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OPERATOR REPRESENTATIONS OF FUNCTION ALGEBRAS AND FUNCTIONAL CALCULUS

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Abstract. This paper deals with some operator representations Φ of a weak*-Dirichlet algebra A, which can be extended to the Hardy spaces $H^p(m)$, associated to A and to a representing measure m of A, for $1 \leq p \leq \infty$. A characterization for the existence of an extension Φ_p of Φ to $L^p(m)$ is given in the terms of a semispectral measure F_Φ of Φ. For the case when the closure in $L^p(m)$ of the kernel in A of m is a simply invariant subspace, it is proved that the map $\Phi_p|H^p(m)$ can be reduced to a functional calculus, which is induced by an operator of class C_ρ in the Nagy-Foiaş sense. A description of the Radon-Nikodym derivative of F_Φ is obtained, and the log-integrability of this derivative is proved. An application to the scalar case, shows that the homomorphisms of A which are bounded in $L^p(m)$ norm, form the range of an embedding of the open unit disc into a Gleason part of A.

Keywords: weak*-Dirichlet algebra, Hardy space, operator representation, semispectral measure.

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1. INTRODUCTION AND PRELIMINARIES

Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex continuous functions on X. Denote by A a function algebra on X, that is a closed subalgebra of $C(X)$ which contains the constant functions and separates the points of X. $\mathcal{M}(A)$ stands for the set of all non zero complex homomorphisms (or Gelfand spectrum) of A. The equivalence classes of $\mathcal{M}(A)$ induced by the relation: $\gamma \sim \varphi$ iff $\|\gamma - \varphi\| < 2$ for $\gamma, \varphi \in \mathcal{M}(A)$, are the Gleason parts of A (see [2, 21]).

For $\gamma \in \mathcal{M}(A)$, A_γ means the kernel of γ, and M_γ designates the set of all representing measures m for γ, that is m is a probability Borel measure on X satisfying

$\gamma(f) = \int f dm$, $f \in A$. For a subspace $B \subset C(X)$, we put $\overline{B} = \{\overline{f} : f \in B\}$. Notice that the homomorphism γ can be naturally extended to $A + \overline{A}$ by

$$\gamma(f + \overline{g}) = \gamma(f) + \overline{\gamma(g)}, \qquad f, g \in A.$$

In this paper we consider A to be a function algebra on X which is *weak*-Dirichlet* in $L^\infty(m)$, that is $A + \overline{A}$ is weak* dense in $L^\infty(m)$, for some fixed $m \in M_\gamma$ and $\gamma \in \mathcal{M}(A)$. This concept introduced in [20] is weaker than one of *Dirichlet algebra*, which means that $A + \overline{A}$ is dense in $C(X)$. For example, the *standard algebra* $A(\mathbb{T})$ of all continuous functions f on the unit circle $\mathbb{T}$ which have analytic extensions $\widetilde{f}$ to the open unit disc $\mathbb{D}$, is a Dirichlet algebra on $\mathbb{T}$. On the other hand, the subalgebra $A_1(\mathbb{T})$ of $A(\mathbb{T})$ of those functions f satisfying $f(1) = \widetilde{f}(0)$ is a weak*-Dirichlet algebra in $L^\infty(m_0)$, m_0 being the normalized Lebesgue measure on $\mathbb{T}$, and $A_1(\mathbb{T})$ is not a Dirichlet algebra.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$.

Any bounded linear and multiplicative map Φ of A in $\mathcal{B}(\mathcal{H})$ with $\Phi(1) = I$ (the identity operator on $\mathcal{H}$) is called a *representation* of A on $\mathcal{H}$. When $\|\Phi\| \leq 1$ one says that Φ is *contractive.* Here, we only consider a representation Φ for which there exist a scalar $\rho > 0$ and a system $\{\mu_x\}_{x\in\mathcal{H}}$ of positive measures on X with $\|\mu_x\| = \|x\|^2$ such that

$$\langle \Phi(f)x, x\rangle = \int [\rho f + (1-\rho)\gamma(f)] d\mu_x$$

for any $f \in A$ and $x \in \mathcal{H}$. Such a μ_x is called a *weak ρ-spectral measure for Φ attached to x by γ.* It is known ([8,9]) that the existence of a system of measures $\{\mu_x\}_{x\in\mathcal{H}}$ as above, is equivalent to the fact that Φ satisfies a weaker von Neumann inequality of the form

$$w(\Phi(f)) \leq \|\rho f + (1-\rho)\gamma(f)\| \qquad (f \in A), \tag{1.1}$$

where $w(T)$ means the numeric radius of $T \in \mathcal{B}(\mathcal{H})$.

In [10] it was proved that if the representation Φ of A on $\mathcal{H}$ admits a system $\{\mu_x\}_{x\in\mathcal{H}}$ of weak ρ-spectral measures attached by γ such that μ_x is $m - a.c.$ for any $x \in \mathcal{H}$, then Φ has a γ-spectral ρ-dilation, that is there exists a contractive representation $\widetilde{\Phi}$ of $C(X)$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ satisfying the relation

$$\Phi(f) = \rho P_{\mathcal{H}} \widetilde{\Phi}(f)|\mathcal{H} \qquad (f \in A_\gamma), \tag{1.2}$$

where $P_{\mathcal{H}}$ is the orthogonal projection on $\mathcal{H}$. Moreover, in this case there exists a unique semispectral measure $F_\Phi : Bor(X) \to \mathcal{B}(\mathcal{H})$ such that $\langle F_\Phi(\cdot)x, x\rangle = \mu_x$, or equivalently

$$\langle \Phi(f)x, y\rangle = \int [\rho f + (1-\rho)\gamma(f)] d\langle F_\Phi x, y\rangle \qquad (f \in A), \tag{1.3}$$

for any $x, y \in \mathcal{H}$. As usual, $Bor(X)$ denotes the set of all Borel subsets of X. Using the polarization formula, it follows that all measures $\langle F_\Phi(\cdot)x, y\rangle$ for $x, y \in \mathcal{H}$ are $m - a.c.$

The relation (1.2) means that the representation $\widetilde{\Phi}$ is a γ-*spectral* ρ-*dilation* of Φ, and F_Φ is obtained as the compression to $\mathcal{H}$ of the spectral measure of $\widetilde{\Phi}$ (see [21]).

The representations with spectral ρ-dilations was first studied by D. Gaşpar ([4–6]), and recently by T. Nakazi ([15, 16]). Any such representation of the algebra $A(\mathbb{T})$ on $\mathcal{H}$ reduces to the usual functional calculus with the operators of class C_ρ in $\mathcal{B}(\mathcal{H})$ in the sense of Sz. Nagy-Foiaş [22] (i.e. ρ-contractions; [1, 11]). In the general setting of a weak*-Dirichlet algebra A, it is natural to find conditions for a representation Φ of A on $\mathcal{H}$, under which Φ can be reduced to a certain functional calculus with a ρ-contraction. Recall that in [6] was given an example of a contractive representation of a Dirichlet algebra which cannot be reduced to a functional calculus with contractions.

In the sense of [5, 6], the problem of reduction to a functional calculus refers to absolutely continuous representations with respect to representing measures. Thus, we only investigate here the representations Φ which have a system of $m-a.c.$ weak ρ-spectral measures attached by γ. In the sequel $H^p(m)$ stands for the (weak*, for $p=\infty$) closure of A into $L^p(m)$, that is the *Hardy space* associated to A in $L^p(m)$.

