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A SAMPLING THEORY FOR INFINITE WEIGHTED GRAPHS

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Abstract. We prove two sampling theorems for infinite (countable discrete) weighted graphs G; one example being "large grids of resistors" i.e., networks and systems of resistors. We show that there is natural ambient continuum X containing G, and there are Hilbert spaces of functions on X that allow interpolation by sampling values of the functions restricted only on the vertices in G. We sample functions on X from their discrete values picked in the vertex-subset G. We prove two theorems that allow for such realistic ambient spaces X for a fixed graph G, and for interpolation kernels in function Hilbert spaces on X, sampling only from points in the subset of vertices in G. A continuum is often not apparent at the outset from the given graph G. We will solve this problem with the use of ideas from stochastic integration.

Keywords: weighted graph, Hilbert space, Laplace operator, sampling, Shannon, white noise, Wiener transform, interpolation.

Mathematics Subject Classification: 05C22, 68R01, 81T05, 42A99, 47L60, 94A20, 47B35.

1. INTRODUCTION

Sampling here means sampling in the sense of Shannon, see e.g., [21], but Shannon's viewpoint is extended to a general class of infinite weighted graphs. This is non-trivial for several reasons: for example, in the general contexts of infinite graphs we do not have at our disposal traditional tools from Fourier analysis. Our substitute will make use of operator theory and stochastic integration.

1.1. SIGNALS

While Shannon had in mind signals, or time-series, in restating his well known theorem below, we will use the framework of Hilbert space: In one dimension, the ambient Hilbert space is the L^2-space of functions on the real line $\mathbb{R}$. In higher dimension, we work with the Hilbert space $L^2\left(\mathbb{R}^d\right)$. In this context, but not in these words, Shannon proved that if functions f in L^2 are restricted to be band-limited (so a closed subspace in L^2), then the values $f(t)$ for t in $\mathbb{R}^d$, may be interpolated from the sampled values $f(n)$ with n ranging over a rank-d lattice in $\mathbb{R}^d$. There is an interpolation kernel, the well known Shannon kernel. Now in d variables, the rank-d lattices are just considered one example of an infinite graph; graphs arising in networks of resistors, or even as models for the internet.

If we denote the standard integer lattice (of rank d) by $\mathbb{Z}^d$ (so points x in $\mathbb{Z}^d$ will be vertices), then there is a standard way of assigning 2d nearest neighbors to each vertex x. However the variety of weighted graph is far richer than just the finite-rank lattices. But even for the rank-d lattices, there is a variety of applications: For example in electric networks, or in statistical mechanics, weights are assigned to each of the 2d edges, and for each vertex x. This way, a weight of the graph G becomes a positive function defined on the edges of the specified graph G. Below we develop the general theory, illustrate its applications; and we obtain Shannon's result as a special case. An especially attractive statistical mechanics application is [24], and [9].

Now Shannon's view is motivated by signal processing, i.e., engineering of signals, see e.g., [12]: interpolation of functions (signals) on a continuum, determining band-limited functions defined on a continuum from their discrete samples. But in some cases outside the study of lattice graphs, one might only have available the particular given (countably discrete!) graph G; no ambient continuum; and indeed there might well be a variety of choices for an ambient continuum. This is the viewpoint taken in the present paper.

In more detail, starting with a fixed infinite graph G, it will be convenient for us to denote the set of vertices $G^{(0)}$, and the edges $G^{(1)}$. And we will study functions on both sets; more precisely, Hilbert spaces obtained by completion in certain quadratic forms; as well as the interconnections between spaces of functions on $G^{(0)}$, and on $G^{(1)}$.

1.2. ELECTRICAL NETWORKS

In electrical network theory, for example we typically consider configurations of voltages and of currents. In this case, the interconnections between spaces may be understood with the use of the two rules for electrical networks, Ohm's law, and Kirchhoff's sum-rules for current flow. In this case, therefore functions on $G^{(0)}$ represent voltages, for example voltage potentials; while functions on $G^{(1)}$ can be a configuration of Kirchhoff-current (measured in Amps).

In a variety of frameworks, we will develop this analysis on weighted graphs; and these graphs in turn arise in a host of applications, the network theory being just one of them.

To make our initial summary more visual, we will illustrate the concepts with reference to "grids of resistors", i.e., networks and systems of resistors. And indeed, there are already a number of separate applications of the same theory. Nonetheless we are not aware of theorems that allow for realistic ambient spaces X for a fixed infinite weighted graph G; as well as an interpolation in function spaces on X, from sampling points in the vertices of G. One reason for this is that, starting with G, the choice of such an ambient continuum X is not apparent from the given graph G.

We will solve this problem with the use of ideas from stochastic integration.

Now the study of weighted graphs is also a part of the wider subject of random walk analysis. In this wider context, however, we are really dealing with an analysis of weighted graphs. We are looking at a positive function c (the notation c is for conductance, and $c = 1/\text{resistance}$). We define a weighted graph to be a pair (G, c) where c is a fixed positive function defined on the set of (unordered) edges $G^{(1)}$.

The network point of view is developed in recent papers by the present authors and Erin Pearse. This work in turn is motivated by electrical engineering, probability, and statistical mechanics.

We start with a given large systems of resistors (G, c), G for graph, and $c = 1/\text{resistance}$ for conductance. Since the resistance is independent of direction, it is therefore natural to restrict attention to undirected graphs. Recall that the resistance between two vertices, linked by an edge, is a positive number measured in Ohm.

The following thought-experiment is illuminating: Insert an amp at one fixed vertex x, and extract it at another vertex y. This will generate a current, and a voltage distribution in the entire graph. Now the current is directed; and as a result we get an induced directed graph. But with the direction on the edges depending on choices: A different amp-in amp-out experiment, will induce a different current configuration on G.

We might even extract the one amp at a point at infinity, so this then entails a subtle analysis of boundaries of infinite graphs.

Take for example the case when G is an infinite tree, then one easily sees that the boundary of G, bdG should be a Cantor set; but to make this precise we must introduce a Laplace operator on G (this will depend on the choice of conductance function c); a suitable Hilbert space in such a way that bdG acquires properties in the present discrete context, otherwise familiar from classical (continuous) harmonic analysis, for example the harmonic functions on G must have an integral representation, with an integral kernel on the Cartesian product $G \times bdG$, much like the more familiar Poisson kernel from the study of harmonic functions on an open domain in the complex plane.

1.3. THE GRAPH LAPLACIAN

We then turn to the development of formula for the Laplace operator of such a given weighted graph (G, c), and we introduce a Hilbert space $\mathcal{H}(G, c)$. The Laplace operator will be Hermitian as a densely defined linear operator in $\mathcal{H}(G, c)$; depending on the context, it could be bounded or unbounded. It might even have non-zero deficiency indices in the sense of von Neumann.

Now this Hilbert space $\mathcal{H}(G, c)$ must account for the essential ingredients in a harmonic analysis. While it might be natural to try the more simple looking L^2 sequence space $\ell^2\left(G^{(0)}\right)$ as a candidate for Hilbert space, this will not work. The papers by Jorgensen and Pearse make the case that "the right" Hilbert space is $\mathcal{H}(G, c)$; called the energy Hilbert space. The study of these Hilbert spaces also entails notions of resistance distance, see [20].

This $\mathcal{H}(G,c)$ turns out to be a reproducing kernel Hilbert space (RKHS), but not in the traditional sense of the notion of RKHS, see [7,8]. It turns out that the reproducing kernel in $\mathcal{H}(G,c)$ is a function depending on a pair of points in $G^{(0)}$; not just one point, which is the traditional notion of RKHS; see e.g., [17].

In a number of recent papers, two different approaches to analysis on graphs have been used. While separate teams of authors, ask closely related questions about the graphs under study, the tools are different. In this paper we combine results from both, and use them in the solution of two questions.

The first one is based on algebras of operators, their representations, and on harmonic analysis of groupoid actions. The second approach has been focused more on analysis of networks, typically very large (infinite) networks. In mathematical terms the object here is weighted graphs. In our discussion below, we say that a graph G consists of two sets, vertices and edges. In the simplest case we may view edges as pairs of vertices.

1.4. THE LITERATURE

Both for groupoid actions (see [10,11]) and for networks [20], we study functions on vertices, and on edges. And both approaches make use of combinatorial tools, as well as symbolic dynamics computations in words on edges. A finite word in an alphabet A is thought as a finite path in G, and we shall be considering infinite paths as well.

A finite word w consists of a finite number of edges lined up so pairs of edges in w are linked by a common vertex. Two words w and w' may be concatenated forming the word ww'. It is possible when the terminal vertex of w agrees with the initial vertex of w'. This in this natural way we get a groupoid. Also note that concatenation of words w and w' forming the new concatenated word ww' is possible when w' is an infinite word. The result is understood naturally in the form of a groupoid action; a boundary action.

