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THE HARDY POTENTIAL AND EIGENVALUE PROBLEMS

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Abstract. We establish the existence of principal eigenfunctions for the Laplace operator involving weighted Hardy potentials. We consider the Dirichlet and Neumann boundary conditions.

Keywords: Dirichlet and Neumann problems, Hardy potential, principal eigenfuctions.

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1. INTRODUCTION

In this paper we investigate the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda \frac{m(x)}{|x|^2} u & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega,\ u > 0 \text{ on } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with a smooth boundary $\partial\Omega$ and $B(u) = \frac{\partial u}{\partial \nu}$ (the Neumann boundary conditions) or $B(u) = u$ (the Dirichlet boundary conditions). We make the following assumption on the weight function m:

(M) $m \in C(\bar{\Omega})$, $\int_\Omega \frac{m(x)}{|x|^2}\, dx < 0$, $m^+ \neq 0$, where $m^+(x) = \max(m(x), 0)$.

According to this assumption, $m(x)$ is positive on a proper subset of Ω.

In the case of the Dirichlet boundary conditions, we also consider the above problem assuming that $m(x) \geq 0$ and $\neq 0$ on $\bar{\Omega}$. In this paper we are concerned with the existence of the principal eigenvalues and corresponding eigenfunctions. Solutions to this problem are sought in the Sobolev space $H^1(\Omega)$ in the case of the Neumann boundary conditions, while for the Dirichlet boundary conditions, in the space $H^1_\circ(\Omega)$. We recall that $H^1(\Omega)$ and $H^1_\circ(\Omega)$ are the Sobolev spaces equipped with norms

$$\|u\|_N^2 = \int_\Omega \left(|\nabla u|^2 + u^2\right) dx \quad \text{and} \quad \|u\|_D^2 = \int_\Omega |\nabla u|^2\, dx,$$

respectively. We use the decomposition of the space $H^1(\Omega)$

$$H^1(\Omega) = V \oplus \mathbb{R},\ V = \Big\{v \in H^1(\Omega);\ \int_\Omega v(x)\,dx = 0\Big\}.$$

This decomposition yields the following equivalent norm on $H^1(\Omega)$

$$\|u\|_V^2 = \||\nabla v|\|_2^2 + t^2,\ v \in V,\ \ t \in \mathbb{R}.$$

Problem (1.1) is related to the Hardy inequality, which in the space $H^1_\circ(\Omega)$ takes the form

$$\Lambda_N \int_\Omega \frac{u^2}{|x|^2}\,dx \leq \int_\Omega |\nabla u|^2\,dx,\ \Lambda_N = \Big(\frac{N-2}{2}\Big)^2,\ N \geq 3. \tag{1.2}$$

It is known that Λ_N is an optimal constant. Moreover, there is no nontrivial function changing this inequality into equality [8]. Inequality (1.2) is no longer true in the space $H^1(\Omega)$. However, inequality (1.2) can be extended to the subspace V: there exists a constant $A_N > 0$ such that

$$A_N \int_\Omega \frac{v^2}{|x|^2}\,dx \leq \int_\Omega |\nabla v|^2\,dx \tag{1.3}$$

for every $v \in V$. In this article we need the following extension of the Hardy inequality: let $0 \in \bar{\Omega}$, then for every $\delta > 0$ there exists a constant $A = A(\delta, \Omega)$ such that

$$\int_\Omega \frac{u^2}{|x|^2}\,dx \leq \Big(\frac{1}{\Lambda_N} + \delta\Big) \int_\Omega |\nabla u|^2\,dx + A \int_\Omega u^2\,dx \tag{1.4}$$

for every $u \in H^1(\Omega)$ (see [10]).

The paper is organized as follows. In Section 2 we consider the eigenvalue problem with the Neumann boundary conditions. Section 3 is devoted the the eigenvalue problem with the Dirichlet boundary conditions. In the final Section 4, we examine the behaviour of eigenfunctions around a singular point 0. Section 5 is devoted to an eigenvalue problem with multiple singularities of a Hardy type. The main results of this paper are concerned with the Neumann problem and are presented in Sections 2, 4 and 5. As a byproduct of our approach to the Neumann problem, described n Section 2, we formulate parallel results for the Dirichlet problem in Section 3. We point out that Theorems 3.1 and 3.4 of Section 3 can also be deduced from a general abstract result in paper [25].

In recent years, eigenvalue problems for the Laplacian or more generally for the p-Laplacian, have attracted considerable interest. These problems have been investigated under various boundary conditions: the Dirichlet, Neumann, Robin or Steklov boundary conditions. We refer to papers [2, 5, 13, 18, 30] and [3], where further bibliographical references can be found. The common feature of these papers is the fact that weight functions generate functionals which are completely continuous. The main point of this paper is to consider weights whose corresponding functionals are only continuous.

We also mention papers [1,26] and [27] where the existence of principal eigenfunctions has been investigated, however, for the Dirichlet problem and for the Hardy--Sobolev operators. Finally, the papers [25,28] and [29] deal with the weighted Hardy potentials in the case of the Dirichlet boundary conditions. In particular, the approach in paper [25] is based on a very general concentration -compactness lemma. By contrast in this paper we give an exact upper bound of the principal eigenvalue which allows to prove the existence of a principal eigenfunction.

Throughout this paper, in a given Banach space we denote strong convergence by "$\to$" and weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \le p \le \infty$, are denoted by $\|\cdot\|_p$.

2. THE NEUMANN BOUNDARY CONDITIONS

In this section we consider the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda \frac{m(x)}{|x|^2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Obviously $\lambda = 0$ is an eigenvalue and the corresponding eigenfunctions are constant functions. We show the existence of a second principal eigenvalue which is positive. This eigenvalue is given by

$$\lambda_1^N(m) = \inf\Big\{ \int_\Omega |\nabla u|^2\,dx;\ u \in H^1(\Omega),\ \int_\Omega \frac{m(x)u^2}{|x|^2}\,dx = 1 \Big\}. \tag{2.2}$$

Lemma 2.1. *Suppose* (**M**) *holds. Then* $\lambda_1^N(m) > 0$.

Proof. Arguing by contradiction, assume $\lambda_1^N(m) = 0$. Then there exists a sequence $\{u_k\} \subset H^1(\Omega)$ such that $\int_\Omega |\nabla u_k|^2\,dx \to 0$ as $k \to \infty$ and $\int_\Omega \frac{m(x)u_k^2}{|x|^2}\,dx = 1$ for every k. We now use the decomposition

$$u_k = v_k + t_k,\ v_k \in V,\ t_k \in \mathbb{R}.$$

It follows from (1.3) that $v_k \to 0$ in $L^2(\Omega)$. We now show that the sequence $\{t_k\}$ is bounded. In the contrary case we may assume that $t_k \to \infty$. The case $t_k \to -\infty$ can be treated in a similar way. We have

$$\int_\Omega \frac{m(x)u_k^2}{|x|^2} dx = t_k^2 \int_\Omega \frac{m(x)}{|x|^2}\big(1 + \frac{v_k}{t_k}\big)^2\,dx =$$

$$= t_k^2 \left[\int_\Omega \frac{m(x)}{|x|^2}\,dx + \frac{2}{t_k}\int_\Omega \frac{m(x)}{|x|^2} v_k\,dx + \frac{1}{t_k^2}\int_\Omega \frac{m(x)}{|x|^2} v_k^2\,dx \right].$$