In Section 2 we characterize in terms of F_Φ the representations Φ which have bounded linear extensions Φ_p to the space $L^p(m)$ for $1 \le p \le \infty$. In Section 3 we prove the main result which says that, under some hypothesis on an invariant subspace of $H^p(m)$ when $1 \le p \le 2$, the map $\Phi_p|H^p(m)$ is given by a functional calculus with a ρ-contraction with the spectrum in $\mathbb{D}$, the functional calculus being induced by a Hoffman type [7] naturally associated to the corresponding invariant subspace. In this case, the Radon-Nikodym derivative of F_Φ is an essentially bounded function on X and its logarithm belongs to $L^1(m)$. The scalar case is considered in Section 4 where we refer to the homomorphisms in $\mathcal{M}(A)$ which are bounded in the $L^p(m)$-norm. Our main result is a version of Wermer's embedding theorem ([1,7,21]) for weak*-Dirichlet algebras, which prove that the set of above quoted homomorphisms corresponds to an analytic disc in the Gleason part which contains γ.

2. EXTENSION OF A REPRESENTATION TO THE SPACE $L^p(m)$

We characterize below some representations Φ of A on $\mathcal{H}$ which can be linearly and boundedly extended to the space $L^p(m)$ for $1 \le p \le \infty$. Our characterization is given in the terms of the Radon-Nikodym derivative with respect to m of the corresponding $\mathcal{B}(\mathcal{H})$-valued semispectral measure F_Φ. In the sequel we put $\varphi_{x,y}dm = d\langle F_\Phi(\cdot)x, y\rangle$ for $x, y \in \mathcal{H}$.

Theorem 2.1. *Let Φ be a representation of A on $\mathcal{H}$ which admits a system of $m-a.c.$ weak ρ-spectral measures attached by γ. Then Φ has a bounded linear extension Φ_p from $L^p(m)$ into $\mathcal{B}(\mathcal{H})$ for $1 \le p \le \infty$, if and only if $\varphi_{x,y} \in L^q(m)$ and there exists a constant $c > 0$ such that*

$$\|\varphi_{x,y}\|_q \le c\|x\|\|y\| \qquad (x, y \in \mathcal{H}), \tag{2.1}$$

where $\frac{1}{p}+\frac{1}{q}=1$. *In this case,* Φ_p *is uniquely determined and it satisfies for* $h \in L^p(m)$ *and* $x, y \in \mathcal{H}$ *the relation*

$$\langle \Phi_p(h)x, y\rangle = \int [\rho h + (1-\rho)\int h dm]\varphi_{x,y} dm. \tag{2.2}$$

Furthermore, for $h \in L^2(m)$ *and* $x \in \mathcal{H}$ *we have the inequality*

$$\|\Phi_2(h)x\|^2 \leq \int |\rho h + (1-\rho)\int h dm|^2 \varphi_{x,x} dm. \tag{2.3}$$

Hence, if $\{h_\alpha\} \subset L^\infty(m)$ *is a bounded net such that* $\{h_\alpha\}$ *converges a.e.* (m) *to* $h \in L^\infty(m)$, *then* $\{\Phi_p(h_\alpha)\}$ *strongly converges to* $\Phi_p(h)$ *in* $\mathcal{B}(\mathcal{H})$, *for* $p \geq 2$.

Proof. Suppose firstly that $\varphi_{x,y} \in L^q(m)$ and that the inequality (2.1) is satisfied. Since for $f \in A$, $g \in A_\gamma$ and $x, y \in \mathcal{H}$ we have

$$\langle (\Phi(f) + \Phi(g)^*)x, y\rangle = \int [\rho(f+\overline{g}) + (1-\rho)\gamma(f+\overline{g})]\varphi_{x,y} dm,$$

we infer that

$$|\langle (\Phi(f) + \Phi(g)^*)x, y\rangle| \leq \rho|\int (f+\overline{g})\varphi_{x,y} dm| + |(1-\rho)\int (f+\overline{g})dm \cdot \int \varphi_{x,y} dm| \leq$$
$$\leq (\rho + |1-\rho|)\|f+\overline{g}\|_p \|\varphi_{x,y}\|_q.$$

Since $A + \overline{A}_\gamma$ is weak* dense in $L^\infty(m)$, the closure of $A + \overline{A}_\gamma$ in $L^p(m)$ is just $L^p(m)$, for $1 \leq p < \infty$, (see [20]). Thus, the previous relations prove that for any $x, y \in \mathcal{H}$ there exists a bounded linear functional $\Phi_{x,y}$ on $L^p(m)$ satisfying for $f \in A$, $g \in A_\gamma$, $h \in L^p(m)$,

$$\Phi_{x,y}(f+\overline{g}) = \langle (\Phi(f) + \Phi(g)^*)x, y\rangle,$$

and

$$\Phi_{x,y}(h) = \int [\rho h + (1-\rho)\int h dm]\varphi_{x,y} dm.$$

Also we have $\Phi_{x,y} = \overline{\Phi_{y,x}}$ and using (2.1) we obtain

$$\|\Phi_{x,y}\| \leq c(\rho + |1-\rho|)\|x\|\|y\|.$$

It follows that for every $h \in L^p(m)$, the map $(x, y) \mapsto \Phi_{x,y}(h)$ is a bounded bilinear functional on $\mathcal{H} \times \mathcal{H}$, hence there exists an operator $\Phi_p(h) \in \mathcal{B}(\mathcal{H})$ such that

$$\langle \Phi_p(h)x, y\rangle = \Phi_{x,y}(h), \quad x, y \in \mathcal{H}$$

and

$$\|\Phi_p(h)\| \leq c(\rho + |1-\rho|)\|h\|.$$

Then $\Phi_p : h \mapsto \Phi_p(h)$ is a bounded linear map from $L^p(m)$ into $\mathcal{B}(\mathcal{H})$, which extends Φ and also satisfies the relation (2.2). Using also (2.2) with $h = f + \overline{g}$ for $f \in A$, $g \in A_\gamma$ one can see that Φ_p is the unique bounded linear extension of Φ to $L^p(m)$.

Now, let $\widetilde{\Phi}$ be the γ-spectral ρ-dilation of Φ (from (1.2)), corresponding to the Naimark dilation (as a spectral measure) of the semispectral measure F_Φ (see [6,21]). Then for $f \in A$, $g \in A_\gamma$ and $x \in \mathcal{H}$ we have $\Phi(g)^* x = \rho P_{\mathcal{H}} \widetilde{\Phi}(\overline{g})x$ and

$$\begin{aligned}\|\Phi_2(f+\overline{g})x\|^2 &= \|(\Phi(f)+\Phi(g)^*)x\|^2 = \|P_{\mathcal{H}}\widetilde{\Phi}(\rho(f+\overline{g})+(1-\rho)\gamma(f+\overline{g}))x\|^2 \leq \\ &\leq \langle \widetilde{\Phi}(|\rho(f+\overline{g})+(1-\rho)\gamma(f+\overline{g})|^2)x,x\rangle = \\ &= \int |\rho(f+\overline{g})+(1-\rho)\gamma(f+\overline{g})|^2 \varphi_{x,x} dm.\end{aligned}$$

Since $A+A_\gamma$ is dense in $L^2(m)$, by the continuity of Φ_2 one obtains from this inequality just the inequality (2.3).