There is a recent renewed interest in sampling techniques, motivated in part by quantization requirements in digital signal processing, and by contrast to our present approach, much of this is based on harmonic analysis tools, see e.g., [1,2,6,13,14], and other investigations on Markov processes [15,18–20,22,23,26].

2. REPRODUCING KERNEL HILBERT SPACES

The notion of reproducing kernel Hilbert spaces (RKHSs) comes up in a host of applications in solutions to PDE problems, and in signal processing. It is almost ubiquitous: We only need the structure of a Hilbert space $\mathcal{H}$ of functions on some set X, discrete or otherwise, so a Hilbert space-inner product on some function space.

Detailed citations are in the Introduction above, and in the detailed discussion below. The idea is that for f in $\mathcal{H}$, the values $f(x)$ must be represented by inner product with a vector in H, depending on the point x. In graph analysis, one studies functions representing voltage differences, and it is therefore natural to ask instead that differences $f(x) - f(y)$ be represented by inner product with a vector in $\mathcal{H}$, now depending on both of the point x and y.

As outlined in section 1 above, we shall need two classes of Reproducing Kernel Hilbert Spaces (RKHSs), abbreviated as *monopole-kernels* and *dipole-kernels.*

Definition 2.1.

(a) A Hilbert space $\mathcal{H}$, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$, $u \in \mathcal{H}$, is called a Reproducing Kernel Hilbert Space (RKHS), if there is a set X such that the following conditions hold:
  (i) Elements u in $\mathcal{H}$ are functions on X, i.e., $u : X \to \mathbb{C}$;
  (ii) For every x in X, there is a (unique) function $k_x : X \to \mathbb{C}$ such that $k_x \in \mathcal{H}$, and
$$u(x) = \langle k_x, u\rangle \tag{2.1}$$
holds for all $u \in \mathcal{H}$.

(b) Let $\mathcal{H}$, X be a RKHS, and set
$$k(x, y) := k_y(x), \ \forall (x, y) \in X \times X. \tag{2.2}$$
Then $\langle k_x, k_y\rangle =: k(x, y)$ is called the reproducing kernel for $(\mathcal{H}, X)$.

(c) Consider a pair $(\mathcal{H}, X)$ where $\mathcal{H}$ is a Hilbert space of functions on a set X. We say that it is a *relative* RKHS if for every pair of points $(x, y) \in X \times X$ there is a (unique) function
$$v_{x,y} : X \to \mathbb{C}$$
such that
$$u(x) - u(y) = \langle v_{x,y}, u\rangle \tag{2.3}$$
holds for all $u \in \mathcal{H}$. In this case the family $(v_{x,y})_{x,y\in X\times X}$ will be called a *relative* reproducing kernel; or a system of *dipoles* for $(\mathcal{H}, X)$.

(d) In the framework of (c), it will be convenient to work with equivalence classes of functions on X. We say that two functions u_1, and u_2, or X are *equivalent* if they differ by a constant function, i.e., if there is a constant $c \in \mathbb{C}$, such that
$$u_1(x) - u_2(x) = c, \ \forall x \in X. \tag{2.4}$$

Below we illustrate all the points (a)–(d) by examples.

Example 2.2. The following three examples serve to illustrate the abstract definitions; and they will all play a key role in our subsequent discussion:

— L^2_B: *band-limited* L^2-functions u defined in the whole real line $\mathbb{R}$. The condition "band-limited" refers to the Fourier transform
$$\hat{u}(\xi) := \int_{\mathbb{R}} e^{-i\xi x} u(x)\ dx, \ \forall \xi \in \mathbb{R} \tag{2.5}$$

and we assume that $\hat{u}(\cdot)$ is supported in the compact interval (frequency band) $[-\pi, \pi]$.

Claim: L^2_B is a RKHS with kernel function

$$k(x,y) = \frac{\sin \pi (x-y)}{\pi (x-y)}, \ \forall (x,y) \in \mathbb{R} \times \mathbb{R}. \tag{2.6}$$

— $\mathcal{W}_2$: Here the inner-product and norm is different from that of L^2_B. We set

$$\|u\|^2_w = \int_0^1 |u'(x)|^2 \, dx, \tag{2.7}$$

where $u' = \frac{du}{dx}$. Note that the Hilbert space $\mathcal{W}_2$, defined by $\|u\|^2_w = \int_0^1 |u'(x)|^2 \, dx < \infty$, only defines functions on $I = [0,1]$ up to equivalence, see [4]. In this case, the dipole-kernels are as follows:

$$v_{x,0}(y) = x \wedge y = \min(x,y), \ \forall (x,y) \in I \times I. \tag{2.8}$$

— $\mathcal{H}_E = \mathcal{H}(G,c)$ = the Hilbert space of all finite-energy functions defined on the vertices of some weighted graph (G,c).

Recall a weighted graph (G,c) is a system of two sets $G^{(0)}$: vertices; $G^{(1)}$: edges, where $G^{(1)} \subset G^{(0)} \times G^{(0)}$; and $c: G^{(1)} \to \mathbb{R}_+$ (the positive reals) such that:

(i) $c(x,y) = c(y,x), \ \forall (x,y) \in G^{(1)}$ and
(ii) $\forall x \in G^{(0)}$ set $\left\{ y \in G^{(0)} \,|\, c(x,y) > 0 \right\}$ is finite.

If $(x,y) \in G^{(1)}$ satisfies (ii), then we write $x \sim y$.

By analogy with [7], for functions u on $G^{(0)}$, we set

$$\|u\|^2_E = \frac{1}{2} \sum \sum_{x \sim y} c(x,y) \, |u(x) - u(y)|^2 . \tag{2.9}$$

We say that $u \in \mathcal{H}_E$ (=: $\mathcal{H}(G,c)$) if $\|u\|_E < \infty$.

We will further assume that all weighted graphs are *connected*. By this we mean that for any two vertices $x, y \in G^{(0)}$, there is a finite path

$$x_0, x_1, \ldots, x_n$$

of edges

$$(x_{i-1}, \ x_i) \in G^{(1)}, i = 1, 2, \ldots, n \text{ such that } x_0 = x_1, \text{ and } x_n = y. \tag{2.10}$$

Finally, in our consideration of systems (G,c) we shall exclude self-loops (Fig. 1). By this we mean that for every $(y,z) \in G^{(1)}$ it will always be the case that $y \neq z$.

Finally, it will be convenient to select a base point $o \in G^{(0)}$. In this case the dipole $v_{x,0}$ will simply be denoted v_x. The family of all dipoles $\{v_x\}$ will be indexed by the punctured set $G^{(0)} \setminus \{o\}$; and we chose the normalization

$$v_x(o) = 0. \tag{2.11}$$

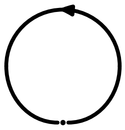

Fig. 1. Self-loops

Lemma 2.3. *Let $\mathcal{H}_E = \mathcal{H}(G, c)$, $o \in G^{(0)}$ fixed, be as above; then there exists a unique real valued $v_x \in \mathcal{H}_E$, such that*

$$u(x) - u(o) = \langle v_x, u \rangle_E = \frac{1}{2} \sum_{y \sim z} \sum_{in\ G^{(1)}} c(y, z)(v_x(y) - v_x(z))(u(y) - u(z)) \quad (2.12)$$

holds for all $u \in \mathcal{H}_E$; see (2.3).

Proof. Let (G, c) be as above. A simple Schwarz-estimate shows that for all $x \in G^{(0)}$, there is a finite constant $A = A_x$ such that

$$|u(x) - u(o)| \leq A \|u\|_E \quad (2.13)$$

holds for all $n \in \mathcal{H}_E$. If the points $(x_i)_{i=1}^n$ as in (2.10) are chosen such that $x_0 = o$ and $x_n = x$, then we may take

$$A = \left(\sum_{i=1}^{n} \frac{1}{c(x_{i-1}, x_i)} \right)^{\frac{1}{2}}, \quad (2.14)$$

but this is generally not at all the best constant A in (2.13); see [20]. □

Proof of (2.6). Even though the formula for the kernel $k(x, y)$ in (2.6) is well known, see e.g., [21], we shall need the reasoning, and we sketch the arguments in outline, referring to [21], or other books for details.