It is easy to see that the expression in brackets on the right-hand side tends to $\int_\Omega \frac{m(x)}{|x|^2}\,dx$ as $k\to\infty$. Hence

$$\lim_{k\to\infty}\int_\Omega \frac{m(x)}{|x|^2}u_k^2\,dx \to -\infty,$$

which is impossible. Therefore the sequence $\{u_k\}$ is bounded in $H^1(\Omega)$. We may assume that, up to a subsequence, $u_k \rightharpoonup u$ in $H^1(\Omega)$ and $u_k \to u$ in $L^2(\Omega)$. Using the decomposition $u_k = v_k + t_k$, we see that $v_k \to 0$ in $H^1(\Omega)$ and $t_k \to t$. Hence $u_k \to t$ in $H^1(\Omega)$. This yields

$$\int_\Omega \frac{m(x)}{|x|^2}u_k^2\,dx \to t^2 \int_\Omega \frac{m(x)}{|x|^2}\,dx \le 0.$$

Since $\int_\Omega \frac{m(x)}{|x|^2}u_k^2\,dx = 1$ for each k, we have arrived at a contradiction. Hence $\lambda_1^N(m) > 0$. □

The assumption $\int_\Omega \frac{m(x)}{|x|^2}\,dx < 0$ is essential in this lemma. In fact, one can easily check that $\lambda_1^N(m) = 0$ if $\int_\Omega \frac{m(x)}{|x|^2}\,dx \ge 0$.

Lemma 2.2. *Suppose that* (**M**) *holds and that* $m(0) > 0$. *Then*

$$0 < \lambda_1^N(m) \le \frac{\Lambda_N}{m(0)}. \tag{2.3}$$

Proof. Given $\delta \in (0, m(0))$, we choose $r > 0$ such that $m(x) \ge m(0) - \delta$ for $x \in B(0,r)$. Let $u \in H^1_\circ(B(0,r)) \setminus \{0\}$. Then

$$\lambda_1^N(m) \le \frac{\int_{B(0,r)} |\nabla u|^2\,dx}{\int_{B(0,r)} \frac{m(x)}{|x|^2}u^2\,dx} \le \frac{\int_{B(0,r)} |\nabla u|^2\,dx}{(m(0)-\delta)\int_{B(0,r)} \frac{u^2}{|x|^2}\,dx}.$$

Taking the infimum over $H^1_\circ(\Omega)$ on the right-hand side, we get $\lambda_1^N(m) \le \frac{\Lambda_N}{m(0)-\delta}$. Since $\delta > 0$ is arbitrary, inequality (2.3) follows. □

Let $\mathcal{M} = \{u \in H^1(\Omega);\ \int_\Omega u^2\,dx = 1\}$.

Theorem 2.3. *Suppose that* (**M**) *holds. If* $m(0) > 0$ *and*

$$\lambda_1^N(m) < \frac{\Lambda_N}{m(0)}, \tag{2.4}$$

then there exists a minimizer for $\lambda_1^N(m)$, *which is positive on* $\Omega \setminus \{0\}$.

Proof. Let $\{u_k\} \subset \mathcal{M}$ be a minimizing sequence for $\lambda_1^N(m)$. Let $u_k = v_k + t_k$ with $v_k \in V$ and $t_k \in \mathbb{R}$. It is clear that $\{v_k\}$ is bounded in $H^1(\Omega)$. As in Lemma 2.1 we show that the sequence $\{t_k\}$ is bounded. Hence the sequence $\{u_k\}$ is bounded in $H^1(\Omega)$. Therefore, up to a subsequence, $u_k \rightharpoonup u$ in $H^1(\Omega)$ and $u_k \to u$ in $L^2(\Omega)$. By the P.L. Lions' concentration – compactness principle [17] there exist constants $\mu_\circ, \nu_\circ \in [0, \infty)$ and nonnegative measures μ and ν such that

$$|\nabla u_k|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \mu_\circ \delta_0 \tag{2.5}$$

and

$$\frac{u_k^2}{|x|^2} \rightharpoonup \nu = \frac{u^2}{|x|^2} + \nu_\circ \delta_0 \tag{2.6}$$

in the sense of measures, where δ_0 is the Dirac measure assigned to 0. The constants $\mu_\circ$ and $\nu_\circ$ satisfy the inequality

$$\Lambda_N \nu_\circ \leq \mu_\circ. \tag{2.7}$$

First we show that $u \not\equiv 0$. Arguing by contradiction, assume $u \equiv 0$. Since $\{u_k\} \subset \mathcal{M}$ we deduce from (2.6) that $\nu_\circ m(0) = 1$. Hence $\mu_\circ \geq \frac{\Lambda_N}{m(0)}$ and by (2.5) we deduce $\lambda_1^N(m) \geq \frac{\Lambda_N}{m(0)}$, which is impossible. To complete the proof it suffices to show that $\nu_\circ = 0$. Arguing by contradiction assume $\nu_\circ > 0$. We distinguish three cases: (i) $\int_\Omega \frac{m(x)}{|x|^2} u^2\, dx = 0$, (ii) $\int_\Omega \frac{m(x)}{|x|^2} u^2\, dx < 0$ and (iii) $\int_\Omega \frac{m(x)}{|x|^2} u^2\, dx > 0$. The case (i) cannot occur, by the first part of the proof. In case (ii) we have

$$1 < 1 - \int_\Omega \frac{m(x)}{|x|^2} u^2\, dx = \nu_\circ m(0).$$

So by (2.7) we get $\mu_\circ > \frac{\Lambda_N}{m(0)}$. Then (2.5) yields $\lambda_1^N(m) > \frac{\Lambda_N}{m(0)}$, which is impossible. Finally, in the case (iii) we have

$$1 = \int_\Omega \frac{m(x)}{|x|^2} u^2\, dx + m(0)\nu_\circ > \int_\Omega \frac{m(x)}{|x|^2} u^2\, dx \quad \text{and} \quad \frac{1}{m(0)}\left(1 - \int_\Omega \frac{m(x)}{|x|^2} u^2\, dx\right) = \nu_\circ.$$

Then from (2.5) and (2.7), we derive

$$\begin{aligned} \lambda_1^N(m) &\geq \int_\Omega |\nabla u|^2\, dx + \frac{\Lambda_N}{m(0)}\left(1 - \int_\Omega \frac{m(x)}{|x|^2} u^2\, dx\right) \geq \\ &\geq \lambda_1^N(m) \int_\Omega \frac{m(x)}{|x|^2} u^2\, dx + \frac{\Lambda_N}{m(0)}\left(1 - \int_\Omega \frac{m(x)}{|x|^2} u^2\, dx\right). \end{aligned}$$

From this we deduce

$$\lambda_1^N(m) - \frac{\Lambda_N}{m(0)} \geq \left(\lambda_1^N(m) - \frac{\Lambda_N}{m(0)}\right) \int_\Omega \frac{m(x)}{|x|^2} u^2\, dx$$

implying that $\int_\Omega \frac{m(x)}{|x|^2} u^2\, dx \geq 1$. This contradiction completes the proof. Since $|u|$ is also a minimizer, we may assume that $u \geq 0$. By the Harnack inequality, $u > 0$ on $\Omega \setminus \{0\}$ (see Theorem 8.20 in [14] or [23]). It follows from Proposition 4.2 in Section 4, that $\lim_{x\to 0} u(x) = \infty$. □

In Proposition 2.4, we give conditions on m guaranteeing the validity of inequality (2.4). We introduce notation $\Omega^+ = \{x \in \Omega;\ m(x) > 0\}$.