Next, let $\{h_\alpha\} \subset L^\infty(m)$, be a bounded net which converges a.e. (m) to $h \in L^\infty(m)$. Then using (2.3) we obtain

$$\begin{aligned}&\|(\Phi_2(h_\alpha)-\Phi_2(h))x\|^2 \leq \\ &\leq \int |\rho(h_\alpha - h) + (1-\rho)\int (h_\alpha - h)dm|^2 \varphi_{x,x} dm \leq \\ &\leq 2[\rho^2 \int |h_\alpha - h|^2 \varphi_{x,x} dm + |1-\rho|^2 \int | \int (h_\alpha - h) dm|^2 \varphi_{x,x} dm] \leq \\ &\leq 2[\rho^2 \int |h_\alpha - h|^2 \varphi_{x,x} dm + |1-\rho|^2 \int |h_\alpha - h|^2 dm \cdot \int \varphi_{x,x} dm] = \\ &= 2\int |h_\alpha - h|^2 (\rho^2 \varphi_{x,x} + |1-\rho|^2 \|x\|^2) dm \longrightarrow_\alpha 0.\end{aligned}$$

The convergence to 0 is assured by Lebesgue's theorem, because $\mu = \varphi_x^{(\rho)} m$ is a $m - a.c.$ positive measure on X, where $\varphi_x^{(\rho)} = \rho^2 \varphi_{x,x} + |1-\rho|^2 \|x\|^2$. We infer that $\Phi_2(h_\alpha)x \longrightarrow \Phi_2(h)x$ in $\mathcal{H}$ for any $x \in \mathcal{H}$, and since $\Phi_p = \Phi_2|L^p(m)$ we have that $\{\Phi_p(h_\alpha)\}$ strongly converges to $\Phi_p(h)$ in $\mathcal{B}(\mathcal{H})$, for $p \geq 2$ (including and the case $p = \infty$ because $\Phi_\infty = \Phi_p|L^\infty(m)$ for $p < \infty$).

For the converse statement, we suppose now that Φ admits a bounded linear extension Ψ to $L^p(m)$ with $1 \leq p < \infty$. For $x, y \in \mathcal{H}$ the functional $\langle \Psi(\cdot)x, y\rangle$ is bounded linear on $L^p(m)$, so there exists $\psi_{x,y} \in L^q(m)$ such that

$$\langle \Psi(h)x, y\rangle = \int h\psi_{x,y} dm \qquad (h \in L^p(m)).$$

Since $\Psi|A = \Phi$ we have for $f \in A$ and $g \in A_\gamma$,

$$\begin{aligned}\int (f+\overline{g})\psi_{x,y} dm &= \langle \Psi(f+\overline{g})x, y\rangle = \langle (\Phi(f)+\Phi(g)^*)x, y\rangle = \\ &= \int [\rho(f+\overline{g}) + (1-\rho)\gamma(f+\overline{g})]\varphi_{x,y} dm = \\ &= \int (f+\overline{g})(\rho\varphi_{x,y} + (1-\rho)\langle x, y\rangle) dm.\end{aligned}$$

Using the weak* density of $A + \overline{A}_\gamma$ in $L^\infty(m)$ we obtain

$$\int h\psi_{x,y}dm = \int h(\rho\varphi_{x,y} + (1-\rho)\langle x, y\rangle)dm$$

for any $h \in L^\infty(m)$, hence $\psi_{x,y} = \rho\varphi_{x,y} + (1-\rho)\langle x, y\rangle$. This implies $\varphi_{x,y} \in L^q(m)$ and also

$$\|\varphi_{x,y}\|_q = \frac{1}{\rho}\|\psi_{x,y} + (\rho-1)\langle x, y\rangle\|_q \leq \Big(\frac{1}{\rho}\|\Psi\| + |1 - \frac{1}{\rho}|\Big)\|x\|\|y\|,$$

for any $x, y \in \mathcal{H}$. Thus, $\varphi_{x,y}$ satisfies (2.1) and this proves the converse statement when $p < \infty$. If $p = \infty$ that is we assume that Φ has a bounded linear extension Ψ to $L^\infty(m)$, then clearly we have

$$\langle \Psi(h)x, y\rangle = \int (\rho h + (1-\rho)\int hdm)\varphi_{x,y}dm$$

for all $h \in L^\infty(m)$ and $x, y \in \mathcal{H}$. Since $\varphi_{x,y} \in L^1(m)$ we get

$$\begin{aligned}
\|\varphi_{x,y}\|_1 &= \sup_{g \in L^\infty(m), \|g\| \leq 1} \left| \int g\varphi_{x,y}dm \right| = \\
&= \sup_{g \in L^\infty(m), \|g\| \leq 1} \left| \langle \Psi\left(\frac{1}{\rho} + \left(1 - \frac{1}{\rho}\right)\int hdm\right)x, y\rangle \right| \leq \\
&\leq \|\Psi\| \left(\frac{1}{\rho} + \left|1 - \frac{1}{\rho}\right|\right)\|x\|\|y\|,
\end{aligned}$$

and so $\varphi_{x,y}$ also satisfies (2.1) when $p = \infty$. This ends the proof. □

Remark 2.2. The equivalent conditions of Theorem 2.1 imply

$$\|\Phi\|_p := \sup_{f \in A, \|f\|_p \leq 1} \|\Phi(f)\| < \infty. \tag{2.4}$$

It is easy to see that the condition (2.4) is equivalent to the existence of a bounded linear extension $\widehat{\Phi}_p$ of Φ to $H^p(m)$. In this case, $\widehat{\Phi}_p$ is uniquely determined and it satisfies the relation (2.2) for $g \in H^p(m)$. In addition, the following property holds.

Proposition 2.3. *Let Φ be a representation of A on $\mathcal{H}$ as in Theorem 2.1 such that $\|\Phi\|_p < \infty$. Then*

$$\widehat{\Phi}_p(fg) = \widehat{\Phi}_p(f)\widehat{\Phi}_p(g) \quad (f \in H^\infty(m),\ g \in H^p(m)) \tag{2.5}$$

and, in particular, $\widehat{\Phi} := \widehat{\Phi}_p|H^\infty(m)$ is a representation of $H^\infty(m)$ on $\mathcal{H}$. Moreover, if $\{f_\alpha\} \subset H^\infty(m)$ is a bounded net which converges a.e. (m) to $f \in H^\infty(m)$, then $\{\widehat{\Phi}(f_\alpha)\}$ strongly converges to $\widehat{\Phi}(f)$ in $\mathcal{B}(\mathcal{H})$.

Proof. Let $g \in H^p(m)$ and $f, g_n \in A$ such that $g_n \to g$ in $L^p(m)$. Then $fg_n \to fg$ in $L^p(m)$, so

$$\widehat{\Phi}_p(fg) = \lim_n \Phi(fg_n) = \Phi(f)\Phi_p(g).$$

Now, if $f \in H^\infty$ and $\{f_\alpha\} \subset A$ is a net which converges to f in the weak* topology of $L^\infty(m)$ then for g, g_n as above and $x, y \in \mathcal{H}$ one has

$$\begin{aligned}
\langle \widehat{\Phi}_p(fg)x, y\rangle &= \int (\rho fg + (1-\rho)\int fg dm)\varphi_{x,y} dm = \\
&= \lim_n \lim_\alpha \int (\rho f_\alpha g_n + (1-\rho)\int f_\alpha g_n dm)\varphi_{x,y} dm = \\
&= \lim_n \lim_\alpha \langle \Phi(f_\alpha g_n)x, y\rangle = \lim_n \lim_\alpha \langle \Phi(f_\alpha)\Phi(g_n)x, y\rangle = \\
&= \lim_n \lim_\alpha \int (\rho f_\alpha + (1-\rho)\gamma(f_\alpha))\varphi_{\Phi(g_n)x,y} dm = \\
&= \lim_n \int (\rho f + (1-\rho)\int f dm)\varphi_{\Phi(g_n)x,y} dm = \\
&= \lim_n \langle \widehat{\Phi}_p(f)\Phi(g_n)x, y\rangle = \langle \widehat{\Phi}_p(f)\widehat{\Phi}_p(g)x, y\rangle.
\end{aligned}$$

So, property (2.5) is proved. This also gives that $\widehat{\Phi}_p$ is multiplicative on $H^\infty(m)$, therefore $\widehat{\Phi} := \widehat{\Phi}_p | H^\infty(m)$ is a representation of $H^\infty(m)$ on $\mathcal{H}$.