Now let $u \in L^2_B$, i.e., $u\colon \mathbb{R} \to \mathbb{C}$ is in $L^2(\mathbb{R})$, and

$$\operatorname{supp}(\hat{u}) \subseteq [-\pi, \pi]. \quad (2.15)$$

Then for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
\left\langle \frac{\sin \pi (x-\cdot)}{\pi (x-\cdot)}, u \right\rangle_B &= \int_{\mathbb{R}} \frac{\sin \pi (x-y)}{\pi (x-y)} u(y) \ dy = \\
&=_{(\text{Fourier})} \int_{\mathbb{R}} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi(x-y)} \ d\xi \ u(y) \ dy = \\
&=_{(\text{Fubini})} \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\mathbb{R}} e^{-i\xi y} \ u(y) \ dy \ e^{i\xi x} \ d\xi = \\
&=_{(\text{by } (2.5))} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{u}(\xi) e^{i\xi x} \ d\xi = \\
&=_{(\text{by } (2.15))} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}(\xi) e^{i\xi x} \ d\xi = \\
&=_{(\text{by inverse Fourier transform})} u(x),
\end{aligned}$$

which is the desired reproducing kernel formula; see (2.1) and (2.6). □

Proof (sketch) of (2.8). We now prove that the family of functions (k_x), indexed by $x \in I$, from (2.8) are di-pole reproducing kernels in the Hilbert space $\mathcal{W}_2$ from the list in Example 2.2. This is also the Hilbert space which defines the correlations of the standard Brownian motion on the line $\mathbb{R}$, or rather restricted to the unit interval $I = [0,1]$ as outlined above. With $k_x(\cdot)$ given by (2.8), and the $\mathcal{W}_2$-inner product by (2.7), we now get $(u \in \mathcal{W}_2, \ 0 < x < 1)$:

$$\begin{aligned}
\langle k_x, u \rangle_W &=_{\left(\substack{\text{distribution}\\ \text{derivative}}\right)} \int_0^1 \frac{d}{dy} k_x(y) u'(y) \ dy = \\
&=_{(\text{by parts})} - \int_0^1 k_x(y) u''(y) \ dy + x u'(1) = \\
&= - \int_0^x y u''(y) \, dy - x \int_x^1 u''(y) \ dy + x u'(1) = \\
&=_{(\text{by parts})} \int_0^x u'(y) \ dy - x u'(x) - x (u'(1) - u'(x)) + x u'(1) = \\
&=_{(\text{by cancellation})} \int_0^x u'(y) \ dy = \\
&=_{(\text{calculus})} u(x) - u(0);
\end{aligned}$$

which is the desired conclusion (2.3); i.e., the property which characterizes di-poles. □

3. SHANNON'S FORMULA

In the early days of signal processing, Shannon introduced classes of band-limited functions, and proved that these functions admit interpolation from discrete sub-samples, for examples sampling from integral multiples of some fixed sampling-rate; see precise details below. While Shannon's theorem lies at the root of early information theory, its scope and applications are typically shaped by tools from harmonic analysis. In our present context the setting of arbitrary infinite graphs, and their function spaces, does not invite Fourier duality. For one thing, there is not a group associated with the questions we address. Nonetheless we will prove that there is a host of function theoretic problems on arbitrary infinite graph which are amenable to precisely the kind of interpolation envisioned by Shannon.

We begin with the familiar version of Shannon's interpolation formula in the context of the Hilbert space L_B^2 in Example 2.2; but note that it holds much more generally, see the discussion in section 1.

Our present purpose is to extend Shannon's formula to the graph Hilbert spaces $\mathcal{H}_E = \mathcal{H}(G, c)$ defined for the infinite weighted graphs introduced in sections 1 and 2.

Lemma 3.1 (Shannon)**.** *If $u \in L_B^2$ and $x \in \mathbb{R}$, then the following interpolation formula holds:*

$$u(x) = \sum_{n \in \mathbb{Z}} u(n) \frac{\sin \pi (x-n)}{\pi (x-n)}, \tag{3.1}$$

where the summation in (3.1) *is over the subgroup of the integers $\mathbb{Z} \subset \mathbb{R}$, and is absolutely convergent.*

Proof sketch. In this version, Shannon's formula is very well known; see e.g. [21], but we include a proof outline because the basic idea will be needed in our extension of the formula to arbitrary infinite weighted graphs (G, c).

First recall that the assumption on the function u entails the support restriction in (2.15). Hence we may use a Fourier series expansion for the function $\hat{u}(\xi)$, $\xi \in [-\pi, \pi]$; recall $\hat{u}(\cdot) = 0$ in $\mathbb{R} \setminus [-\pi, \pi]$. We have

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} u(n) \frac{\sin \pi(x-n)}{\pi(x-n)} &=_{\text{(Fourier series on } \hat{u})} \sum_{n \in \mathbb{Z}} \underbrace{\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\xi} \hat{u}(\xi)\, d\xi \right)}_{u(n)} \frac{\sin \pi(x-n)}{\pi(x-n)} = \\
&=_{\text{(Fubini)}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{u}(\xi) \sum_{n \in \mathbb{Z}} e^{in\xi} \frac{\sin \pi(x-n)}{\pi(x-n)} = \\
&=_{\text{(by distribution theory)}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{u}(\xi)\, e^{i\xi x}\, d\xi = \\
&=_{\text{(by (2.15))}} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}(\xi)\, e^{i\xi x}\, d\xi = \\
&=_{\text{(by inverse Fourier transform)}} u(x);
\end{aligned}$$

which is the desired formula (3.1). ∎

4. WEIGHTED GRAPHS

We now turn to our present setting of arbitrary infinite graphs, and their function spaces. It is quite general and there is not a group associated with the questions we address. Nonetheless we will prove that there is a host of function theoretic problems on arbitrary infinite graph which are amenable to precisely the kind of interpolation envisioned by Shannon. We show that there are important function spaces amenable to precisely the kind of interpolation, discussed in the previous sections, and envisioned by Shannon in the context of Fourier analysis.

Following notation from section 2, we will denote a weighted graph (G, c), but as noted, it consists of three items, two sets $G^{(0)}$ and $G^{(1)}$, and a function

$$c\colon G^{(1)} \to \mathbb{R}^+.$$

As outlined in the third example in Examples 2.2, the three are assumed to satisfy the axiom system 2.2 (i) and (ii); see also (2.9)–(2.11). In electrical models of configurations on sets of vertices, we have the following interpretation:

Axiom 4.1 (Axiom System)**.**

— $G^{(0)}$: a discrete set of vertices,
— $G^{(1)} \subset G^{(0)} \times G^{(0)}$: a specified set of edges; and
— $c\colon G^{(1)} \to \mathbb{R}^+$ some prescribed conductance function.

The axioms are:

1. For pairs of vertices $x, y \in G^{(0)}$, we have

$$(xy) \in G^{(1)} \Leftrightarrow (yx) \in G^{(1)}$$

2. If $(xy) \in G^{(1)}$, written $x \sim y$, then $x \neq y$; no self-loops (see Fig. 1).
3. If $x \in G^{(0)}$, then the set $\left\{y \in G^{(0)} \,|\, y \sim x\right\}$ is non-empty and *finite*.
4. G is assumed connected.
5. For $(xy) \in G^{(1)}$, we have $c\,(x, y) = c\,(y, x)$.

Definition 4.2. Let (G, c) be a weighted graph (with fixed conductance function c); and let $\mathcal{F}\left(G^{(0)}\right)$ be the vector space of all functions on $G^{(0)}$, i.e.,

$$\begin{cases} \mathcal{F}\left(G^{0}\right) = \prod\limits_{G^{(0)}} \mathbb{C} \ \text{(Cartesian product)} = \\ \qquad\qquad\qquad = \mathbb{C}^{G^{(0)}}. \end{cases} \tag{4.1}$$

On $\mathcal{F} = \mathcal{F}\left(G^{(0)}\right)$, we define the following operator Δ (the (G, c)-Laplace operator):

$$(\Delta u)\,(x) := \sum_{y \sim x} c\,(x, y)\,(u\,(x) - u\,(y))\,. \tag{4.2}$$

Further, set $c\,(x) := \sum_{y \sim x} c\,(x, y)$.

Lemma 4.3. *Let (G,c) be a weighted graph; let $o \in G^{(0)}$ be a choice of base-point in the vertex-set $G^{(0)}$; let $\left\{v_x \middle| \, x \in G^{(0)} \setminus \{o\}\right\}$ be the system of di-poles from Lemma 2.3; and let Δ be the Laplace operator in* (4.2). *Then the following two rules hold:*

$$\langle \delta_x, u \rangle_E = (\Delta u)(x), \tag{4.3}$$

where $\langle \cdot, \cdot \rangle_E$ is the energy-quadratic form (*see Lemma 2.3*).

$$\langle u_1, u_2 \rangle_E := \frac{1}{2} \underset{\substack{s \sim t \\ (edges\ of\ vertex\ pairs)}}{\sum\sum} c(s,t) \left(\overline{u_1(s)} - \overline{u_1(t)} \right) (u_2(s) - u_2(t)) \tag{4.4}$$

and

$$\Delta v_x = \delta_x - \delta_o, \tag{4.5}$$

where the functions on the RHS in (4.5) *denote the usual Dirac masses, i.e.,*

$$\delta_x(y) := \delta_{x,y}, \ \forall x, y \in G^{(0)} \times G^{(0)}. \tag{4.6}$$

Proof of (4.3). We compute the LHS in (4.3) with the help of (4.4). Note that all summations are finite by virtue of Axiom System 4.1.