Proposition 2.4. *Let m satisfy* (**M**). *Moreover assume that there exists $\overline{B(x_M, r)} \subset \Omega^+$ such that $m(x) \geq m(x_M)$ for all $x \in B(x_M, r)$, $0 \notin \overline{B(x_M, r)}$ and that $m(0) > 0$. If*

$$\frac{m(0)}{m(x_M)} < \frac{r^2(N-2)^2}{2(r+|x_M|)^2(N+1)(N+2)}, \tag{2.8}$$

then $\lambda_1^N(m) < \frac{\Lambda_N}{m(0)}$.

Proof. Let $u \in H^1_\circ\big(B(x_M, r)\big) \setminus \{0\}$. Then

$$\int\limits_{B(x_M,r)} \frac{m(x)}{|x|^2} u^2\, dx \geq m(x_M) \int\limits_{B(x_M,r)} \frac{u^2}{|x|^2}\, dx \geq \frac{m(x_M)}{(r+|x_M|)^2} \int\limits_{B(x_M,r)} u^2\, dx.$$

Hence

$$\frac{\int_{B(x_M,r)} |\nabla u|^2\, dx}{\int_{B(x_M,r)} \frac{m(x)}{|x|^2} u^2\, dx} \leq \frac{(r+|x_M|)^2 \int_{B(x_M,r)} |\nabla u|^2\, dx}{m(x_M) \int_{B(x_M,r)} u^2\, dx}. \tag{2.9}$$

Since $H^1_\circ\big(B(x_M, 0)\big) \setminus \{0\} \subset \{u \in H^1(\Omega);\ \int_\Omega \frac{m(x)}{|x|^2} u^2\, dx > 0\}$, we derive from (2.9) that

$$\lambda_1^N(m) \leq \frac{(r+|x_M|)^2}{m(x_M)} \lambda_1^D(B(x_M, r)), \tag{2.10}$$

where $\lambda_1^D(B(x_M, r))$ denotes the first eigenvalue for "$-\Delta$" in $B(x_M, r)$ with the Dirichlet boundary conditions. We now estimate $\lambda_1^D := \lambda_1^D\big(B(x_M, r)\big)$. We test λ_1^D with $v(x) = r - |x - x_M|$ for $x \in B(x_M, r)$. We have

$$\int\limits_{B(x_M,r)} v^2\, dx = \int\limits_{B(0,r)} (r-|x|)^2\, dx = \omega_N \int\limits_0^r (r-s)^2 s^{N-1}\, ds = \frac{2\omega_N r^{N+2}}{N(N+1)(N+2)}$$

and

$$\int\limits_{B(x_M,r)} |\nabla v|^2\, dx = \frac{\omega_N r^N}{N}.$$

Hence

$$\lambda_1^D \leq \frac{\int_{B(x_M,r)} |\nabla v|^2\, dx}{\int_{B(x_M,r)} v^2\, dx} = \frac{(N+1)(N+2)}{2r^2}.$$

Combining this with (2.10) we derive

$$\lambda_1^N(m) \leq \frac{(N+1)(N+2)(r+|x_M|)^2}{2r^2 m(x_M)}.$$

Therefore $\lambda_1^N(m) < \frac{\Lambda_N}{m(0)}$ if (2.8) holds. $\square$

We now give another example of a weight function satisfying (2.4). Let $M \in C(\bar{\Omega})$ satisfy the following conditions: $M^+ \neq 0$, $M(0) = 0$, $\int_\Omega \frac{M(x)}{|x|^2}\,dx < 0$ and $M(x) > 0$ for $x \in \overline{B(0,r)} \setminus \{0\}$, where $\overline{B(0,r)} \subset \Omega$. We now check that $m_A(x) = M(x) + A$, with $A > 0$ small enough, satisfies (2.4). Indeed, let $u \in H^1_\circ(B(0,r)) \setminus \{0\}$. We then have

$$\frac{\int_{B(0,r)} |\nabla u|^2\,dx}{\int_{B(0,r)} \frac{M(x)}{|x|^2} u^2\,dx + A\int_{B(0,r)} \frac{u^2}{|x|^2}\,dx} \leq \frac{\int_{B(0,r)} |\nabla u|^2\,dx}{\int_{B(0,r)} \frac{M(x)}{|x|^2} u^2\,dx}.$$

Since $u \in \{v \in H^1(\Omega);\ \int_\Omega \frac{m_A(x)}{|x|^2} v^2\,dx > 0\}$, we see that $\lambda_1^N(m_A) \leq \lambda_1^D(M, B(0,r))$, where $\lambda_1^D(M, B(0,r))$ is the principal eigenvalue for the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda \frac{M(x)}{|x|^2} u & \text{in } B(0,r),\\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B(0,r). \end{cases}$$

The existence of $\lambda_1^D(M, B(0,r))$ follows from Theorem 3.4 in Section 3. We choose $A > 0$ small so that $\lambda_1^D(M, B(0,r)) < \frac{\Lambda_N}{A}$ and $\int_\Omega \frac{m_A(x)}{|x|^2}\,dx < 0$.

In the above examples the value of a weight function m at 0 is rather small. One can construct a weight function with $m(0)$ large. Let M be a function from the above example. We put $m_{B,A}(x) = BM(x) + A$. As in the above example we show that $\lambda_1^N(m_{B,A}) \leq \frac{1}{B}\lambda_1^D(M, B(0,r))$. Given $A > 0$ we choose $B > 0$ sufficiently large so that $\frac{1}{B}\lambda_1^D(M, B(0,r)) < \frac{\Lambda_N}{A}$ and $\int_\Omega \frac{m_{B,A}(x)}{|x|^2}\,dx < 0$. With this choice of A, condition (2.4) is satisfied.

We now consider the case $m(0) \leq 0$.

Theorem 2.5. *Suppose that* (**M**) *holds and that* $m(0) \leq 0$. *Then there exists a minimizer for* $\lambda_1^N(m)$.

Proof. If $m(0) = 0$, then the functional $J(u) := \int_\Omega \frac{m(x)}{|x|^2} u^2\,dx$ is completely continuous on $H^1(\Omega)$. So the existence of a minimizer in this case is obvious. Therefore we only consider the case $m(0) < 0$. Let $\{u_k\} \subset \mathcal{M}$ be a minimizing sequence for $\lambda_1^N(m)$. By Lemma 2.1 the sequence $\{u_k\}$ is bounded in $H^1(\Omega)$. By the P.L. Lions' concentration-compactness principle [17] there exist nonnegative constants $\mu_\circ$ and $\nu_\circ$ such that the relations (2.5), (2.6) and (2.7) hold. Since $m(0) < 0$, $u \neq 0$. To show that $\nu_\circ = 0$ we now distinguish cases (i), (ii) and (iii), as in the proof of Theorem 2.3. Since $m(0) < 0$, the cases (i) and (ii) must be excluded. So it remains to consider case (iii), that is, $\int_\Omega \frac{m(x)}{|x|^2} u^2\,dx > 0$. By (2.6) we have $\int_\Omega \frac{m(x)}{|x|^2} u^2\,dx > 1$. Since $\int_\Omega \frac{m(x)}{|x|^2}\,dx < 0$, u cannot be a constant function. Hence $\int_\Omega |\nabla u|^2\,dx > 0$. It then follows from (2.5) that

$$\lambda_1^N(m) \geq \int_\Omega |\nabla u|^2\,dx + \mu_\circ > \lambda_1^N(m) \int_\Omega \frac{m(x)}{|x|^2} u^2\,dx > \lambda_1^N(m)$$

and we have arrived at a contradiction. $\square$

If $m(x) > 0$ on Ω, then problem (2.1) does not have a positive solution for $\lambda > 0$, that is, 0 is the only principal value. However, in this case we can define the second eigenvalue, namely,

$$\lambda_2^N(m) = \inf_{u\in H^1(\Omega)\setminus\{0\}, \int_\Omega \frac{m(x)}{|x|^2}u\,dx=0} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega \frac{m(x)}{|x|^2}u^2\,dx}.$$

The existence of minimizers for $\lambda_2^N(m)$ with $m \equiv 1$ on Ω has been investigated in paper [11].