The second statement of the proposition can be infered as in the previous proof. □

Remark 2.4. If the representation Φ in Theorem 2.1 is contractive, that is $\rho = 1$ and $\|\Phi\| = 1$ (because $\Phi(1) = I$), then its extension Φ_p is also contractive, in the case when it exists. Indeed, if $\widetilde{\Phi}$ is as in the proof of Theorem 2.1, we have for $f \in A$, $g \in A_\gamma$ and $x, y \in \mathcal{H}$,

$$\begin{aligned}
\left| \int (f + \overline{g})\varphi_{x,y} dm \right| &= |\langle (\Phi(f) + \Phi(g)^*)x, y\rangle| = |\langle P_{\mathcal{H}}\widetilde{\Phi}(f + \overline{g})x, y\rangle| \leq \\
&\leq \|\widetilde{\Phi}(f + \overline{g})\| \|x\| \|y\| \leq \|f + \overline{g}\| \|x\| \|y\|,
\end{aligned}$$

because $\widetilde{\Phi}$ is a contractive representation of $C(X)$. From this inequality we infer by the density of $A + \overline{A}_\gamma$ in $L^p(m)$ that

$$\left| \int h\varphi_{x,y} dm \right| \leq \|h\|_\infty \|x\| \|y\| \qquad (h \in L^p(m)),$$

hence $\|\varphi_{x,y}\|_q \leq \|x\| \|y\|$. Thus, we can take $c = 1$ in (2.1) and from the proof of Theorem 2.1 we deduce (the case $\rho = 1$) that $\|\Phi_p\| \leq 1$, and finally $\|\Phi_p\| = 1$ because $\Phi_p(1) = I$.

3. REDUCTION TO FUNCTIONAL CALCULUS

In the sequel we denote by $H_0^p(m)$ the closure (weak*, if $p = \infty$) of A_γ in $L^p(m)$, that is

$$H_0^p(m) = \left\{ f \in H^p(m) : \int f dm = 0 \right\}.$$

We say ([17, 20, 21]) that $H_0^p(m)$ is *simply invariant* if the closure of $A_\gamma H_0^p(m)$ in $L^p(m)$ is strictly contained in $H_0^p(m)$. By Theorem 4.1.6 [20] (see also [17, 21]) if $H_0^p(m)$ is simply invariant then there exists a function $Z \in H_0^\infty(m)$ with $|Z| = 1$ a.e. (m) such that $H_0^p(m) = ZH^p(m)$.

As in Theorem 3 [14] one can prove that, if m_0 is the normalized Lebesgue measure on $\mathbb{T}$, there exists an isometric $*$-isomorphism τ of $L^p(m_0)$ onto a closed subspace of $L^p(m)$, taking $H^p(m_0)$ onto a closed subspace of $H^p(m)$, for $1 \le p \le \infty$. In fact, τ is defined by

$$(\tau h)(s) = h(Z(s))$$

for $h \in L^p(m_0)$ and a.e. (m) $s \in X$.

The following main result shows that under the simple invariance of $H_0^p(m)$ with $1 \le p \le 2$, the representations from Theorem 2.1 and their extensions to $H^p(m)$ can be reduced to functional calculus. For this we need to define the operator $S : H^p(m) \to L^p(m)$ by

$$Sg = \overline{Z}(g - \int gdm) \qquad (g \in H^p(m)). \tag{3.1}$$

Also, for $T \in \mathcal{B}(\mathcal{H})$ we denote by $r(T)$ the spectral radius of T.

Theorem 3.1. *Suppose that $H_0^p(m)$ is a simply invariant subspace for $1 \le p < \infty$, and let Φ be a representation of A on $\mathcal{H}$ satisfying Theorem 2.1. Then $r(\widehat{\Phi}(Z)) < 1$, and if $1 \le p \le 2$ one has*

$$\widehat{\Phi}_p(g) = \sum_{n=0}^{\infty} \widehat{g}(n) \widehat{\Phi}(Z)^n \qquad (g \in H^p(m)), \tag{3.2}$$

where $\widehat{g}(n) = \int \overline{Z}^n gdm$ for $n \in \mathbb{N}$, the series being absolutely convergent in $\mathcal{B}(\mathcal{H})$. Moreover, the relation (3.2) is also true when $2 < p < \infty$, for $g \in H^p(m)$ such that $\{S^n g\}$ is a bounded sequence in $H^p(m)$, S being the operator from (3.1).

Proof. The assumption on Φ means that $\varphi_{x,y}$ satisfies (2.1) for any $x, y \in \mathcal{H}$. As a bounded linear functional on $L^p(m)$, $\varphi_{x,y}$ induces, by the isomorphism τ, a bounded linear functional on $L^p(m_0)$, that is there exists $\varphi_{x,y}^0 \in L^q(m_0)$ satisfying

$$\int h\varphi_{x,y}^0 dm_0 = \int (\tau h)\varphi_{x,y} dm \qquad (h \in L^p(m_0)). \tag{3.3}$$

Since τ is an isometry we find

$$\|\varphi_{x,y}^0\|_q = \sup_{\|h\|_p=1} \left| \int h\varphi_{x,y}^0 dm_0 \right| = \sup_{\|\tau h\|_p=1} \left| \int (\tau h)\varphi_{x,y} dm \right| \le \|\varphi_{x,y}\|_q \le c\|x\|\|y\|,$$

with c as in (2.1).

Now from (3.3) and (2.2) we infer, for any analytic polynomial P, that

$$\int [\rho P + (1-\rho)P(0)]\varphi^0_{x,y} dm_0 = \int [\rho(P\circ Z) + (1-\rho)P(0)]\varphi_{x,y} dm = \\ = \langle \Phi_p(P\circ Z)x, y\rangle = \langle P(\Phi_p(Z))x, y\rangle.$$

So, using the previous inequality we get

$$|\langle P(\Phi_p(Z))x, y\rangle| \le \|\rho P + (1-\rho)P(0)\|_p \|\varphi^0_{x,y}\|_q \le \\ \le c(\rho + |1-\rho|)\|P\|_p \|x\|\|y\|,$$

and putting $c_\rho = c(\rho + |1-\rho|)$ one obtains

$$\|P(\Phi_p(Z))\| \le c_\rho \|P\|_p.$$

This means that the operator $\Phi_p(Z)$ is polynomially bounded. On the other hand, taking $P(\lambda) = \lambda^n$ for $n \in \mathbb{N}$ in the above equality, we obtain

$$\langle \Phi_p(Z)^n x, y\rangle = \rho \int \lambda^n \varphi^0_{x,y} dm_0$$

and so it follows that for $x, y \in \mathcal{H}$ there exists $\psi_{x,y} \in L^q(m_0)$ such that

$$\langle \Phi_p(Z)^{*n} x, y\rangle = \int \overline{\lambda}^n \psi_{x,y} dm_0 \qquad (n \in \mathbb{N}).$$

This yields that the operator $\Phi_p(Z)^*$ is absolutely continuous, and since $\psi_{x,y} \in L^q(m_0)$ with $q > 1$ (by the choose of p), from Lebow's theorem [13] we infer that $r(\Phi_p(Z)) < 1$.