Now

$$\begin{aligned}
\langle \delta_x, u \rangle_E &=_{(\text{by } (4.4))} \frac{1}{2} \underset{s \sim t}{\sum\sum} c(s,t) (\delta_x(s) - \delta_x(t)) (u(s) - u(t)) = \\
&= \frac{1}{2} \left(\sum_t c(x,t)(u(x) - u(t)) - \sum_s c(s,x)(u(s) - u(x)) \right) = \\
&= \sum_{t \sim x} c(x,t)(u(x) - u(t)) = \\
&=_{(\text{by } (4.2))} (\Delta u)(x),
\end{aligned}$$

which proves (4.3). □

Proof of (4.5). It is enough to prove the formula

$$\langle v_y, \Delta v_x \rangle_E = \langle v_y, \delta_x - \delta_o \rangle_E, \ \forall (x,y) \in G^{(0)} \times G^{(0)} \setminus \{(o,o)\}. \tag{4.7}$$

Version 1:

$$\begin{aligned}
(LHS)_{(4.7)} &=_{(\text{by }(2.3))} (\Delta v_x)(y) - (\Delta v_x)(o) = \\
&=_{(\text{by }(4.2))} \sum_s c(y,s)(v_x(y) - v_x(o)) - \sum_t c(o,t)(v_x(o) - v_x(t)) = \\
&= c(y)\, v_x(y) + \sum_t c(o,t)\, v_x(t) = \\
&= c(x)\, v_x(y) - \sum_s c(x,s)\, v_y(s) + \sum_t c(o,t)\, v_y(t) = \\
&= \sum_s c(x,s)(v_y(x) - v_y(s)) - \sum_t c(o,t)(v_y(o) - v_y(t)) = \\
&=_{(\text{by }(4.2))} (\Delta v_y)(x) - (\Delta v_y)(o) = \\
&=_{(\text{by }(4.3))} \langle v_y, \delta_x - \delta_o \rangle_E = \\
&= (RHS)_{(4.7)}.
\end{aligned}$$

Version 2:

$$\begin{aligned}
(LHS)_{(4.7)} &= (\Delta v_x)(y) - (\Delta v_x)(o) = \\
&=_{(\text{by }(4.3))} \langle \delta_y, v_x \rangle_E - \langle \delta_o, v_x \rangle_E = \\
&=_{(\text{by }(2.3))} \delta_y(x) - \delta_y(o) - (\delta_o(x) - \delta_o(o)) = \\
&= \delta_{x,y} + 1 = \\
&=_{(\text{by }(2.3))} (\delta_x(y) - \delta_x(o)) - (\delta_o(y) - \delta_o(o)) = \\
&= (\Delta v_y)(x) - (\Delta v_y)(o) = \\
&=_{(\text{by }(4.3))} (RHS)_{(4.7)}.
\end{aligned}$$

□

The following result shows that the delta functions (δ_x) are finite linear combinations of the dipole vectors

$$(v_x)_{x \in G^{(o)} \setminus \{o\}}.$$

Lemma 4.4. *Consider a given weighted graph (G, c) as in the previous lemma. Then the following formula holds: For every $x \in G^{(0)} \setminus \{o\}$, we have*

$$\delta_x = c(x)\, v_x - \sum_{y \sim x} c(x,y)\, v_y. \tag{4.8}$$

Proof. Both sides in (4.8) vanish at the base-point o. We now evaluate the RHS in (4.8) at a point in $G^{(0)} \setminus \{o\}$:

$$\begin{aligned}
c(x)\, v_x(z) - \sum_{y \sim x} c(x,y)\, v_y(z) &=_{(\text{symmetry})} c(x)\, v_z(x) - \sum_{y \sim x} c(x,y)\, v_z(y) = \\
&=_{(\text{by }(4.2))} (\Delta v_z)(x) = \\
&= (\delta_z - \delta_o)(x) = \\
&= \delta_{x,z} = \\
&= \delta_x(z),
\end{aligned}$$

thus proving the desired formula (4.8). □

Example 4.5(a) Set $(G, c) = (\mathbb{Z}, 1)$. By this we mean that the vertex set $G^{(0)}$ is the set of integers $\mathbb{Z}$, the edges $G^{(1)}$ are nearest neighbors $(x, x \pm 1)$ for all $x \in \mathbb{Z}$, i.e.,

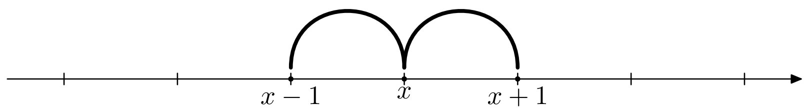

Fig. 2. Nearest neighbors

For base-points we have $o = 0$.

Example 4.5(b) Set $(G, c) = (\mathbb{Z}_+ \cup \{0\}, 1)$, the same as in (a), but with vertex set only consisting of the positive integers union the base-point.

In both cases, we have the following formula for the Laplace operator Δ:

$$(\Delta u)(x) = 2u(x) - u(x+1) - u(x-1) \tag{4.9}$$

for $x \in \mathbb{Z}$, and for functions u on $\mathbb{Z}$, or on $\mathbb{Z}_+$.

Lemma 4.6. *Let G and Δ be as in Examples 4.5, and consider Fourier series for 2π-periodic functions $\tilde{u}(\theta)$, $\theta \in [-\pi, \pi]$, or $\theta \in \mathbb{R}/2\pi\mathbb{Z}$:*

$$\tilde{u}(\theta) = \sum_{x \in \mathbb{Z}} u(x) e^{ix\theta}, \tag{4.10}$$

with Parseval's identity:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{u}(\theta)|^2 \, d\theta = \sum_{x \in \mathbb{Z}} |u(x)|^2. \tag{4.11}$$

Then

$$\left(\tilde{\Delta}\tilde{u}\right)(\theta) = 4 \sin^2(\theta/2)\, \tilde{u}(\theta). \tag{4.12}$$

Proof. A direct computation. □

Lemma 4.7. *The dipoles in the two examples are as follows:*
Example 4.5 (a) *If $x \in \mathbb{Z}_+$, then*

$$v_x(y) = \begin{cases} x \wedge y \ (\textit{minimum}) & \textit{if } y \geq 0, \\ 0 & \textit{if } y < 0. \end{cases} \tag{4.13}$$

If $x \in \mathbb{Z}_-$, then set

$$v_x(y) = v_{-x}(-y), \tag{4.14}$$

(*see Figs. 3 and 4*).

Proof. Apply Lemma 4.3. □

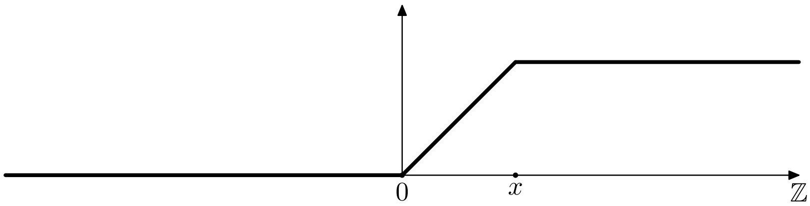

Fig. 3. v_x for $x > 0$

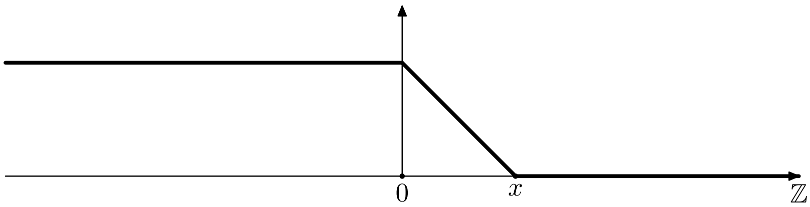

Fig. 4. v_x for $x < 0$

Corollary 4.8. *For the $\mathbb{Z}$-graph examples in Examples 4.5, the following allows us to compute the $\mathcal{H}_E$-norm (energy norm) via* (4.10):

$$\|u\|_E^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \sin^2\left(\frac{\theta}{2}\right) |\tilde{u}(\theta)|^2 \; d\theta. \tag{4.15}$$

Proof. Left to the reader! □

Corollary 4.9. *Consider the Laplace operator Δ in Example 4.5(a), i.e.,* (4.9)*, and for $t \in \mathbb{R}$; set*

$$\tilde{\xi}_t(\theta) = \sum_{x \in \mathbb{Z}} \xi_t(x)\, e^{ix\theta} = \frac{1}{4} \sec^2(\theta/2)\, e^{it\theta}. \tag{4.16}$$

Then

$$(\Delta \xi_t)(x) = \frac{\sin \pi (t - x)}{\pi (t - x)} \tag{4.17}$$

holds for $(t, x) \in \mathbb{R} \times \mathbb{Z}$.

Proof. Note that the RHS in (4.17) is the Shannon-kernel (2.6) in Example 2.2.