Theorem 2.6. *Suppose that $m \in C(\bar{\Omega})$, $m \geq 0$ and $\neq 0$ on Ω.*

(i) *If $m(0) > 0$ and $\lambda_2^N(m) < \frac{\Lambda_N}{m(0)}$, then $\lambda_2^N(m)$ has a minimizer.*
(ii) *If $m(0) = 0$, then $\lambda_2^N(m)$ has a minimizer.*

Obviously $\lambda_2^N(m) > 0$. The proofs of (i) and (ii) are similar to those of Theorem 2.3 and 2.5 and are omitted.

We now give an example of a nonnegative weight function satisfying $m(0) > 0$ and $\lambda_2^N(m) < \frac{\Lambda_N}{m(0)}$. Let $\Omega = B(0,1)$ and let $M(|x|)$ be a continuous radial function on $B(0,1)$ such that $M(|x|) \geq 0$, $\neq 0$ on $B(0,1)$ and $M(0) = 0$. We are going to find the range of A for which the perturbation $m_A(|x|) = M(|x|) + A$, $A > 0$, of $M(|x|)$ satisfies condition (i) of Theorem 2.6. Towards this end, we observe that the coordinate function x_j satisfies $\int_{B(0,1)} \frac{m_A(x)}{|x|^2}x_j\,dx = 0$. Hence

$$\begin{aligned}
\lambda_2^N(m_A) &\leq \frac{\int_{B(0,1)} |\nabla(x_j)|^2\,dx}{\int_{B(0,1)} \frac{M(|x|)}{|x|^2}x_j^2\,dx + A\int_{B(0,1)} \frac{x_j^2}{|x|^2}\,dx} = \\
&= \frac{\frac{\omega_N}{N}}{\frac{1}{N}\int_{B(0,1)} M(|x|)\,dx + \frac{A}{N}\int_{B(0,1)}\,dx} = \\
&= \frac{1}{\int_0^1 M(r)r^{N-1}\,dx + \frac{A}{N}}.
\end{aligned}$$

Therefore $\lambda_2^N(m_A) < \frac{\Lambda_N}{A}$ provided $\frac{1}{\int_0^1 M(r)r^{N-1}\,dx + \frac{A}{N}} < \frac{\Lambda_N}{A}$. If $\Lambda_N \geq N$, that is $N \geq 8$, then condition (i) of Theorem 2.6 holds for every $A > 0$. If $\Lambda_N < N$, that is $N = 3, \ldots, 7$, then condition (i) of Theorem 2.6 holds for $A < \frac{N\Lambda_N}{N-\Lambda_N}\int_0^1 M(r)r^{N-1}\,dx$.

3. EIGENVALUES WITH THE DIRICHLET BOUNDARY CONDITIONS

It is well-known that the minimization problem

$$\Lambda_N = \inf_{u\in H_\circ^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega \frac{u^2}{|x|^2}\,dx}$$

has no solution. On the other hand the weighted eigenvalue problem

$$\begin{cases} -\Delta u = \lambda \frac{m(x)}{|x|^2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

under appropriate assumptions on a weight function m, has a principal eigenfunction. Let us define

$$\lambda_1^D(m) = \inf_{u \in H_\circ^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega \frac{m(x)}{|x|^2} u^2 \, dx},$$

where $m \in C(\bar{\Omega})$.

Theorem 3.1. *Suppose that $m \in C(\bar{\Omega})$, $m \geq 0$ and $\neq 0$ on Ω.*

(i) *If $m(0) > 0$ and*

$$\lambda_1^D(m) < \frac{\Lambda_N}{m(0)}, \tag{3.2}$$

then a minimization problem for $\lambda_1^D(m)$ has a solution.

(ii) *If $m(0) = 0$, then the minimization problem for $\lambda_1^D(m)$ has a solution.*

The proof is similar to that of Theorem 2.3 and is omitted.

An example of a weight function satisfying (3.2) follows from Proposition 2.4: assume additionally that $\overline{B(x_M, r)} \subset \bar{\Omega}$, $0 \notin \overline{B(x_M, r)}$ and that $m(x) \geq m(x_M) > 0$ for $x \in B(x_M, r)$ and $m(0) > 0$. If

$$\frac{m(0)}{m(x_M)} < \frac{r^2(N-2)^2}{2(r + |x_M|)^2(N+1)(N+2)},$$

then (3.2) holds.

As another example, we fix a function $M \in C(\bar{\Omega})$, $M(x) \geq 0$ and $\neq 0$ on Ω and $M(0) = 0$. We show that a small perturbation $m_A(x) = M(x) + A$, $A > 0$, of M produces a weight function satisfying (3.2). Indeed, let $v \in H_\circ^1(\Omega)$. Then

$$\lambda_1^D(m_A) \leq \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega \frac{M(x)}{|x|^2} v^2 \, dx + A \int_\Omega \frac{v^2}{|x|^2} \, dx} \leq \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega \frac{M(x)}{|x|^2} v^2 \, dx}.$$

Taking the infimum over $H_\circ^1(\Omega)$ we get $\lambda_1^D(m_A) \leq \lambda_1^D(M)$. We now choose the largest $A_\circ > 0$ such that $\lambda_1^D(M) \leq \frac{\Lambda_N}{A_\circ}$. Then m_A satisfies (3.2) for $A \in (0, A_\circ)$.

We point out here that in the papers [4] and [19] the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda w(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with the weight function w belonging to a Lorentz space, has been investigated. The authors of these papers proved the existence of the principal eigenfunction for w belonging the Lorentz space $L^{(\frac{N}{2}, q)}(\Omega)$ for some $1 < q < \infty$. This has been extended in the paper [4], by constructing a larger space of admissible weight functions. This space, denoted by $\mathcal{F}_{\frac{N}{2}}$, has been obtained as a completion of the space $C_\circ^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{(\frac{N}{2},\infty)}$ of the Lorentz space $L^{(\frac{N}{2},\infty)}(\Omega)$. This space has the property $L^{(\frac{N}{2},p)}(\Omega) \subset \mathcal{F}_{\frac{N}{2}} \subset L^{(\frac{N}{2},\infty)}(\Omega)$ for every $1 < p < \infty$. For more details we refer to the paper [4]. The weight function $\frac{m(x)}{|x|^2}$ from Theorem 3.1, in general, does not belong to the space $\mathcal{F}_{\frac{N}{2}}$. Another example of a weight function not belonging to $\mathcal{F}_{\frac{N}{2}}$ can be found in [29] in the case of an eigenvalue problem on $\mathbb{R}^N$.

Remark 3.2. We always have $\lambda_1^D(m) \leq \frac{\Lambda_N}{m(0)}$ (see Lemma 2.2). If $m(x) \leq m(0)$, then $\lambda_1^D(m) = \frac{\Lambda_N}{m(0)}$. This is an easy consequence of the Hardy inequality. In this case $\lambda_1^D(m)$ does not admit a minimizer. Indeed, assuming that u is a minimizer we would get

$$\int_\Omega |\nabla u|^2\,dx \leq \lambda_1^D(m)m(0)\int_\Omega \frac{u^2}{|x|^2}\,dx < \lambda_1^D(m)m(0)\Lambda_N^{-1}\int_\Omega |\nabla u|^2\,dx,$$

implying that $\lambda_1^D(m) > \frac{\Lambda_N}{m(0)}$, which is impossible.