The assumption that $H^0_p(m) = ZH^p(m)$ assures that the range of operator S from (3.1) is contained in $H^p(m)$, so $S \in \mathcal{B}(H^p(m))$. In addition, for $g \in H^p(m)$ we have

$$\int Sg dm = \int \overline{Z} g dm = \widehat{g}(1),$$

therefore $S^2 g = \overline{Z}(Sg - \widehat{g}(1))$, or $Sg = \widehat{g}(1) + Z(S^2 g)$. This also gives

$$g = \int g dm + Z(Sg) = \widehat{g}(0) + \widehat{g}(1)Z + Z^2(S^2 g).$$

Assume now that $g = \sum_{j=0}^{n-1} \widehat{g}(j)Z^j + Z^n(S^n g)$ for $n > 1$. Then

$$S^n g = \overline{Z}^n g - \sum_{j=0}^{n-1} \widehat{g}(j)\overline{Z}^{n-j},$$

whence we get $\int S^n g dm = \widehat{g}(n)$. So, we have $S^{n+1} g = \overline{Z}(S^n g - \widehat{g}(n))$, or $S^n g = \widehat{g}(n) + Z(S^{n+1} g)$, and by our assumption on g we obtain

$$g = \sum_{j=0}^{n} \widehat{g}(j)Z^j + Z^{n+1}(S^{n+1} g) \qquad (g \in H^p(m)). \tag{3.4}$$

Considering the extension $\widehat{\Phi}_p = \Phi_p | H^p(m)$ of Φ to $H^p(m)$ (as in Proposition 2.3) we get by (3.4) that

$$\begin{aligned}\|\widehat{\Phi}_p(g) - \sum_{j=0}^{n} \widehat{g}(j)\widehat{\Phi}(Z)^j\| = \|\widehat{\Phi}(Z^{n+1})\widehat{\Phi}_p(S^{n+1}g)\| \leq \\ \leq \|\widehat{\Phi}_p\|\|S^{n+1}g\|_p\|\widehat{\Phi}(Z)^{n+1}\|,\end{aligned} \tag{3.5}$$

for any $g \in H^p(m)$.

If $p = 2$, the operator S is a contraction on $H^2(m)$ that is

$$\|Sg\|_2 = \|g - \int gdm\|_2 \leq \|g\|_2,$$

because $g - \int gdm$ is the orthogonal projection of g on $H_0^2(m)$ for $g \in H^2(m)$. In this case, in (3.5) we have $\|S^{n+1}g\|_2 \leq \|g\|_2$ for any $n \in \mathbb{N}$, and since $\widehat{\Phi}(Z)^n \to 0$ $(n \to \infty)$ by a remark before, it follows that the representation (3.2) holds true for $g \in H^2(m)$.

Suppose now $1 \leq p < 2$. As $H^2(m)$ is dense in $H^p(m)$, for $g \in H^p(m)$ and every $\varepsilon > 0$ there exists $g_\varepsilon \in H^2(m)$ with $\|g - g_\varepsilon\|_p < \varepsilon$. Since $|\widehat{g}(n)| \leq \|g\|_p$ for $n \in \mathbb{N}$, the series from (3.2) is absolutely convergent in $\mathcal{B}(\mathcal{H})$ and applying the previous remark to g_ε we obtain

$$\begin{aligned}\|\sum_{n=0}^{\infty} \widehat{g}(n)\widehat{\Phi}(Z)^n - \Phi_p(g)\| \leq \|\sum_{n=0}^{\infty}(\widehat{g}(n) - \widehat{g}_\varepsilon(n))\widehat{\Phi}(Z)^n\| + \|\Phi_p(g_\varepsilon - g)\| \leq \\ \leq \|g - g_\varepsilon\|_p(\|\Phi_p\| + \sum_{n=0}^{\infty}\|\widehat{\Phi}(Z)^n\|) < \varepsilon M\end{aligned}$$

for some constant $M > 0$. Thus, the representations (3.2) occurs for any $g \in H^p(m)$, if $p \leq 2$. When $p > 2$, from the inequality (3.5) we infer that the equality (3.2) is also true for $g \in H^p(m)$ for which $\{S^n g\}$ is a bounded sequence in $H^p(m)$. The proof is finished. □

Remark 3.2. By (3.4) we have that the sequence $\{S^n g\}_n$ is bounded if and only if the sequence $\{\sum_{j=0}^{n} \widehat{g}(j)Z^j\}_n$ is bounded in $H^p(m)$, and in particular, this happens if S is a power bounded operator in $\mathcal{B}(H^p(m))$. But, even if the second sequence before converges, its limit is not necessary the function g. In fact, one has (by (3.4)) $g = \sum_{j=0}^{\infty} \widehat{g}(j)Z^j$ in $H^p(m)$ if and only if $S^n g \to 0$ $(n \to \infty)$; but this condition is false, in general, as we can see in the following

Example 3.3. Let A be the algebra of all continuous functions f on $\mathbb{T}^2$ having the Fourier coefficients

$$c_{ij} = \int_{\mathbb{T}^2} \overline{\lambda}^i \overline{w}^j f(\lambda, w) dm_2 \quad (i, j \in \mathbb{Z})$$

such that $c_{ij} = 0$ if either $j < 0$, or $j = 0$ and $i < 0$. Then A is a Dirichlet algebra on $\mathbb{T}^2$, while the normalized Lebesgue measure m_2 on $\mathbb{T}^2$ is the representing measure for the homomorphism of evaluation in $(0,0)$ of A. Here the function $Z \in H_0^\infty(m_2)$ is given by $Z(\lambda, w) = \lambda$, $\lambda, w \in \mathbb{T}$. On the other hand, for the function $g_0 \in H^\infty(m_2)$ defined by $g_0(\lambda, w) = w$, we have $(S^n g_0)(\lambda, w) = \lambda^n w$ and $\|S^n g_0\|_p = 1$, for any $n \in \mathbb{N}$, $\lambda, w \in \mathbb{T}$. Hence $\{S^n g_0\}$ is a bounded sequence which is not convergent to 0, in $H^p(m_2)$ for $1 \leq p \leq \infty$. Clearly, $\widehat{g_0}(n) = 0$ for any $n \geq 0$, therefore $\sum_{j=0}^n \widehat{g_0}(j)\lambda^j = 0$ for $n \geq 0$, what justifies the last assertion of Remark 3.2.

This example also provides that, in general under the hypothesis of Theorem 3.1, the space $H^p(m)$ is not spanned by $\{Z^n\}_{n\in\mathbb{N}}$, even if the operator S is power bounded. For instance, S is always a contraction on $H^2(m)$, but $\{Z^n\}_{n\in\mathbb{N}}$ becomes an orthonormal basis in $H^2(m)$ if and only if $H^\infty(m)$ is a maximal weak* closed algebra in $L^\infty(m)$, when m is the unique representing measure for γ, while $\{\gamma\}$ is not a Gleason part of A (see [1,6]).

If $H^p(m)$ is spanned by $\{Z^n\}_{n\in\mathbb{N}}$ then for any $g \in H^p(m)$ the representation (3.2) holds (by Remark 3.2), which means that the map Φ_p is reduced to a functional calculus. Theorem 3.1 shows that this fact occurs for Φ satisfying (2.1) for $2 \leq q \leq \infty$, but we cannot prove (3.2) in the case $2 < p \leq \infty$ (when $1 \leq q < 2$), the boundedness condition (2.1) for $\varphi_{x,y}$, being weakened in this case.

We see now that, from the point of view of the semispectral measure F_Φ, the cases when p belongs to the range $1 \leq p \leq 2$ are not essentially different, in Theorem 3.1.

Theorem 3.4. *Suppose $1 \leq p \leq 2$ and that $H_0^p(m)$ is a simply invariant subspace in $H^p(m)$. Let Φ be a representation of A on $\mathcal{H}$ satisfying Theorem 2.1. Then the semispectral measure F_Φ has the form $F_\Phi = \theta(\cdot)m$ where the function $\theta : X \to \mathcal{B}(\mathcal{H})$ is given by*

$$\theta(s) = \sum_{n=-\infty}^{\infty} \overline{Z}^n(s)\widehat{\Phi}(Z)_\rho^{(n)}, \tag{3.6}$$

while the series converges absolutely and uniformly a.e. (m) for $s \in X$. Moreover, θ is a bounded function a.e. (m) on X.