Using (4.10) we get

$$\sum_{x \in \mathbb{Z}} \frac{\sin \pi(t - x)}{\pi (t - x)} e^{ix\theta} = e^{it\theta}. \tag{4.18}$$

Using this, and formula (4.12) for the Laplace operator in $(\mathbb{Z}, 1)$, the desired conclusion (4.17) follows. □

Corollary 4.10. *Let $u : \mathbb{Z} \to \mathbb{R}$ satisfy the following: For all $p \in \mathbb{Z}_+$, there is a constant $A > \infty$ such that*

$$|u(x)| \leq \frac{A}{1+|x|^p}, \ \forall x \in \mathbb{Z}. \tag{4.19}$$

Then

$$\langle u, \xi_t \rangle_E = \sum_{x\in\mathbb{Z}} u(x)(\Delta\xi_t)(x) = \sum_{x\in\mathbb{Z}} u(x) \frac{\sin \pi(t-x)}{\pi(t-x)}. \tag{4.20}$$

Proof. With the family

$$\{v_x | \, x \in \mathbb{Z} \setminus \{0\}\} \subset \mathcal{H}_E \tag{4.21}$$

of dipoles from (4.13) in Lemma 4.7, we set

$$\varepsilon_x \colon = v_x - v_{x-1}, \tag{4.22}$$

and note that then

$$\{\varepsilon_x | \, x \in \mathbb{Z}\setminus \{0\}\} \tag{4.23}$$

is an orthonormal basis (ONB) in the Hilbert space $\mathcal{H}_E$, i.e., the following hold:

(i) $\langle \varepsilon_x, \varepsilon_y \rangle_E = \delta_{x,y}, \ \forall x, y \in \mathbb{Z} \setminus \{0\}$.
(ii) If $u \in \mathcal{H}_E$ satisfies $\langle \varepsilon_x, u \rangle_E = 0, \ \forall x \in \mathbb{Z} \setminus \{0\}$, then $u = 0$ in $\mathcal{H}_E$.

Now assume $u \in \mathcal{H}_E$ satisfies the estimate (4.19). Then the following summations are absolutely convergent; and we have; see (4.20) above. The first step uses (i) and (ii) combined with Parsevel:

$$\begin{aligned}
\langle u, \xi_t \rangle_E &= \sum_{x\in\mathbb{Z}\setminus\{0\}} \langle u, \varepsilon_x \rangle_E \langle \varepsilon_x, \xi_t \rangle_E = \\
&=_{(\text{by (4.22)})} \sum_x \langle u, v_x - v_{x-1} \rangle_E \langle v_x - v_{x-1} \xi_t \rangle_E = \\
&=_{(\text{by (2.3)})} \sum_x (u(x) - u(x-1))(\xi_t(x) - \xi_t(x-1)) = \\
&=_{\left(\begin{array}{l}\text{by rearranging}\\ \text{summations}\\ \text{and (2.4)}\end{array}\right)} \sum_x u(x)(2\xi_t(x) - \xi_t(x+1) - \xi_t(x-1)) = \\
&=_{(\text{by (4.9)})} \sum_x u(x)(\Delta\xi_t)(x) = \\
&=_{(\text{by (4.16)})} \sum_{x\in\mathbb{Z}} u(x) \frac{\sin \pi(t-x)}{\pi(t-x)}.
\end{aligned}$$

□

5. GELFAND TRIPLES

One of the ways an infinite graph G might be embedded in a continuum arises in the study of stochastic processes. In signal processing, it often happens that the points in G index a stochastic process on a continuous probability space, or sampling space. We show below that this happens naturally in such a matter that G is embedded in the probability space. And so this is one of the ways the interpolation question arises.

In the next section we will turn to a different context of such an interpolation.

The purpose of this section is to turn a weighted graph (G, c) into a stochastic process. Starting with (G, c) and a choice of base-point $o \in G^{(0)}$, we get a system of dipoles

$$\left\{v_x | \, x \in G^{(0)} \setminus \{o\}\right\} \subset \mathcal{H}_E$$

such that each $v_x \in \mathcal{H}_E$ satisfies (2.3), i.e.,

$$u\,(x) - u\,(o) = \langle v_x, u \rangle_E \tag{5.1}$$

for all $u \in \mathcal{H}_E$.

Our aim is to exhibit a measure space

$$\left(S', \mathcal{B}, W\right), \tag{5.2}$$

where S' is a set, equipped with a sigma-algebra $\mathcal{B}$ of subsets of S', and W is a probability measure defined on $\mathcal{B}$. Further, we require that S' is the dual of a Fréchet space S which is contained in $\mathcal{H}_E$, and

$$v_x \in S \text{ for all } x \in G^{(0)} \setminus \{o\}\,. \tag{5.3}$$

When this is accomplished, we get a pair of containments

$$S \hookrightarrow \mathcal{H}_E \hookrightarrow S'. \tag{5.4}$$

In considering the possibilities for solutions to (5.2)–(5.4), it is convenient to restrict attention to *real valued* functions on $G^{(0)}$.

If $f \in S$ and $\xi \in S'$, we set

$$\tilde{f}\,(\xi) = \langle f, \xi \rangle_E. \tag{5.5}$$

Definition 5.1. If $(S', \mathcal{B}, W)$ is a Borel-probability space subject to condition (5.3) above, then we denote the corresponding expectation by $\mathbb{E}$, i.e., if f is a measurable function on (5.2), we set

$$\mathbb{E}\,(f) = \int\limits_{S'} f(\xi)\, dW(\xi). \tag{5.6}$$

From (5.2), we get the Hilbert space

$$L^2\,(S') = L^2\,(S', \mathcal{B}, W)\,. \tag{5.7}$$

We aim for an isometric isomorphism (a Wiener-transform)

$$T : \mathcal{H}_E \to L^2(S') \tag{5.8}$$

such that

$$\mathbb{E}\left(T(v_x)\,T(v_y)\right) = \langle v_x, v_y\rangle_E, \tag{5.9}$$

$$\mathbb{E}\left(T(v_x)\right) = 0, \tag{5.10}$$

and

$$\text{each } T(v_x) \text{ is a Gaussian random variable.} \tag{5.11}$$

We will accomplish this with the use of Wiener chaos, and for details we rely on the presentation in [30] of the theory of Brownian motion, white noise, functional integrals, and Minlos' construction.

To do this, we need that the embedding $S \hookrightarrow \mathcal{H}_E$ in (5.4) is *nuclear* (see [30]). Here S as a linear subspace is equipped with a Fréchet space topology, and $\mathcal{H}_E$ with its $\|\cdot\|_E$-norm topology. With the usual identification of $\mathcal{H}_E$ with its dual, we therefore get the double inclusion sketched in (5.4).

Lemma 5.2 (Minlos)**.** *There is a solution to the problem in Definition 5.1 such that*

$$\mathbb{E}\left(e^{i\langle f,\cdot\rangle_E}\right) = e^{-\frac{1}{2}\|f\|_E^2} \textit{ holds for all } f \in S. \tag{5.12}$$

Proof. See e.g., [30]. $\square$

Remark 5.3. Expanding the power-series in (5.12), we get:

$$\mathbb{E}\left(\langle f,\cdot\rangle_E^k\right) = 0 \text{ if } k \in \mathbb{Z}_+ \text{ is odd;} \tag{5.13}$$

and

$$\mathbb{E}\left(\langle f,\cdot\rangle_E^{2n}\right) = \frac{(2n)!}{n!}\|f\|_E^{2n} \text{ for all } f \in S. \tag{5.14}$$

Proof. The idea is to first construct a Gelfand-triple for the easy case of $\ell^2 = \ell^2(\mathbb{Z}_+)$ the usual Hilbert space of square-summable sequence. A choice of ONB in $\mathcal{H}_E$ then sets up an isometric isomorphism between the ℓ^2-construction and (5.4) for $\mathcal{H}_E$.

Our choice of ONB in $\mathcal{H}_E$ is as follows: Select an ordering in $G^{(0)}$

$$o = x_0, x_1, x_2, x_3, \ldots, \tag{5.15}$$

and then apply Gram-Schmidt to

$$v_{x_1}, v_{x_2}, v_{x_3}, \ldots. \tag{5.16}$$

The result is an ONB in $\mathcal{H}_E$ as well as a solution to the Gelfand-triple problem in Definition 5.1. $\square$

Lemma 5.4. *Let*

$$\{\varepsilon_1, \varepsilon_2, \varepsilon_{3,} \ldots\} \subset \mathcal{H}_E \tag{5.17}$$

obtained from (5.16) *by Gram-Schmidt. Then*

$$v_{x_k} = \sum_{j=1}^{k} \varepsilon_j\left(x_k\right)\ \varepsilon_j. \tag{5.18}$$

Proof. Substitute into the $\mathcal{H}_E$-inner products:

$$\langle \varepsilon_i, v_{x_k} \rangle_E = \sum_{j=1}^{k} \varepsilon_j\left(x_k\right)\ \delta_{i,j} = \varepsilon_i\left(x_k\right)$$

which is the desired conclusion. □

6. FUNCTION SPACES

In this section we introduce the function spaces need in our analysis of a general class of weighted graphs (G, c); i.e., G a given graph with specified sets of vertices and edges, and c a positive function defined on the edges. In the electrical network model, the function c represents conductance. For general discrete models, there is no Fourier duality available.

