\mathcal{G}_1})}(T)) = T, \quad \mathcal{O}^{(\mathcal{G},\widetilde{\mathcal{G}})}(I_{\ell^2}) = I_H.$$

(v) *The assignment* $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}$ *is a *- morphism, i.e. the matrix* $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}(T^*)$ *is the Hermitian adjoint to the matrix* $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}(T)$ *for any bounded linear operator* $T : H \to H$, *if and only if the frame* $\mathcal{G}$ *is tight.*

(vi) $T \in \mathcal{B}(H)$ *belongs to the Schatten-von Neumann ideal* $\mathcal{S}_p$ *if and only if so does its matrix* M. *Moreover, the following equality takes place in the normalised tight frame case:*
$$\|T\|_p = \|M\|_p$$
for any finite number $p \geq 1$.

One can ask, if the trace itself can be recovered from that of the related matrix. If we just want to compare up to some constants, the trace norm can be estimated from the product $C_{\mathcal{G}_2} T C_{\mathcal{G}_1}^*$ from the above theorem in both directions (here by $\|C_{\mathcal{G}_2}\| \|T\|_1 \|C_{\mathcal{G}_1}^*\|$, where the operator norm of the analysis operator is the (square root of) the Bessel constant. But the exact value will not follow in general, even for Parseval frames, a clear consequence of the overcompleteness. It is therefore useful to note the following

Proposition 3.2. *If the frame* $\mathcal{G}$ *is a Riesz basis and* $\widetilde{\mathcal{G}}$ *is its biorthogonal sequence, then*
$$\mathtt{tr}(T) = \mathtt{tr}(\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}(T))$$
for any trace-class operator T.

Indeed, let $\delta_{mk} \in \{0, 1\}$ denote the Kronecker symbol. If e_i are orthonormal basic vectors, then using the equalities
$$e_i = \sum_m \langle e_i, \tilde{g}_m \rangle g_m = \sum_k \langle e_i, g_k \rangle \tilde{g}_k$$
and
$$\sum_i \langle e_i, \tilde{g}_m \rangle \langle g_k, e_i \rangle = \langle \tilde{g}_m, g_k \rangle = \delta_{mk},$$
we compute $\mathtt{tr}(T)$ as equal to
$$\sum_i \langle T e_i, e_i \rangle = \sum_i \sum_m \sum_k \langle \langle e_i, \tilde{g}_m \rangle T g_m, \langle e_i, g_k \rangle \tilde{g}_k \rangle =$$
$$= \sum_{i,m,k} \langle e_i, \tilde{g}_m \rangle \langle g_k, e_i \rangle \langle T g_m, \tilde{g}_k \rangle.$$

Now recognising the sum over i as δ_{mk}, we see, that the triple sum reduces to $\sum_m \langle T g_m, \tilde{g}_m \rangle$, which is the trace of our matrix $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}(T)$. $\qquad\square$

It turns out that some of these results have extension for unbounded operators. This requires special care with domains and with taking closures. Since in general, there can be infinitely many nonzero coefficients of a frame vector w.r.to the dual frame, the summability of each column may not suffice for the frame vectors to belong to the domain of the operator formally associated with the matrix.

If we make it an assumption, then more can be said:

Theorem 3.3. *Let $\mathcal{G}$ be a tight frame in a Hilbert space H.*

(i) *If $\mathcal{G}$ is contained in the domain of a closed symmetric operator T in H, then $\mathrm{Matr}(T)$ is a Hermitian matrix.*

(ii) *If the matrix A is hermitian, closed as an operator on ℓ^2 and such that $\sum_j |a_{jk}|^2 < \infty$ for any k, then the associated operator $T = \mathcal{O}^{(\mathcal{G},\mathcal{G})}(A)$ is closed and symmetric, containing $\mathcal{G}$ in its domain.*

Proof. (i) is an easy consequence of the defining formula, so it remains to show (ii). The arguments employed in the case of orthonormal base decompositions in [1](p. 100) show that the formally defined operator by a matrix is closed on its maximal domain and if a Hermitian matrix has square-summable columns, then it defines a closed symmetric operator. The square sumability of $\langle f, g_k \rangle$ is used together with our assumption on the columns to obtain dense domains for the related operator (and of its adjoint). The last assumption is needed to replace the missing biorthogonality. This assumption is easy to verify by direct computation in the concrete case (2.5) of the exponents frame in the Fock space. $\square$

Let us finally remark that the assumption does not imply the boundedness of T. If one has a uniform bound on ℓ^1-norms of columns and rows, then boundedness follows from the Schur criterion. Another instance of boundedness follows from the Closed Mapping Theorem:

Corollary 3.4. *If the domain of $T = \mathcal{O}^{(\mathcal{G},\mathcal{G})}(A)$ for the matrix A satisfying the assumptions of* (ii) *above is closed, then this operator is bounded.*

(Note that in view of the density of the linear span of frame vectors, the assumed closedness means just the equality of the domain of T to the whole space H.)