Proof. Since $r(\widehat{\Phi}(Z)) < 1$ (by Theorem 3.1) one can define the function

$$\theta_+(s) = \sum_{n=0}^{\infty} \overline{Z}^n(s)\widehat{\Phi}(Z)^n,$$

the series being absolutely and uniformly convergent a.e. (m) for $s \in X$. In addition, one has

$$\|\theta_+(s)\| \leq \sum_{n=0}^{\infty} \|\widehat{\Phi}(Z)^n\| \quad \text{(a.e. } (m)\ s \in X).$$

Then for $g \in H^p(m)$ and $x \in \mathcal{H}$ the function $g\langle\theta_+(\cdot)x, x\rangle$ belongs to $L^p(m)$, and we have (by (3.2) and (2.2)),

$$\int g\langle\theta_+(\cdot)x, x\rangle dm = \sum_{n=0}^{\infty} \widehat{g}(n)\langle\widehat{\Phi}(Z)^n x, x\rangle = \langle\sum_{n=0}^{\infty} \widehat{g}(n)\widehat{\Phi}(Z)^n x, x\rangle =$$
$$= \langle\Phi_p(g)x, x\rangle = \int (\rho g + (1-\rho)\int g dm)\varphi_{x,x} dm.$$

Equivalently, taking $\frac{1}{\rho}g + (1 - \frac{1}{\rho})\int g dm$ instead of g in this relation, we obtain

$$\int g\varphi_{x,x} dm = \int \left[\frac{1}{\rho}g(s) + \left(1 - \frac{1}{\rho}\right)\int g dm\right]\langle\theta_+(s)x, x\rangle dm =$$
$$= \int g(s)\left[\frac{1}{\rho}\langle\theta_+(s)x, x\rangle + \left(1 - \frac{1}{\rho}\right)\int\langle\theta_+(s)x, x\rangle dm\right] dm =$$
$$= \int g(s)\left[\frac{1}{\rho}\langle\theta_+(s)x, x\rangle + \left(1 - \frac{1}{\rho}\right)\|x\|^2\right] dm =$$
$$= \left(1 - \frac{1}{\rho}\right)\|x\|^2 + \frac{1}{\rho}\int\sum_{n=0}^{\infty} g\overline{Z}^n\langle\widehat{\Phi}(Z)^n x, x\rangle dm =$$
$$= \left(1 - \frac{1}{\rho}\right)\|x\|^2 + \frac{1}{\rho}\int g(s)\langle\theta(s)x, x\rangle dm,$$

where the function θ is defined as in (3.6), that is

$$\theta(s) = \theta_+(s) + \theta_+(s)^* - I \quad \text{(a.e. } (m)\ s \in X).$$

Clearly, we used before that $\int gZ^n dm = 0$ for $n > 0$.

Since $\varphi_{x,x}$ and $\langle\theta(\cdot)x, x\rangle$ are real functions, we get that

$$\int (f + \overline{g})\varphi_{x,x} dm = \int (f + \overline{g})\langle\theta(\cdot)x, x\rangle dm$$

for $f \in A$, $g \in A_\gamma$, and this gives $\varphi_{x,x} = \langle\theta(\cdot)x, x\rangle$ because A is weak* Dirichlet in $L^\infty(m)$. Hence θ is the Radon-Nikodym derivative of F_Φ with respect to m, and θ is bounded a.e. (m) on X, in fact

$$\|\theta(s)\| \leq 1 + \frac{2}{\rho}\sum_{n=0}^{\infty}\|\widehat{\Phi}(Z)^n\| \quad \text{(a.e. } (m)\ s \in X).$$

This ends the proof. □

From this theorem it follows that, for Φ as in Theorem 2.1, the $L^q(m)$-boundedness of $\varphi_{x,y}$ in the sense of (2.1) for any $x, y \in \mathcal{H}$ and some q in the range $2 \leq q \leq \infty$, is equivalent to the fact that the Radon-Nikodym derivative of F_Φ is a bounded function a.e. (m) on X, if $H_0^p(m)$ is simply invariant. In this last case, Φ can be extended to whole $L^1(m)$ as in Theorem 2.1 and one has $\Phi_p = \Phi_1|L^p(m)$ for $1 < p \leq \infty$. Moreover, if $1 \leq p \leq r \leq \infty$ then $\widehat{\Phi}_r = \widehat{\Phi}_p|H^r(m)$. Hence we infer from Theorem 3.1 the following

Corollary 3.5. *Suppose that for some $p \in [1,2]$ the subspace $H_0^p(m)$ is simply invariant, and let Φ be a representations of A on $\mathcal{H}$ satisfying Theorem 2.1. Then the relation* (3.2) *holds for $\widehat{\Phi}_r$ and any $g \in H^r(m)$ with $p \leq r \leq \infty$.*

Notice that the above results extend some facts from [12] where only the case $p = 2$ was considered. Remark also that the assertion $r(\widehat{\Phi}(Z) < 1$ in the corresponding version in [12] of Theorem 3.1 before was obtained in a different way, adapting an argument of M. Schreiber [19].

In turn the Theorem 3.4 shows that the semispectral measure F_Φ can be described by the operator $\widehat{\Phi}(Z)$. Conversely, $\widehat{\Phi}(Z)$ can be retrieved from F_Φ as follows.

Proposition 3.6. *Suppose that $H_0^p(m)$ is a simple invariant subspace for some $p \in [1,2]$, and let Φ be a representation of A on $\mathcal{H}$ satisfying Theorem 2.1. Then $\widehat{\Phi}(Z)$ is a ρ-contraction on $\mathcal{H}$ and we have*

$$\widehat{\Phi}(Z)_\rho^{(n)} = \int Z^n(s)\theta(s)dm \quad (n \in \mathbb{Z}), \tag{3.7}$$

where θ is function defined in (3.6)*.*

Moreover, if there exists $s_0 \in X$ and $\lambda \in \mathbb{C}$ such that $\theta(s_0) = \lambda I$ then $\widehat{\Phi}(Z)$ is a normal strict contraction.

Proof. The relation (3.7) follows immediately because we may integrate the series of θ term by term (by uniform convergence in norm), having in view that $\int Z dm = 0$. From (3.7) we infer for any analytic polynomial P and $x \in \mathcal{H}$ that

$$\begin{aligned}\langle P(\widehat{\Phi}(Z))x, x\rangle &= \int [\rho(P \circ Z)(s) + (1-\rho)P(0)]\langle\theta(s)x, x\rangle dm = \\ &= \int [\rho(P \circ Z)(s) + (1-\rho)P(0)]\varphi_{x,x}(s)dm,\end{aligned}$$

the last equality being ensured by Theorem 3.4. So, we obtain

$$\begin{aligned}|\langle P(\widehat{\Phi}(Z))x, x\rangle| &\leq \sup_{|\lambda|=1} |\rho P(\lambda) + (1-\rho)P(0)| \int \varphi_{x,x} dm = \\ &= \|\rho P + (1-\rho)P(0)\| \|x\|^2,\end{aligned}$$

whence

$$\sup_{\|x\|=1} |\langle P(\widehat{\Phi}(Z))x, x\rangle| \leq \|\rho P + (1-\rho)P(0)\|.$$

This last inequality just means that $\widehat{\Phi}(Z)$ is a ρ-contraction on $\mathcal{H}$ (see [1, 4, 6, 22]).