The above remark suggests that the weighted Hardy inequality can be improved. For this we need the following improved Hardy inequality (see [8] Theorem 4.1): let $N \geq 3$, then for every $u \in H_\circ^1(\Omega)$ we have

$$\Lambda_N\int_\Omega \frac{u^2}{|x|^2}\,dx + H\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega u^2\,dx \leq \int_\Omega |\nabla u|^2\,dx, \tag{3.3}$$

where H is the first eigenvalue of the Laplacian "$-\Delta$" in the unit ball under the Dirichlet boundary conditions with $N = 2$.

Proposition 3.3. *Suppose that $m(x) \leq m(0)$ for $x \in \Omega$. Then*

$$\int_\Omega \frac{m(x)u^2}{|x|^2}\,dx + m(0)\frac{H}{\Lambda_N}\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega u^2\,dx \leq \frac{1}{\lambda_1^D(m)}\int_\Omega |\nabla u|^2\,dx$$

for every $u \in H_\circ^1(\Omega)$.

Proof. It follows from (3.3) that

$$\begin{aligned}
&\int_\Omega \frac{m(x)u^2}{|x|^2}dx + m(0)\frac{H}{\Lambda_N}\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega u^2\,dx \leq \\
&\leq m(0)\left[\int_\Omega \frac{u^2}{|x|^2}\,dx + \frac{H}{\Lambda_N}\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}\int_\Omega u^2\,dx\right] \leq \\
&\leq \frac{m(0)}{\Lambda_N}\int_\Omega |\nabla u|^2\,dx.
\end{aligned}$$

Since $\lambda_1^D(m) = \frac{\Lambda_N}{m(0)}$, the result follows. $\square$

It is clear that the improved Hardy inequality in Proposition 3.3 yields

$$\int\limits_\Omega \frac{m(x)u^2}{|x|^2}\,dx + \frac{H}{\Lambda_N}\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \int\limits_\Omega m(x)u^2\,dx \le \frac{1}{\lambda_1^D(m)} \int\limits_\Omega |\nabla u|^2\,dx.$$

For weights functions changing sign, we can formulate results parallel to Theorems 2.3 and 2.5:

Theorem 3.4. *Suppose that* (**M**) *holds.*

(i) *If* $\lambda_1^D(m) < \frac{\Lambda_N}{m(0)}$, $(m(0) > 0)$, *then there exists a minimizer for* $\lambda_1^D(m)$.

(ii) *If* $m(0) \le 0$, *then there exists a minimizer for* $\lambda_1^D(m)$.

Remark 3.5. We mention here papers [21] and [22] dealing with eigenvalue problems involving weighted Hardy potentials with the Dirichlet boundary conditions. In these papers the following eigenvalue problem has been investigated

$$\begin{cases} -\Delta u - \lambda\eta(x)\frac{u}{|x|^2} = \mu\frac{u}{|x|^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $\eta \in C^\alpha(\bar{\Omega})$, $\alpha \in (0,1)$, $\eta \ge 0$, $\eta \not\equiv 0$ on $\bar{\Omega}$ and $\lambda, \mu \in \mathbb{R}$. The principal eigenvalue μ_λ is defined by

$$\mu_\lambda = \inf_{H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \lambda\eta(x)\frac{u^2}{|x|^2}\right)dx}{\int_\Omega \frac{u^2}{|x|^2}\,dx}.$$

The main result of papers [21] and [22] asserts: if $\mu_\lambda < \Lambda_N$, then there exists a minimizer for μ_λ which is a solution to problem (3.4) with $\mu = \mu_\lambda$. The condition $\mu_\lambda < \Lambda_N$ is satisfied for $\lambda > \lambda^*$ for some $\lambda^* > 0$. Parallel results for eigenvalue problems with Hardy potential involving the distance to the boundary can be found in papers [6, 7, 20, 24].

4. BEHAVIOUR AROUND A SINGULAR POINT

We commence with a higher integrability property of the principal eigenfunction for the eigenvalue problem (2.1). In what follows we always assume that the principal eigenfunction is chosen to be positive.

Lemma 4.1. *Let* $\lambda_1^N(m) < \frac{\Lambda_N}{m(0)}$. *Then the principal eigenfunction* φ_1 *of* (2.1) *belongs to* $L^{2^*(1+\delta)}(\Omega)$ *for some* $\delta > 0$.

Proof. We set $\lambda = \lambda_1^N(m)$, $u = \varphi_1$ and $v = u\min(u,L)^{p-2} = uu_L^{p-2}$, where $p > 2$ and $L > 0$. Testing equation (2.1) with v, we obtain

$$\int\limits_\Omega |\nabla u|^2 u_L^{p-2}\,dx + (p-2)\int\limits_\Omega \nabla u \nabla u_L u_L^{p-2}\,dx = \lambda \int\limits_\Omega \frac{m(x)}{|x|^2} u^2 u_L^{p-2}\,dx. \tag{4.1}$$

We also have

$$\int_\Omega |\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\,dx = \int_\Omega |\nabla u|^2 u_L^{p-2}\,dx + \frac{(p-2)^2}{4}\int_\Omega |\nabla u_L|^2 u_L^{p-2}\,dx + + (p-2)\int_\Omega \nabla u \nabla u_L u_L^{p-2}\,dx = = \int_\Omega |\nabla u|^2 u_L^{p-2}\,dx + \frac{p^2-4}{4}\int_\Omega |\nabla u_L|^2 u_L^{p-2}\,dx.$$

Multiplying (4.1) by $\frac{p+2}{4}$ observing that $\frac{p+2}{4} > 1$ we obtain

$$\int_\Omega |\nabla u|^2 u_L^{p-2}\,dx + \frac{p^2-4}{4}\int_\Omega \nabla u\nabla u_L u_L^{p-2}\,dx \le \frac{\lambda(p+2)}{4}\int_\Omega \frac{m(x)}{|x|^2}u^2 u_L^{p-2}\,dx. \tag{4.2}$$

Hence

$$\int_\Omega |\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\,dx \le \frac{\lambda(p+2)}{4}\int_\Omega \frac{m(x)}{|x|^2}u^2u_L^{p-2}\,dx. \tag{4.3}$$

Since $\frac{\lambda m(0)}{\Lambda_N} < 1$, we can choose $\epsilon_1 > 0$ so that $\frac{\lambda}{\Lambda_N}(m(0)+\epsilon_1) < 1$. By the continuity of m there exists $r_1 > 0$ such that $m(x) \le m(0) + \epsilon_1$ for $x \in B(0, r_1)$. Applying inequality (1.4) with $\delta = \epsilon$ we obtain

$$\begin{aligned}
\frac{\lambda(p+2)}{4}\int_\Omega \frac{m(x)}{|x|^2}u^2u_L^{p-2}\,dx &\le \frac{\lambda(p+2)}{4}\int_{B(0,r_1)} \frac{m(0)+\epsilon_1}{|x|^2}u^2u_L^{p-2}\,dx+\\
&+ \frac{\lambda(p+2)}{4}\frac{\|m\|_\infty}{r_1^2}\int_\Omega u^2u_L^{p-2}\,dx \le\\
&\le \frac{\lambda(p+2)}{4}(m(0)+\epsilon_1)\Big(\frac{1}{\Lambda_N}+\epsilon\Big)\int_{B(0,r_1)} |\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\,dx+\\
&+ A(\epsilon,\Omega)\frac{\lambda(p+2)}{4}(m(0)+\epsilon_1)\int_{B(0,r_1)} \big(uu_L^{\frac{p}{2}-1}\big)^2\,dx+\\
&+ \frac{\lambda(p+2)}{4}\frac{\|m\|_\infty}{r_1^2}\int_\Omega u^2u_L^{p-2}\,dx.
\end{aligned} \tag{4.4}$$