Indeed, the comparison to the classical case of Shannon interpolation and sampling in signal processing (see sections 3 and 4) is subtle since a general graph (G, c) typically is not endowed with a group structure that invites any kind of Fourier duality. Hence, in selecting a notion of band-limiting functions for the general case of (G, c), other tools must be brought to bear. We resolve this here with the use of Gelfand-triples over graphs (see section 5 above), and with the introduction of the Wiener transform.

In this section we introduce the precise spaces, and in the next we narrow down on a useful notion of band-limited functions.

Let (G, c) be a weighted graph. The interesting case is when the vertex-set $G^{(0)}$ is *infinite*. We select a base-point $o \in G^{(0)}$, and set

$$V := G^{(0)} \setminus \{o\}\,. \tag{6.1}$$

The dipoles from Definition 2.1(d) will be denoted $\{v_x \mid x \in V\}$, with the abbreviation

$$v_x := v_{x,o}. \tag{6.2}$$

Definition 6.1. (i) Since $G^{(0)}$ is countable, we may select a bijection

$$V \xrightarrow{b} \mathbb{Z}_+. \tag{6.3}$$

With b chosen, we set

$$(x \leq y \text{ in } V) \underset{Def.}{\Longleftrightarrow} \left(b\left(x\right) \leq b\left(y\right)\ \text{ in } \mathbb{Z}_+\right). \tag{6.4}$$

(ii) Make the normalization $v_x\left(o\right) = 0$.

Using Lemma 5.4 above, we get:

Lemma 6.2. *With the notation and definitions from above, we have:*

$$v_x(\cdot) = \sum_{y \leq x} \varepsilon_y(x)\ \ \varepsilon_y(\cdot),\ \textit{and} \tag{6.5}$$

$$\varepsilon_x(\cdot) = \sum_{y \leq x} (\Delta\varepsilon_x)(y) v_y(\cdot). \tag{6.6}$$

Proof. Note that (6.5) is contained in Lemma 5.4.

Let $z \in V$, and consider (6.6) for some $x \in V$:

$$\begin{aligned}
\langle \delta_z, \mathrm{RHS}_{(6.6)} \rangle_E &= \sum_{y \leq x} (\Delta\varepsilon_x)(y) \langle \delta_z, v_y \rangle_E = \\
&=_{(\text{by (4.3)})} \sum_{y \leq x} (\Delta\varepsilon_x)(y) (\delta_z(y) - \delta_z(o)) = \\
&=_{(\text{by (4.5)})} \sum_{y \leq x} (\Delta\varepsilon_x)(y)\, \delta_{z,y} = \\
&= (\Delta\varepsilon_x)(z) = \\
&=_{(\text{by (4.3)})} \langle \delta_z, \Delta\varepsilon_x \rangle_E = \\
&= \langle \delta_z, LHS_{(\text{by (6.6)})} \rangle_E.
\end{aligned}$$

□

Definition 6.3. Let ξ be any function defined in $G^{(0)}$. For $x \in V$, set

$$\tilde{v}_x(\xi) := \xi(x) - \xi(o) = \langle v_x, \xi \rangle_E \text{ (by abuse of notation; see (2.3));} \tag{6.7}$$

and

$$\tilde{\varepsilon}_x(\xi) := \sum_{y \leq x} (\Delta\varepsilon_x)(y)\ \tilde{v}_y(\xi). \tag{6.8}$$

Definition 6.4. (i) Let the data be as in Definition 6.1, i.e.,

$$b : V \to \mathbb{Z}_+ \text{ fixed,}$$

and the two systems

$$\begin{aligned}
&\{v_x |\ x \in V\} \subset \mathcal{H}_E, \text{ and} \\
&\{\varepsilon_x |\ x \in V\} \subset \mathcal{H}_E.
\end{aligned}$$

Then the operation

$$(\mathcal{H}_E \ni u) \xmapsto{B} (\langle \varepsilon_x, u \rangle_E)_{x \in V} \in \ell^2(V) \tag{6.9}$$

is an isometric isomorphism, i.e.,

$$\|u\|_E^2 = \sum_{x \in V} |\langle \varepsilon_x, u \rangle_E|^2. \tag{6.10}$$

(ii) Via (6.3) and (6.4), the two sets V and $\mathbb{Z}_+$ are order-isomorphic. So via (6.9) function spaces defined on sequences carry over to $V = G^{(0)} \setminus \{o\}$.

Set
$$S(\mathbb{Z}_+) = \left\{ (s_n)_{n\in\mathbb{Z}} \mid \forall p \in \mathbb{Z}_+,\ \exists A < \infty \text{ s.t. } |s_n| \leq An^{-p} \right\}, \tag{6.11}$$
and
$$S'(\mathbb{Z}_+) = \left\{ (t_n)_{n\in\mathbb{Z}} \mid \exists p \in \mathbb{Z}_+,\ \exists C < \infty \text{ s.t. } |t_n| \leq Cn^{p} \right\}. \tag{6.12}$$

Note that with the semi-norms in (6.11)
$$\|s\|_p = \sup_{n\in\mathbb{Z}_+} n^p |s_n|, \tag{6.13}$$

$S(\mathbb{Z}_+)$ turns into a Fréchet space, and $S'(\mathbb{Z}_+)$ is the dual in this Fréchet topology.

We have
$$S(\mathbb{Z}_+) \overset{i}{\hookrightarrow} \ell^2(\mathbb{Z}_+) \underset{dual}{\hookrightarrow} S'(\mathbb{Z}_+), \tag{6.14}$$
where the embedding i in (6.14) is nuclear.

(iii) With B as in (6.9), set
$$\begin{aligned} S(G,c)&:= B^{-1}(S(\mathbb{Z}_+)), \text{ and} \\ S'(G,c)&:= B^{-1}(S'(\mathbb{Z}_+)). \end{aligned}$$

We get
$$S(G,c) \underset{i_G}{\hookrightarrow} \mathcal{H}_E(G,c) \hookrightarrow S'(G,c) \tag{6.15}$$
with nuclear embedding i_G in (6.15) carried over from (6.14).

7. SAMPLING AND INTERPOLATION

We return to the Shannon sampling question from section 5. Our purpose is to show that by the use of the energy Hilbert space, graph Laplacian, and extensions to the bigger space from our Gelfand triple (section 6) we are able to derive a formula for the interpolation kernel which accomplishes interpolation from discrete samples to the ambient continuum.

In this section we discuss versions of *band-limited* functions f in the vertices $G^{(0)}$ in a fixed weighted graph (G,c) such that the following interpolation formula holds:
$$\tilde{f}(\xi) = \sum_{x\in G^{(0)}} f(x)(\Delta\xi)(x). \tag{7.1}$$

Definition 7.1. To appreciate the meaning of (7.1), we ask the reader to compare with Shannon's formula as we present it in section 3 above, as well as its graph variation in Corollary 4.10.

Two issues must be addressed for making (7.1) precise:

(i) We must select a continuum, say X, containing $G^{(0)}$, or $G^{(0)} \setminus \{o\}$, as a subset.

(ii) With the choice made in (i), we must select a linear space $\mathcal{A}$ of functions f in $G^{(0)}$ such that every $f \in \mathcal{A}$ admits a "natural" extension $\tilde{f}$, i.e., extending f as a function on $G^{(0)}$ to a function $\tilde{f}$ on X.

Remark 7.2. Using Definition 6.4 above, we note that the pair $(\mathcal{H}_E, S')$ is admissible in the sense of (i) and (ii) of Definition 7.1.

Specifically, we may take $X = S'$, and $\mathcal{A} = \mathcal{H}_E$.
To see this, we will realize the embedding

$$G^{(0)} \setminus (o) \hookrightarrow S' \tag{7.2}$$

as follows:

$$G^{(o)} \ni x \mapsto v_x \in S \hookrightarrow \mathcal{H}_E \hookrightarrow S'. \tag{7.3}$$

If $\langle u, \xi \rangle_E$ denotes the energy-pairing between $u \in \mathcal{H}_E$ and $\xi \in S'$, we then have

$$\tilde{u}(\xi) = \overline{\langle u, \xi \rangle}_E. \tag{7.4}$$

This is consistent since

$$\tilde{u}(v_x) = \overline{\langle u, v_x \rangle}_E = u(x) - u(o). \tag{7.5}$$

Note that because of the term $u(o)$ in (7.5), care must be given to normalizations at the base-point o.

Our next result shows that the Dirac delta functions δ_x play a special role. For each $x \in G^{(0)}$ we view δ_x as a function on the vertex set $G^{(0)}$:

$$\delta_x(y) = \begin{cases} 1 & y = x, \\ 0 & y \neq x. \end{cases} \tag{7.6}$$

For functions f on $G^{(0)}$, set

$$(M_f)(y) := f(y)\, u(y), \; y \in G^{(0)};$$

i.e., the multiplication operator.