Suppose now that there exists $s_0 \in X$ and $\lambda \in \mathbb{C}$ such that $\theta(s_0) = \lambda I$. We write $\theta(s_0) = I + T + T^*$ where

$$T = \frac{1}{\rho}\sum_{n=1}^{\infty} \overline{Z}^n \widehat{\Phi}(Z)^n.$$

Then our assumption yields $TT^* = (\lambda - 1)T - T^2 = T^*T$, hence T is a normal operator. Since one has

$$\rho T = [I - \overline{Z}(s_0)\widehat{\Phi}(Z)]^{-1} - I,$$

we get

$$\widehat{\Phi}(Z) = Z(s_0)[I - (I + \rho T)^{-1}],$$

therefore $\widehat{\Phi}(Z)$ is a normal operator. This also gives $\|\widehat{\Phi}(Z)\| = r(\widehat{\Phi}(Z)) < 1$, that is $\widehat{\Phi}(Z)$ is a strict contraction. This ends the proof. □

The converse statement fails for the second assertion of Proposition 3.6, even in the case $\rho = 1$, and this fact was proved in [19, p.189], concerning the contractive representations of the disc algebra.

Theorem 3.4 can be also completed as follows.

Theorem 3.7. *Suppose that $H_0^p(m)$ is a simply invariant subspace for some $p \in [1, 2]$ and that $H^\infty(m)$ coincides to the weak* closure of the system $\{Z^n\}_{n\in\mathbb{N}}$. Let Φ be a representation of A on $\mathcal{H}$ satisfying Theorem 2.1. Then the semispectral measure F_Φ is mutually absolutely continuous with respect to m, and for every $x \in \mathcal{H}$, $x \neq 0$, the function $\log\langle\theta(\cdot)x, x\rangle$ belongs to $L^1(m)$, where θ is defined in* (3.6).

Proof. Since F_Φ is absolutely continuous with respect to m, it remains to prove the converse assertion.

Notice firstly that for $g \in H^\infty(m)$ one has $g = \sum_{n=0}^{\infty} \widehat{g}(n)Z^n$, and that $\{Z^n\}_{n\in\mathbb{N}}$ forms an orthogonal basis in $H^2(m)$. Since $L^2(m) = H^2(m) \oplus \overline{H_0^2(m)}$ (the bar meaning the complex conjugate), the isomorphism τ applies $L^2(m_0)$ onto $L^2(m)$, and $L^\infty(m_0)$ onto $L^\infty(m)$ too.

Let $\sigma \in Bor(X)$ and $0 \neq x \in \mathcal{H}$ such that $\langle F_\Phi(\sigma)x, x\rangle = 0$. By (3.3) we have ($\chi_\sigma$ being the characteristic function of σ)

$$\int (\tau^{-1}\chi_\sigma)(\tau^{-1}\langle\theta(\cdot)x, x\rangle)dm_0 = \int \chi_\sigma\langle\theta(\cdot)x, x\rangle)dm = \langle F_\Phi(\sigma)x, x\rangle = 0.$$

Since one has

$$(\tau^{-1}\langle\theta(\cdot)x, x\rangle)(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n \widehat{\Phi}(Z)_\rho^{(n)} \qquad (|\lambda| = 1),$$

this function is just the Radon-Nikodym derivative of the semispectral measure $\widehat{F}$ of $\widehat{\Phi}(Z)$ with respect to m_0 ($\widehat{\Phi}(Z)$ being a uniformly stable ρ-contraction, by Theorem 3.1 and Proposition 3.6). So, we have $\int(\tau^{-1}\chi_\sigma)d\langle\widehat{F}x, x\rangle = 0$, and since the measures m_0 and $\langle\widehat{F}x, x\rangle$ are equivalent (see [18]), while $\tau^{-1}\chi_\sigma$ is a positive function ($(\tau^{-1}\chi_\sigma)^2 = \tau^{-1}\chi_\sigma^2 = \tau^{-1}\chi_\sigma \geq 0$), it follows $\int(\tau^{-1}\chi_\sigma)dm_0 = 0$. Then we obtain

$$m(\sigma) = \int \chi_\sigma dm = \int (\tau^{-1}\chi_\sigma)dm_0 = 0,$$

hence the measures m and $\langle F_\Phi x, x\rangle$ are equivalent.

Now, by (3.3) we also have for $g \in H_0^\infty(m)$,

$$\int |1 - g(s)|^p \langle \theta(s)x, x\rangle dm = \int |1 - (\tau^{-1}g)(s)|^p d\langle \widehat{F}x, x\rangle.$$

But $\tau^{-1}H_0^\infty(m) = H_0^\infty(m_0)$, and so taking the infimum for $g \in H_0^\infty(m)$ in the previous equality we obtain by Szegö's Theorem 4.2.2 [20] that

$$\exp \int \log\langle \theta(s)x, x\rangle dm = \exp \int \log \tau^{-1}\langle \theta(\cdot)x, x\rangle dm_0.$$

Since the ρ-contraction $\widehat{\Phi}(Z)$ is completely non unitary, the right side of this equality cannot be 0 (by Theorem 3.8 [18]), hence $\log\langle \theta(\cdot)x, x\rangle \in L^1(m)$. The proof is finished. □

Note that the hypothesis on $H^\infty(m)$ in Theorem 3.7 is not verified for the algebra A in Example 3.3., as was proved in [6]. In the case that $H^\infty(m)$ is the weak* closure of $\{Z^n\}_{n\in\mathbb{N}}$, then for any $g \in H^\infty(m)$ we have $g = \sum\limits_{n=0}^{\infty} \widehat{g}(n)Z^n$ in $H^2(m)$. In this case, for every Φ as above, $\widehat{\Phi}(g) = \widehat{\Phi}_2(g)$ is given by (3.2), and it is easy to see that this means that the representations $\widehat{\Phi}$ of $H^\infty(m)$ on $\mathcal{H}$ is reduced to a functional calculus in the sense of Gaşpar [4, 6]. Finally, let us note that the case $\rho = 1$ of Theorem 3.7 is contained in Theorem 2.3.2 [6].

4. APPLICATION TO THE SCALAR CASE

In this section we consider the case when Φ is a homomorphism of A, this is the one-dimensional case $\mathcal{H} = \mathbb{C}$. In this context, we generalize to a weak* Dirichlet algebra some classical results concerning the function algebra with the uniqueness property for representing measures ([2, 7, 21]).

Theorem 4.1. *Suppose that $H_0^p(m)$ is a simply invariant subspace for some $p \in [1, 2]$. Then for any homomorphism $\varphi \in \mathcal{M}(A)$ with $\|\varphi\|_p < \infty$ we have $|\widehat{\varphi}(Z)| < 1$ and*

$$\varphi_p(g) = \sum_{n=0}^{\infty} \widehat{g}(n)\widehat{\varphi}(Z)^n \quad (g \in H^p(m)), \tag{4.1}$$

where φ_p respectively ($\widehat{\varphi}$) is the bounded linear extension of φ to $H^p(m)$ (respectively, to $H^\infty(m)$), the series being absolutely convergent.
Moreover, the measure

$$\mu = \frac{1 - |\varphi(Z)|^2}{|Z - \varphi(Z)|^2} m \tag{4.2}$$

is a representing measure for φ.

Proof. Let $\varphi \in \mathcal{M}(A)$ with $\|\varphi\|_p = \sup\{|\varphi(f)| : f \in A, \|f\|_p \leq 1\} < \infty$. Assume, by contrary, that $|\widehat{\varphi}(Z)| = 1$. Since Z is uniquely determined by a scalar λ with $|\lambda| = 1$, one can suppose that $\widehat{\varphi}(Z) = 1$. Then for $n \geq 1$ there exists a function $f_n \in A(\mathbb{T})$ of the form $f_n(\lambda) = \sum\limits_{j=0}^{\infty} c_j \lambda^j$ with $f_n(1) = n$ and $\|f_n\|_p \leq 1$, because 1 is a Choquet point for the standard algebra $A(\mathbb{T})$ ([2, 21]). So, $\tau f_n \in H^p(m)$ and we have

$$\varphi_p(\tau f_n) = \varphi_p(\sum_{j=0}^{\infty} c_j Z^j) = \sum_{j=0}^{\infty} c_j \widehat{\varphi}(Z)^j = \sum_{j=0}^{\infty} c_j = f_n(1) = n$$

and $\|\tau f_n\|_p = \|f_n\|_p \leq 1$, contradicting the fact that φ is bounded on $H^p(m)$. Hence $|\widehat{\varphi}(Z)| < 1$.