Here we have used the Hardy inequality in $H^1(\Omega)$ (see (1.4)). We now put $p = 2+\delta$ and choose $\delta > 0$ and $\epsilon > 0$ so small that

$$\lambda\Big(1+\frac{\delta}{4}\Big)(m(0)+\epsilon_1)\Big(\frac{1}{\Lambda_N}+\epsilon\Big) = \Big(1+\frac{\delta}{4}\Big)\frac{\lambda}{\Lambda_N}(m(0)+\epsilon_1)+\lambda\epsilon\Big(1+\frac{\delta}{4}\Big)(m(0)+\epsilon_1) < 1. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we get

$$\Big(1-\lambda\big(1+\frac{\delta}{4}\big)(m(0)+\epsilon_1)\big(\frac{1}{\Lambda_N}+\epsilon\big)\Big)\int\limits_{B(0,r_1)}|\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\,dx \le$$

$$\le A(\epsilon,\Omega)\frac{\lambda(p+2)}{4}(m(0)+\epsilon_1)\int\limits_{B(0,r_1)}\big(uu_L^{\frac{p}{2}-1}\big)^2\,dx+\frac{\lambda(p+2)\|m\|_\infty}{4r_1^2}\int\limits_{\Omega}u^2u_L^{p-2}\,dx.$$

By the Sobolev inequality, we deduce that

$$S\Big(1-\lambda\big(1+\frac{\delta}{4}\big)(m(0)+\epsilon_1)\big(\frac{1}{\Lambda_N}+\epsilon\big)\Big)\Big(\int\limits_{B(0,r_1)}\big(uu_L^{\frac{p}{2}-1}\big)^{2^*}\,dx\Big)^{\frac{2}{2^*}}\le$$

$$\le C(\epsilon,\epsilon_1,\delta,r_1,\|m\|_\infty)\int\limits_{\Omega}\big(uu_L^{\frac{p}{2}-1}\big)^2\,dx,$$

where S denotes the best Sobolev constant for the embedding of $H^1(\Omega)$ into $L^{2^*}(\Omega)$. Letting $L\to\infty$, we deduce that $u\in L^{2^*(1+\frac{\delta}{2})}(B(0,r_1))$. So the assertion holds with $\delta_\circ=\frac{\delta}{2}$. □

Continuing with the above notations for $\lambda_1^N(m)$ and $u=\varphi_1$, we put $u=|x|^{-s}v$, with $s>0$ to be chosen later. We have

$$\text{div}\,\big(|x|^{-2s}\nabla v\big)=-\lambda|x|^{-s}\frac{m(x)u}{|x|^2}+u\big(-s^2|x|^{-s-2}+sN|x|^{-s-2}-2s|x|^{-s-2}\big).$$

We consider this identity in a small ball $B(0,r)$. Since $\lambda_1^N(m)<\frac{\Lambda_N}{m(0)}$, there exists $r>0$ small enough, such that $\lambda_1^N(m)\max_{B(0,r)}m(x)<\Lambda_N$. Let $s=\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda_1^N(m)\bar{m}_r}$, where $\bar{m}_r=\max_{B(0,r)}m(x)$, then

$$-\text{div}\,\big(|x|^{-2s}\nabla v\big)\le 0\ \text{ in }\ B(0,r). \tag{4.6}$$

On the other hand, if $s=\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda_1^N(m)\underline{m}_r}$, where $\underline{m}_r=\min_{B(0,r)}m(x)$, then

$$-\text{div}\,\big(|x|^{-2s}\nabla v\big)\ge 0\ \text{ in }\ B(0,r). \tag{4.7}$$

Proposition 4.2. *Suppose that* (**M**) *holds and that* $\lambda_1^N<\frac{\Lambda_N}{m(0)}$ *and* $m(0)>0$. *Then there exists* $B(0,r)\subset\Omega$ *such that*

$$M_1|x|^{-\left(\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda_1^N(m)\underline{m}_r}\right)}\le\varphi_1\le M_2|x|^{-\left(\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda_1^N(m)\bar{m}_r}\right)} \tag{4.8}$$

for $x\in B(0,r)$ *and some constants* $M_1>0$ *and* $M_2>0$.

Proof. The lower bound follows from Proposition 2.2 in [12]. To apply it, we need inequality (4.7). To obtain the upper bound, we follow the ideas from paper [15]. Let $0<r<\rho$ and $B(0,\rho)\subset\Omega$. We recall that $u=\varphi_1=|x|^{-s}v$. We use as a test function in (4.6) the function $\phi = \eta^2 v v_l^{2(t-1)} = \eta^2 v \min(v,l)^{2(t-1)}$, where $l, t > 1$ and η is a C^1-function such that $\eta = 1$ on $B(0,r)$, $\eta = 0$ on $\Omega \setminus B(0,\rho)$ and $|\nabla\eta| \leq \frac{4}{\rho - r}$ on Ω. Upon a substitution in (4.6) we obtain

$$\int_\Omega |x|^{-2s}\big(2\eta v v_l^{2(t-1)}\nabla\eta\nabla v + \eta^2 v_l^{2(t-1)}|\nabla v|^2 + 2(t-1)\eta^2 v_l^{2(t-1)}|\nabla v_l|^2\big)\,dx \leq 0, \tag{4.9}$$

where $s = \sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_1^N(m)\bar{m}_r}$. For every $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that

$$2\int_\Omega |x|^{-2s}\eta v v_l^{2(t-1)}\nabla\eta\nabla v\,dx \leq \epsilon \int_\Omega |x|^{-2s}\eta^2 v_l^{2(t-1)}|\nabla v|^2\,dx + \\ + C(\epsilon)\int_\Omega |x|^{-2s}|\nabla\eta|^2 v^2 v_l^{2(t-1)}\,dx.$$

Taking $\epsilon = \frac{1}{2}$, we derive from (4.9) that

$$\begin{aligned} &\int_\Omega |x|^{-2s}\big(\eta^2 v_l^{2(t-1)}|\nabla v|^2 + 2(t-1)\eta^2 v_l^{2(t-1)}|\nabla v_l|^2\big)\,dx \leq \\ &\leq C\int_\Omega |x|^{-2s}|\nabla\eta|^2 v^2 v_l^{2(t-1)}\,dx. \end{aligned} \tag{4.10}$$

In the next step, we use the Caffarelli-Kohn-Nirenberg inequality [9]

$$\left(\int_\Omega |x|^{-bp}|w|^p\,dx\right)^{\frac{2}{p}} \leq C_{a,b}\int_\Omega |x|^{-2a}|\nabla w|^2\,dx \tag{4.11}$$

for every $w \in H^1_\circ\big(\Omega, |x|^{-2a}\,dx\big)$, where $-\infty < a < \frac{N-2}{2}$, $a \leq b \leq a+1$, $p = \frac{2N}{N-2+2(b-a)}$ and $C_{a,b} > 0$ is a positive constant depending on a and b. We choose

$$a = b = \sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda_1^N(m)\bar{m}_r} < \frac{N-2}{2}.$$