Set

$$\mathcal{B} := \{ f \mid M_f : \mathcal{H}_E \to \mathcal{H}_E \} \tag{7.7}$$

defines a bounded operator.

Definition 7.3. For $f \in \mathcal{B}$, set

$$\|f\|_{\mathcal{B}} := \|M_f\|_{op}, \tag{7.8}$$

where $\|\cdot\|_{op}$ on the RHS in (7.8) refers to the $\mathcal{H}_E \to \mathcal{H}_E$ operator norm. Then $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach algebra containing the functions δ_x in (7.6) for all $x \in G^{(0)}$.

We denote the Gelfand-space of $\mathcal{B}$ by $X_{\mathcal{B}}$.

Lemma 7.4. (i) *A function f on $G^{(0)}$ in $\mathcal{B}$ if and only if there exists $b \in \mathbb{R}_+$ such that for every finite subset $F \subset G^{(0)}$, the $\#F \times \#F$ matrix*

$$\left(\left(b - f(x)\overline{f(y)}\right) v_x(y)\right)_{(x,y)\in F\times F} \tag{7.9}$$

is positive semidefinite, with x denoting row-index, and y column index in (7.9).

(ii) *For all $x \in G^{(0)}$, δ_x is in $\mathcal{B}$ and*

$$\|\delta_x\|_{\mathcal{B}} = \sqrt{c(x)\, v_x(x)}, \tag{7.10}$$

where

$$c(x) = \sum_{y\sim x} c(x,y) \tag{7.11}$$

(iii) *We have*

$$M_x := M_{\delta_x} = |\delta_x >< v_x| \tag{7.12}$$

with the RHS in (7.12) *is Dirac formalism*

$$(|u_1 >< u_2|)(u_3) = \langle u_2, u_3\rangle_E u_1 \tag{7.13}$$

and

$$M_x^* M_y = \langle \delta_x, \delta_y\rangle_E \, |v_x >< v_y| \,, \tag{7.14}$$

and

$$M_x M_y^* = v_x(y)\, |\delta_x >< \delta_y| \,. \tag{7.15}$$

Proof. Ad(i). First note that by

$$(v_x, u)_E = u(x) - u(o)\,,\ \forall u \in \mathcal{H}_E; \tag{7.16}$$

and the normalization

$$v_x(o) = 0,\ x \in G^{(0)} \setminus \{o\} \tag{7.17}$$

we have

$$M_f^* v_x = \overline{f(x)} v_x,\ x \in G^{(0)} \setminus \{o\}\,, \tag{7.18}$$

where $*$ denotes adjoint operator relative to the $\mathcal{H}_E$-inner product $\langle\cdot,\cdot\rangle_E$. Applying this to $f = \delta_x$, the formula (7.12) follows. The other two formulas (7.14) and (7.15) one immediate from this, and an application of Dirac's formalism (7.13) for rank-1 operators.

Now, let $f \in \mathcal{B}$, and set

$$b := \|M_f\|_{op}^2\,. \tag{7.19}$$

Pick a finite set $F \subset G^{(0)}$ as in the statement of (i). Then in the ordering of Hermitian operators, we have

$$bI - M_f^* M_f \geq 0,$$

and therefore

$$bI - M_f M_f^* \geq 0. \tag{7.20}$$

Consider $(a_x)\, x \in F$, and

$$u := \sum_{x \in F} a_x v_x. \tag{7.21}$$

Then by (7.20), we have

$$\langle u, \left(bI - M_f M_f^*\right) u \rangle_E \geq 0.$$

But

$$\langle u, M_f M_f^* u \rangle = \sum_x \sum_y \bar{a}_x a_y f(x) \overline{f(y)} \langle v_x, v_y \rangle_E.$$

Since $\langle v_x, v_y \rangle_E = v_x(y)$, the desired conclusion (7.9) follows. Further note that the argument works in reverse, so (7.9) is necessary and sufficient for boundedness of the operator $M_f : \mathcal{H}_E \to \mathcal{H}_E$.

Ad(ii). We already established (7.12), (7.14), and (7.15). Specializing (7.14) to $x = y$ yields:

$$\begin{aligned}
\|M_x\|_{op}^2 &= \|M_x^* M_x\|_{op} = \\
&= \|c(x)\, |v_x >< v_x|\|_{op} = \\
&= c(x)\, \|v_x\|_E^2 = c(x)\, v_x(x),
\end{aligned}$$

which is the desired formula (7.10).

Since (iii) follows from (i), the proof is completed. □

Theorem 7.5. *Let f be a function on $G^{(0)}$, and for every finite subset $F \subset G^{(0)}$, set*

$$f_F(\cdot) = \sum_{x \in F} f(x)\, \delta_x(\cdot). \tag{7.22}$$

$$\lim_{F \to G^{(0)}} f_F = f \tag{7.23}$$

holds in the Fréchet topology on S in Definition 6.4. Then formula (7.1) *holds, with absolute convergence on the RHS, i.e.,*

$$\tilde{f}(\xi) = \sum_{x \in G^{(0)}} f(x)\, (\Delta \xi)(x) \tag{7.24}$$

is valid for all $\xi \in S'$.

Proof. First recall from section 6, that the Fréchet topology on S is given by the system of semi-norms $\|\cdot\|_1, \|\cdot\|_2, \ldots$, countably infinite. The meaning of the assumption (7.23) is this: For every $\varepsilon > 0$, and for every $p \in \mathbb{Z}_+$, there exist F_0 (a finite subset) $\subset G^{(0)}$ such that for all $F \supset F_0$, we have

$$\left\| f - \sum_{x \in F} f(x)\, \delta_x \right\|_p < \varepsilon. \tag{7.25}$$

But $\xi \in S'$ is continuous in the Fréchet topology in S, and so applying ξ to (7.23), we get

$$\tilde{f}(\xi) = \lim_{F \to G^{(0)}} \xi\left(f_F\right) = \lim_{F \to G^{(0)}} \sum_{x \in F} f(x) \langle \xi, \delta_x \rangle_E = \sum_{x \in G^{(0)}} f(x)(\Delta \xi)(x),$$

when in the last step we used (4.3) from Lemma 4.3 above. □

The next Lemma shows that $G^{(0)}$ is contained in the Gelfand space of the Banach algebra $\mathcal{B}$ of all bounded *multipliers* of $\mathcal{H}_E$ (= the energy Hilbert space of an arbitrary weighted graph (G, c)); recall Definition 7.3 for details. By "containment" we mean "up to identification;" see details below:

Lemma 7.6. *Let (G, c) be a weighted graph. Select a base-point o, and a system of dipoles $\left(v_x\right)_{x \in G^{(0)} \setminus \{o\}}$. See section 2, and (7.16) and (7.17) for details. For $x \in G^{(0)} \setminus \{o\}$ set*

$$\psi_x\left(M_f\right) = \frac{\langle v_x, M_f v_x \rangle_E}{v_x\left(x\right)} \tag{7.26}$$

for $f \in \mathcal{B}$.

Then

$$\psi_x\left(M_{f_1} M_{f_2}\right) = \psi_x\left(M_{f_1}\right) \psi_x\left(M_{f_2}\right) \tag{7.27}$$

for all $f_1, f_2 \in \mathcal{B}$; and

$$\psi_x\left(I_{\mathcal{H}_E}\right) = 1; \tag{7.28}$$

in other words ψ_x is in the Gelfand space of $\mathcal{B}$; in fact each ψ_x extends by the RHS in (7.26) to a state on the C^*-algebra generated by $\{M_f : f \in \mathcal{B}\}$.

Proof. We first compute the expression on the LHS in (7.26). The remaining conclusions in the lemma will then follow from the:

$$\begin{aligned}
\psi_x\left(M_f\right) = \frac{\langle v_x M_f v_x \rangle_E}{v_x\left(x\right)} = \frac{\langle M_f^* v_x, v_x \rangle_E}{\|v_x\|_E^2} = \\
=_{(\text{by } (7.17))} \frac{\langle \overline{f\left(x\right)} v_x, v_x \rangle_E}{\|v_x\|_E^2} = f\left(x\right) \frac{\langle v_x, v_x \rangle_E}{\|v_x\|_E^2} = f\left(x\right),
\end{aligned}$$

for all $f \in \mathcal{B}$.