Now, we can apply Theorem 3.1 for φ to obtain (4.1). Next, since $|\widehat{\varphi}(Z)| < 1$, the function

$$\theta_0 = \sum_{n=-\infty}^{\infty} \overline{Z}^n \widehat{\varphi}(Z)^{(n)}$$

is well defined and bounded a.e. (m) on X. In fact, because

$$\begin{aligned}\theta_0 &= \sum_{n=0}^{\infty} \overline{Z}^n \widehat{\varphi}(Z)^{(n)} + \sum_{n=1}^{\infty} Z^n \overline{\widehat{\varphi}(Z)}^n = \\ &= \frac{1}{1 - \overline{Z}\widehat{\varphi}(Z)} + \frac{Z\overline{\widehat{\varphi}(Z)}}{1 - Z\overline{\widehat{\varphi}(Z)}} = \frac{1 - |\widehat{\varphi}(Z)|^2}{|Z - \widehat{\varphi}(Z)|^2},\end{aligned}$$

θ_0 is positive and $\int \theta_0 dm = 1$, hence $\mu = \theta_0 m$ is a probability measure on X. Clearly, we have by (4.1) for $f \in A$,

$$\int f d\mu = \sum_{n=-\infty}^{\infty} \widehat{\varphi}(Z)^{(n)} \int \overline{Z}^n f dm = \sum_{n=0}^{\infty} \widehat{f}(n)\widehat{\varphi}(Z)^n = \varphi(f),$$

that is μ is a representing measure for φ. This ends the proof. □

Remark that only boundedness of φ on $H^p(m)$ assures that φ is $m - a.c.$ that is φ has a $m - a.c.$ representing measure, if $H_0^p(m)$ is simply invariant. In the general setting of Theorem 3.1, we cannot prove $r(\widehat{\Phi}(Z)) < 1$ without assuming that Φ is $m - a.c.$

Concerning the existence of homomorphism of A which are bounded on $H^p(m)$, we give the following result which generalize Theorem 6.4 [21] (or Theorem V 7.1, and Theorem VI 7.2 of [1]) in the context of weak* Dirichlet algebras.

Theorem 4.2. *Suppose that $H_0^p(m)$ is a simple invariant subspace for some $p \in [1, 2]$. Then the set $\Delta_p(m)$ of all homomorphisms of A which are bounded on $H^p(m)$ is not reduced to $\{\gamma\}$, and $\Delta_p(m)$ is contained in the Gleason part of A which contains γ. Moreover, there exists a one to one continuous map Γ from $\mathbb{D}$ into $\mathcal{M}(A)$ such that:*

(i) $\Gamma(\mathbb{D}) = \Delta_p(m)$, $\Gamma(0) = \gamma$,

(ii) *For any $f \in A$, the function $\widehat{f} \circ \Gamma$ is analytic on $\mathbb{D}$, where $\widehat{f}$ is the Gelfand transform of f.*

Proof. Let $\Delta_p(m) := \{\varphi \in \mathcal{M}(A) : \|\varphi\|_{H^p(m)} < \infty\}$. For $\varphi \in \Delta_p(m)$ we have by Theorem 4.1 that $|\widehat{\varphi}(Z)| < 1$ where $Z \in H_0^\infty(m)$, $|Z| = 1$ a.e. (m) such that $H_0^p(m) = ZH^p(m)$. We define the map $\Gamma_0 : \Delta_p(m) \to \mathbb{D}$ by $\Gamma_0(\varphi) = \widehat{\varphi}(Z)$, $\varphi \in \Delta_p(m)$.

Firstly, Γ_0 is one to one because if $\Gamma_0(\varphi_0) = \Gamma_0(\varphi_1)$ for $\varphi_0, \varphi_1 \in \Delta_p(m)$ then by (4.1) we have $\varphi_0(f) = \varphi_1(f)$ for $f \in A$, so $\varphi_0 = \varphi_1$. Γ_0 is also onto $\mathbb{D}$. Indeed, for $z \in \mathbb{D}$ we define the linear functional φ_z on A by

$$\varphi_z(f) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \quad (f \in A).$$

Obviously, one has

$$|\varphi_z(f) \leq \frac{\|f\|_p}{1 - |z|},$$

because $|\widehat{f}(n)| \leq \|f\|_p$ for $f \in A$. It is also easy to see (as in the proof of Theorem 6.4 [21]) that φ_z is multiplicative on A, therefore $\varphi_z \in \mathcal{M}(A)$. From the above estimation we have

$$\|\varphi_z\| \leq \frac{1}{1 - |z|},$$

hence $\varphi_z \in \Delta(m)$, and clearly, $\Gamma_0(\varphi_z) = \widehat{\varphi}_z(Z) = z$ that is Γ_0 is surjective. In addition, by Theorem 4.1 a representing measure for φ_z is m_z given by

$$m_z = \frac{1 - |\widehat{\varphi}_z(Z)|^2}{|Z - \widehat{\varphi}_z(Z)|^2} m = \frac{1 - |z|^2}{|Z - z|^2} m.$$

So, the measures m and m_z are mutually absolutely continuous and their corresponding Radon-Nikodym derivatives are bounded a.e. (m) on X. This means that φ_z belongs to the Gleason part $\Delta(\gamma)$ of A which contains γ (see [2, 21]). As Γ_0 is a bijection from $\Delta_p(m)$ onto $\mathbb{D}$, we infer that

$$\{\gamma\} \subsetneqq \Delta_p(m) = \{\varphi_z : z \in \mathbb{D}\} \subset \Delta(\gamma).$$

Now, $\Gamma = \Gamma_0^{-1}$ is one to one from $\mathbb{D}$ onto $\Delta(m)$ and for $f \in A$ and $z \in \mathbb{D}$ we obtain by (4.1),

$$(\widehat{f} \circ \Gamma)(z) = \widehat{f}(\varphi_z) = \varphi_z(f) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n,$$

hence $\widehat{f} \circ \Gamma$ is an analytic function on $\mathbb{D}$. Finally, Γ is a continuous map on $\mathbb{D}$, relative to the Gelfand topology in $\mathcal{M}(A)$, and $\Gamma(0)(f) = \widehat{f}(0) = \int f dm = \gamma(f)$ for $f \in A$, so $\Gamma(0) = \gamma$. This ends the proof. □

Remark 4.3. If for a function algebra A on X, m is the unique representing measure for $\gamma \in \mathcal{M}(A)$, then A is weak* Dirichlet in $L^\infty(m)$, and any $\varphi \in \Delta(\gamma)$ has a unique representing measure which is bounded absolutely continuous with respect to m ([2], Cor. IV 1.2). This gives $\|\varphi\|_{H^p(m)} < \infty$ for $\varphi \in \Delta(\gamma)$, hence $\Delta(\gamma) = \Delta_p(m) \neq \{\gamma\}$ in this case, if $H_0^p(m)$ is simple invariant for some $p \in [1, 2]$. Furthermore, only assumption $\Delta(\gamma) \neq \{\gamma\}$ assures that $H_0^p(m)$ is simply invariant, in the case of unique representing measure (see Theorem 6.4 [21], or Theorem V 7.2 [1]).

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