In this case we have $p = 2^*$. We then deduce from (4.10) and (4.11) with $w = \eta v v_l^{(t-1)}$ that

$$\begin{aligned} \left(\int_\Omega |x|^{-2^*s}|\eta v v_l^{(t-1)}|^{2^*}\,dx\right)^{\frac{2}{2^*}} &\leq C_{a,b}\int_\Omega |x|^{-2s}|\nabla\big(\eta v v_l^{(t-1)}\big)|^2\,dx \leq \\ &\leq 2C_{a,b}\int_\Omega |x|^{-2s}\big(|\nabla\eta|^2 v^2 v_l^{2(t-1)} + \\ &\quad + \eta^2 v_l^{2(t-1)}|\nabla v|^2 + (t-1)^2\eta^2 v_l^{2(t-1)}|\nabla v_l|^2\big)\,dx \leq \\ &\leq Ct\int_\Omega |x|^{-2^*s}|\nabla\eta|^2 v^2 v_l^{2(t-1)}\,dx. \end{aligned} \tag{4.12}$$

We now observe that

$$\int_\Omega |x|^{-2^*s}|\eta|^{2^*}v^2 v_l^{2^*t-2}\,dx \le \int_\Omega |x|^{-2^*s}|\eta v v_l^{(t-1)}|^{2^*}\,dx.$$

Indeed, to show this we have to check that $v^2 v_l^{2^*t-2} \le v_l^{2^*(t-1)}v^{2^*}$ on Ω. This can be shown by considering the cases $v_l = l$ and $v_l = v$. Using this inequality we can rewrite (4.12) as

$$\left(\int_\Omega |x|^{-2^*s}|\eta|^{2^*}v^2 v_l^{2^*t-2}\,dx\right)^{\frac{2}{2^*}} \le Ct\int_\Omega |x|^{-2^*s}|\nabla\eta|^2 v^2 v_l^{2(t-1)}\,dx.$$

Due to the properties of the function η, the above inequality becomes

$$\left(\int_{B(0,r)} |x|^{-2^*s}v^2 v_l^{2^*t-2}\,dx\right)^{\frac{2}{2^*}} \le \frac{Ct}{(\rho-r)^2}\int_{B(0,\rho)} |x|^{-2^*s}v^2 v_l^{2(t-1)}\,dx. \tag{4.13}$$

To proceed further we observe that the integral on the right hand side of (4.13) is finite. This follows from the fact that $v = |x|^s\varphi_1$, so v has no singularity at 0 and moreover $2^*s - 2s < 2$. We now choose $\frac{N}{N-2} < t^* < (1+\delta_\circ)\frac{N}{N-2}$ and define the sequence $t_j = t^*\left(\frac{2^*}{2}\right)^j$, $j = 0, 1, \ldots$. Letting $t = t_j$ in (4.13) we obtain

$$\left(\int_{B(0,r)} |x|^{-2^*s}v^2 v_l^{2t_{j+1}-2}\,dx\right)^{\frac{1}{2t_{j+1}}} \le \left(\frac{Ct_j}{(\rho-r)^2}\right)^{\frac{1}{2t_j}}\left(\int_{B(0,\rho)} |x|^{-2^*s}v^2 v_l^{2t_j-2}\,dx\right)^{\frac{1}{2t_j}}.$$

We put $r_j = \rho_\circ(1+\rho_\circ^j)$, $j = 0, 1, \ldots$ with $\rho_\circ > 0$ so small that $\overline{B(0,2\rho_\circ)} \subset \Omega$. Substituting in the above inequality $\rho = r_j$ and $r = r_{j+1}$ we obtain

$$\left(\int_{B(0,r_{j+1})}|x|^{-2^*s}v^2 v_l^{2t_{j+1}-2}dx\right)^{\frac{1}{2t_{j+1}}}\le \left(\frac{Ct_j}{(\rho_\circ-\rho_\circ^2)^2\rho_\circ^{2j}}\right)^{\frac{1}{2t_j}}\left(\int_{B(0,r_j)}|x|^{-2^*s}v^2 v_l^{2t_j-2}dx\right)^{\frac{1}{2t_j}}. \tag{4.14}$$

Iterating gives

$$\begin{aligned}&\left(\int_{B(0,r_{j+1})} |x|^{-2^*s}v^2 v_l^{2t_{j+1}-2}\,dx\right)^{\frac{1}{2t_{j+1}}} \le \\ &\le \left(\frac{C}{\rho_\circ-\rho_\circ^2}\right)^{\sum_{j=0}^\infty \frac{1}{t_j}}\rho_\circ^{-\sum_{j=0}^\infty \frac{1}{t_j}}\prod_{j=0}^\infty t_j^{\frac{1}{2t_j}}\left(\int_{B(0,r_\circ)} |x|^{-2^*s}v^2 v_l^{2t^*-2}\,dx\right)^{\frac{1}{2t^*}}.\end{aligned} \tag{4.15}$$

We now notice that the infinite sums and the infinite product in the above inequality are finite. Since $2^* < 2t^* < (1+\delta_\circ)2^*$ we deduce

$$\int\limits_{B(0,r_\circ)} |x|^{-2^*s} v^2 v_l^{2t^*-2}\,dx \le \int\limits_{B(0,r_\circ)} |x|^{(2t^*-2^*)s}|u|^{2t^*}\,dx \le \\ \le d^{(2t^*-2^*)s}\int\limits_{\Omega}|u|^{2t^*}\,dx < \infty, \tag{4.16}$$

where $d =$ diam Ω. We now deduce from (4.15) and (4.16) that

$$\|v_l\|_{L^{2t_{j+1}}(B(0,\rho_\circ))} \le \|v_l\|_{L^{2t_{j+1}}(B(0,r_{j+1}))} \le \\ \le d^{\frac{2^*s}{2t_{j+1}}}\Bigg(\int\limits_{B(0,r_{j+1})} |x|^{-2^*s}v^2 v_l^{t_{j+1}-2}\,dx\Bigg)^{\frac{1}{2t_{j+1}}} \le C,$$

where $C > 0$ is a constant independent of l and j. Letting $t_j \to \infty$, we get $\|v_l\|_{L^\infty(B(0,\rho_\circ))} \le C$. Finally, if $l \to \infty$ we obtain $\|v\|_{L^\infty(B(0,\rho_\circ))} \le C$. Since $v = |x|^s\varphi_1$ the result follows. □

If $m(0) \le 0$, then the principal eigenfunction has no singularity at 0.

Proposition 4.3. *Suppose that* (**M**) *holds.*

(i) *If $m(0) < 0$, then $\phi_1 \in L^p(\Omega) \cap L^p(\Omega, \frac{dx}{|x|^2})$ for every $p \ge 2$.*
(ii) *If $m(0) = 0$, then $\phi_1 \in L^p(\Omega)$ for every $p \ge 2$.*

Proof. If $p = 2$ (i) and (ii) are obvious. So we assume that $p > 2$.