The remaining conclusion follows from this, together with

$$M_{f_1 f_2} = M_{f_1} M_{f_2}, \tag{7.29}$$

see (7.7) in Remark 7.2. □

Theorem 7.7. *Let $f \in \mathcal{B}$, and $\xi \in X_{\mathcal{B}}$ (= the Gelfand space of $\mathcal{B}$), and denote the extension (see Lemma 7.6) of f from $G^{(0)}$ to $X_{\mathcal{B}}$ by $\tilde{f}$; consistent with the notation in Theorem 7.5. Then the interpolation formula (7.24) holds for $f(\xi)$.*

Proof. The proof follows closely the arguments from that of Theorem 7.5; and we will use the same terminology, i.e., (7.22). Note that the functions in (7.22), $f_F \in \mathcal{B}\ for$ all finite subsets $F \subseteq G^{(0)}$; as follows from Lemma 7.4. To explain the term $(\Delta\xi)(x)$ on the RHS in (7.23), we may make use of the Wiener transform T from Definition 5.1 and Lemma 5.2. The function

$$\xi \mapsto (\Delta\xi)(x) \tag{7.30}$$

is a random variable on Wiener space. Indeed, using Lemma 5.2, we get

$$\mathbb{E}_{\xi}\left(((\Delta\xi)(x))^{k}\right) = 0 \text{ if } k \text{ is odd and}$$
$$\mathbb{E}_{\xi}\left(((\Delta\xi)(x))^{2n}\right) = \frac{(2n)!}{n!2^{n}} c(x)^{n}.$$

□

8. CONCLUSIONS

We develop tools for an analysis of infinite graphs. As an application we prove sampling and interpolation theorems in a contest where the sample points are vertices of a fixed infinite weighted graphs. The choice of weights are dictated by a host of applications to concrete problems.

The scope of our analysis includes such infinite graphs G as arise in statistical mechanics, in signal/image processing, and in the analysis of large electrical networks. In more traditional approaches the configuration of sample points arise as discrete subset of the time axis, or more generally as chosen sample points in Euclidean space. These choices admit Fourier analysis, and standard definitions of band-limited functions. In the context of arbitrary graph, tools from Fourier analysis are no longer available, and we develop substitutes. Even earlier work on irregular sampling makes use of Fourier tools, or of frame bases in the underlying spaces of "signal functions." See for example [3–6, 16, 25, 32, 33]. For Shannon's original work, see [27–29].

We proved that infinite weighted graphs are naturally embedded in continua, with the embeddings depending on the given weight function. As a result, sampling questions present themselves: Sampling and interpolation for sets of discrete sampling points in the continuum then naturally arises: The discrete sample points in this context will then be the vertices in the given graph G.

Our main object is analysis and sampling formulas in this context of infinite weighted graphs. In electrical network models, the weights (functions defined on the set of edges) may represent inverse resistance.

We make use, among other things, of the theory of unbounded Hermitian operators in Hilbert space, and interpolation kernels.

While there is a large literature on discrete analysis treating such aspects as potential theory, probability, harmonic functions, and boundary theory (see e.g., [31]); the questions we address here are different.

For example, we obtain extensions of Shannon's theory of interpolation and sampling. Starting with an infinite graph G, and a suitable fixed positive weight function, we show that there are then continua (certain sets X) extending G, and associated formulas, and kernel functions, for interpolation band-limited functions on X from their values on G.

Depending on the applications, we will be making use of several notions of metric (effective resistance metric, etc) and a variety of boundaries.

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REFERENCES

[1] E. Acosta-Reyes, A. Aldroubi, I. Krishtal, *On stability of sampling-reconstruction models*, Adv. Comput. Math. **31** (2009) 1–3, 5–34.

[2] A. Aldroubi, C. Cabrelli, C. Heil, K. Kornelson, U. Molter, *Invariance of a shift--invariant space*, J. Fourier Anal. Appl. **16** (2010) 1, 60–75.

[3] A. Aldroubi, C. Cabrelli, C. Heil, K. Kornelson, U. Molter, *Invariance of a shift--invariant space*, J. Fourier Anal. Appl. **16** (2010) 1, 60–75.

[4] A. Aldroubi, C. Cabrelli, U. Molter, *Optimal non-linear models for sparsity and sampling*, J. Fourier Anal. Appl. **14** (2008) 5–6, 793–812.

[5] A. Aldroubi, C. Leonetti, *Non-uniform sampling and reconstruction from sampling sets with unknown jitter*, Sampl. Theory Signal Image Process. **7** (2008) 2, 187–195.

[6] A. Aldroubi, C. Leonetti, Q. Sun, *Error analysis of frame reconstruction from noisy samples*, IEEE Trans. Signal Process. **56** (2008) 6, 2311–2325.

[7] D. Alpay, D. Levanony, *On the reproducing kernel Hilbert spaces associated with the fractional and bi-fractional Brownian motions*, Potential Anal. **28** (2008) 2, 163–184.

[8] D. Alpay, D. Levanony, *Rational functions associated with the white noise space and related topics*, Potential Anal. **29** (2008) 2, 195–220.

[9] M.-J. Cantero, F. A. Grünbaum, L. Moral, L. Velázquez, *Matrix-valued Szegő polynomials and quantum random walks*, Comm. Pure Appl. Math. **63** (2010) 4, 464–507.

[10] I. Cho, P.E.T. Jorgensen, *Applications of automata and graphs: labeling-operators in Hilbert space. I*, Acta Appl. Math. **107** (2009) 1–3, 237–291.

[11] I. Cho, P.E.T. Jorgensen, *Applications of automata and graphs: labeling operators in Hilbert space. II*, J. Math. Phys. **50** (6) (2009), 063511, 42.

[12] I. Daubechies, R. DeVore, *Approximating a bandlimited function using very coarsely quantized data: a family of stable sigma-delta modulators of arbitrary order*, Ann. of Math. (2) **158** (2003) 2, 679–710.

[13] I. Daubechies, R. DeVore, M. Fornasier, C.S. Güntürk, *Iteratively reweighted least squares minimization for sparse recovery*, Comm. Pure Appl. Math. **63** (2010) 1, 1–38.

[14] D. Dutkay, P.E.T. Jorgensen, *Fourier series on fractals: a parallel with wavelet theory*, [in:] Contemp. Math, Radon transforms, geometry, and wavelets, vol. 464, Amer. Math. Soc., Providence, RI, 2008.

[15] Y. Ephraim, W.J.J. Roberts, *An EM algorithm for Markov modulated Markov processes*, IEEE Trans. Signal Process. **57** (2009) 2, 463–470.

[16] F. Gokpinar, Y.A. Ozdemir, *Generalization of inclusion probabilities in ranked set sampling*, Hacet. J. Math. Stat. **39** (2010) 1, 89–95.

[17] P.E.T. Jorgensen, *Analysis of unbounded operators and random motion*, J. Math. Phys. **50** (2009) 11, 113503, 28.

[18] P.E.T. Jorgensen, M.-S. Song, *Analysis of fractals, image compression, entropy encoding, Karhunen-Loève transforms*, Acta Appl. Math. **108** (2009) 3, 489–508.

[19] P.E.T. Jorgensen, M.-S. Song, *An extension of Wiener integration with the use of operator theory*, J. Math. Phys. **50** (2009) 10, 103502, 11.

[20] P.E.T. Jorgensen, E.P.J. Pearse, *Spectral Operator Theory of Electrical Resistance Networks*, (2009),
URL: http://arxiv.org/PS cache/arxiv/pdf/0806/0806.3881v4.pdf

[21] Y. Katznelson, *An introduction to harmonic analysis*, Cambridge Mathematical Library, third edition, Cambridge University Press, Cambridge, 2004.

[22] D. Kim, Y. Oshima, *Some inequalities related to transience and recurrence of Markov processes and their applications*, J. Theoret. Probab. **23** (2010) 1, 148–168.

[23] B. Kümmerer, *Asymptotic behavior of quantum Markov processes*, [in:] Infinite dimensional harmonic analysis IV, World Sci. Publ., Hackensack, NJ, 2009.

[24] R.T. Powers, *Resistance inequalities for the isotropic Heisenberg ferromagnet*, J. Math. Phys. **17** (1976) 10, 1910–1918.

[25] G.K. Rohde, A. Aldroubi, D.M. Healy Jr., *Interpolation artifacts in sub-pixel image registration*, IEEE Trans. Image Process. **18** (2009) 2, 333–345.

[26] W. Rosenkrantz, *Markov processes and applications: algorithms, networks, genome and finance* [book review of mr2488952], SIAM Rev. **51** (2009) 4, 802–807.

[27] C.E. Shannon, *A mathematical theory of communication*, Bell System Tech. J. **27** (1948), 379–423, 623–656.

[28] C.E. Shannon, *Communication in the presence of noise*, Proc. I.R.E. **37** (1949), 10–21.

[29] C.E. Shannon, W. Weaver, *The Mathematical Theory of Communication*, The University of Illinois Press, Urbana, IL., 1949.

[30] B. Simon, *Functional integration and quantum physics*, vol. 86 of *Pure and Applied Mathematics*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979.

[31] D. Ślęzak, *Approximate Markov boundaries and Bayesian networks: rough set approach*, [in:] Rough set theory and granular computing, vol. 125 of Stud. Fuzziness Soft Comput., Springer, Berlin, 2003.

[32] E.A. Suess, B.E. Trumbo, *Introduction to Probability Simulation and Gibbs Sampling with R*, Springer, New York, 2010.

[33] Q.-W. Xiao, Z.-W. Pan, *Learning from non-identical sampling for classification*, Adv. Comput. Math. **33** (2010) 1, 97–112.

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