4. SOME EXAMPLES

The orthogonal projection P from $L^2(\mu)$ onto the Fock space $F^2(\mathbb{C})$ can be written as an integral operator of the form

$$(Pf)(w) = \int f(\zeta) K_\zeta(w) \, d\mu(\zeta) = \langle f, K_w \rangle.$$

Toeplitz operator with symbol h is defined by

$$T_h f := P(hf), \quad f \in \mathcal{D}(T_h) := \{f \in F^2(\mathbb{C}) : fh \in L^2(\mu)\}.$$

In particular, for $h \in L^\infty(\mu)$ one obtains a bounded, everywhere defined operator. If h is nonconstant, analytic, then T_h is unbounded, but densely defined, if g has an exponential growth. Such operators are important in quantum mechanics models of annihilation and creation operators ([5]) and are used in the Berezin's second quantization technique.

Then the matrix with respect to the frame of normalised kernels $g_j := k_{z_j}$ of the form (2.5) with z_j ranging through an appropriate lattice has its matricial entries $a_{nm} = \langle PT_h g_m, g_n \rangle = \langle hK_{z_m}, K_{z_n} \rangle (\|K_{z_m}\| \|K_{z_n}\|)^{-1}$. In the non-analytic case we have to stop here, but then h can be bounded and nonconstant at one time. If h is analytic and of suitable growth, the latter term is, by the reproducing property, equal to

$$(hK_{z_m})(z_n)(\|K_{z_m}\| \|K_{z_n}\|)^{-1} = h(z_n) \exp(\pi z_n \bar{z}_m - \frac{\pi}{2}(|z_n|^2 + |z_m|^2)).$$

This formula for a_{nm} enables one to rephrase the Shatten-von Neumann ideal membership croterion in terms of h. In the $p = 2$ -case the Hilbert-Schmidt condition becomes the finiteness requirement for the double sum of squares of

$$|\langle hK_{z_m}, K_{z_n} \rangle| (\|K_{z_m}\| \|K_{z_n}\|)^{-1}$$

taken over $(m,n) \in \mathbb{N}^2$. By the frame condition, the latter is finite iff

$$\sum_{m=1}^{\infty} \|hk_{z_m}\|^2 < +\infty.$$

The Berezin symbol function $\tilde{h}(z)$ defined as the inner product $\langle hk_z, k_z \rangle$ corresponds to the diagonal entries (if z is confined to the lattice points z_j). Analogously, one defines the Berezin 2-variable symbol

$$\tilde{h}(z,w) := \langle hk_z, k_w \rangle.$$

One can obtain $h(z,w)$ from the one-variable symbol function $h(\cdot)$ by polarisation. Hence we deduce (in any reproducing kernel Hilbert space, where a given lattice of points yields a frame of normalised reproducing kernels) the following useful observation

Corollary 4.1. *The matrix of T in the frame of normalised reproducing kernels (this time not with respect to its dual frame) has entries simply equal to the values of its two-variable Berezin symbol at the lattice points.*

In this way we have from matricial following characterisation of the Schatten-von Neumann classes the following description, which is not new, but has now almost elementary proof:

Corollary 4.2. *The Toeplitz operator T_h on the Fock space is Hilbert-Schmidt iff for the (chosen as above) lattice points z_j and the corresponding frame of normalised reproducing kernels k_{z_j} the corresponding Berezin symbol satisfies $\sum_{j,n} |\tilde{h}(z_j, z_n)|^2 < \infty$, or equivalently, if $\sum_j |\tilde{h}(z_j)|^2 < \infty$.*

For h nonnegative -the trace equals the trace norm and one can obtain the Berezin symbol descriptions of the $\mathcal{S}_p$-ideal of membership using our matricial form. This is giving some estimates of the $L^p(\mu)$-norms of h in terms of our lattice points, which is interesting in its own sake in the nonanalytic h case. Analogous consequences may be expected in the case of Hankel operators.

Let us mention that the case of $d = \infty$ has physical motivation as the infinite number of degrees of freedom. Here the situation complicates, but some formal estimates apply at least to the $p = 2$ case (the work in progress). The measure-theoretic approach has to be done with extreme care, since in this $d = \infty$ case the Gaussian measure lives on a larger space including H as a set of measure zero, while the analytic structure is based on H (although for a given element f it can be extended to a set (depending, unfortunately on f) of full measure [8]. The related frames of reproducing kernels seem a very interesting field for future study.

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