(i) We use again as a test function $v = u\min(u,L)^{p-2} = uu_L^{p-2}$. We have

$$\int\limits_\Omega |\nabla(uu_L^{\frac{p}{2}-1})|^2\,dx + \frac{\lambda(p+2)}{4}\int\limits_\Omega \frac{m^-(x)}{|x|^2}u^2u_L^{p-2}\,dx \le \frac{\lambda(p+2)}{4}\int\limits_\Omega \frac{m^+(x)}{|x|^2}u^2u_L^{p-2}\,dx.$$

Since $m(0) < 0$, we can find $r, \kappa > 0$ such that $m(x) \le -\kappa$ for $x \in B(0,r)$. We then have

$$\int\limits_\Omega |\nabla(uu_L^{\frac{p}{2}-1})|^2\,dx + \frac{\lambda(p+2)\kappa}{4}\int\limits_{B(0,r)} \frac{u^2u_L^{p-2}}{|x|^2}\,dx \le \frac{\lambda(p+2)\|m\|_\infty}{4r^2}\int\limits_\Omega u^2u_L^{p-2}\,dx.$$

We now apply the Sobolev inequality to obtain

$$S\Bigg(\int\limits_\Omega |uu_L^{\frac{p}{2}-1}|^{2^*}\,dx\Bigg)^{\frac{2}{2^*}} + \frac{\lambda(p+2)\kappa}{4}\int\limits_{B(0,r)} \frac{\big(uu_L^{\frac{p}{2}-1}\big)^2}{|x|^2}\,dx \le \\ \le \Bigg(\frac{\lambda(p+2)\|m\|_\infty}{4r^2}+1\Bigg)\int\limits_\Omega \big(uu_L^{\frac{p}{2}-1}\big)^2\,dx.$$

Letting $p = 2^*$ and $L \to \infty$ we see that $u \in L^{2^* \frac{(N-2)}{2}}(\Omega) \cap L^{2^* \frac{(N-2)}{2}}(\Omega, \frac{dx}{|x|^2})$. We complete the proof by iterating the above procedure.
(ii) We start again with the inequality

$$\int_\Omega |\nabla(uu_L^{\frac{p}{2}-1})|^2\,dx \le \frac{\lambda(p+2)}{4}\int_\Omega \frac{m(x)}{|x|^2}u^2u_L^{p-2}\,dx. \tag{4.17}$$

Given $\epsilon > 0$, there exists $r > 0$ such that $m(x) \le \epsilon$ for $x \in B(0,r)$. We then derive from (4.17) that

$$\int_\Omega |\nabla(uu_L^{\frac{p}{2}-1})|^2\,dx \le \frac{\lambda\epsilon(p+2)}{4}\int_\Omega \frac{u^2u_L^{p-2}}{|x|^2}\,dx + \frac{\lambda\|m\|_\infty(p+2)}{4r^2}\int_\Omega u^2u_L^{p-2}\,dx.$$

Applying the Hardy inequality in $H^1(\Omega)$ (see (1.4)) we get for every $\delta > 0$

$$\int_\Omega |\nabla(uu_L^{\frac{p}{2}-1})|^2\,dx \le \frac{\lambda\epsilon(p+2)}{4}\Big(\frac{1}{\Lambda_N}+\delta\Big)\int_\Omega |\nabla(uu_L^{\frac{p}{2}-1})|^2\,dx+ \\ + \Big(A(\delta,\Omega) + \frac{\lambda\|m\|_\infty(p+2)}{4r^2}\Big)\int_\Omega u^2u_L^{p-2}\,dx.$$

We now choose $\epsilon > 0$ so that

$$\frac{\lambda\epsilon(p+2)}{4}\Big(\frac{1}{\Lambda_N}+\delta\Big) < 1.$$

As in part (ii) we apply the Sobolev inequality, let $p = 2^*$ and $L \to \infty$. To complete the proof we iterate this procedure. □

The results of this section can be extended to the principal eigenfunction with the Dirichlet boundary conditions (3.1).

5. EXTENSION – MULTIPLE SINGULARITIES CASE

Singular eigenvalue problems discussed in Sections 2 and 3 can be extended to to eigenvalue problems with weights having multiple singular points. We restrict ourselves to the following Neumann problem

$$\begin{cases} -\Delta u = \lambda \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega,\ u > 0 \text{ on } \Omega, \end{cases} \tag{5.1}$$

where $x^1, \ldots, x^l$ are distinct points in Ω and $l \ge 2$. It is assumed that $m \in C(\bar{\Omega})$, $m^+(x) \not\equiv 0$ and that

(N) $\int_\Omega \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2}\,dx < 0.$

We now define

$$\tilde{\lambda}_1^N(m,x^1,\dots,x^l)=\inf\Big\{\int_\Omega |\nabla u|^2\,dx;\ u\in H^1(\Omega),\ \int_\Omega\sum_{j=1}^l\frac{m(x)}{|x-x^j|^2}u^2\,dx=1\Big\}.$$

Repeating the proof of Lemma 2.1 one can show that $\tilde{\lambda}_1^N(m,x^1,\dots,x^l)>0$.

Theorem 5.1. *Suppose that* (**N**) *holds and that* $m(x^j)>0$ *for* $j=1,\dots,l$. *If*

$$\tilde{\lambda}_1^N(m,x^1,\dots,x^l)<\frac{\Lambda_N}{\max_j m(x^j)},\tag{5.2}$$

then $\tilde{\lambda}_1^N(m,x^1,\dots,x^l)$ *admits a minimizer, which is the principal eigenfunction for problem* (5.1).

Proof. Let $\{u_k\}$ be a minimizing sequence for $\tilde{\lambda}_1^N(m,x^1,\dots,x^l)$. It is easy to show that $\{u_k\}$ is bounded in $H^1(\Omega)$. By the P.L. Lions' concentration-compactness principle there exist nonnegative constants ν^j, μ^j, $j=1,\dots,l$, and nonnegative measures μ and ν such that

$$|\nabla u_k|^2\,dx\rightharpoonup\mu\ge|\nabla u|^2\,dx+\sum_{j=1}^l\mu^j\delta_{x^j}\tag{5.3}$$

and

$$u_k^2\sum_{j=1}^l\frac{1}{|x-x^j|^2}\,dx\rightharpoonup\ \nu=u^2\sum_{j=1}^l\frac{1}{|x-x^j|^2}\,dx+\sum_{j=1}^l\nu^j\delta_{x^j}\tag{5.4}$$

in the sense of measures, where δ_{x_j} are the Dirac measures assigned to x_j. Moreover, we have

$$\Lambda_N\nu^j\le\mu^j,\ \ j=1,\dots,l.$$

First, we show that $u\ne 0$. In the contrary case, we have by (5.4)

$$1=\sum_{j=1}^l m(x^j)\nu^j.$$

Hence by (5.3) we deduce

$$\tilde{\lambda}_1^N(m,x^1,\dots,x^l)\ge\sum_{j=1}^l\mu_j\ge\Lambda_N\sum_{j=1}^l\nu_j\ge\frac{\Lambda_N}{\max_j m(x^j)}\sum_{j=1}^l m(x_j)\nu_j=\frac{\Lambda_N}{\max_j m(x^j)},$$

which is a contradiction. In the final step of the proof we show that $\nu^j=0$ for $j=1,\dots,l$. Arguing by contradiction, assume that $\nu^{j_\circ}>0$ for some $j_\circ$. We now distinguish three cases:

$$\text{(i)}\ \int_\Omega\sum_{j=1}^l\frac{m(x)}{|x-x^j|^2}u^2\,dx=0,\ \ \text{(ii)}\ \int_\Omega\sum_{j=1}^l\frac{m(x)}{|x-x^j|^2}u^2\,dx<0\ \text{ and}$$

$$\text{(iii)} \quad \int_\Omega \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2} u^2\,dx > 0.$$

By the first part of the proof, we exclude case (i). If (ii) occurs then

$$1 < \sum_{j=1}^{l} m(x^j)\nu^j \le \max_j m(x^j) \sum_{j=1}^{l} \nu^j \le \frac{\max_j m(x^j)}{\Lambda_N} \sum_{j=1}^{l} \mu^j.$$

Then by (5.3) we obtain

$$\tilde{\lambda}_1^N(m, x^1, \dots, x^l) \ge \sum_{j=1}^{l} \mu^j \ge \frac{\Lambda_N}{\max_j m(x^j)},$$

which is impossible. In the case (iii) we have

$$1 - \int\limits_\Omega \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2} u^2\,dx = \sum_{j=1}^{l} m(x^j)\nu^j \le \max_j m(x^j) \sum_{j=1}^{l} \nu_j \le$$

$$\le \frac{\max_j m(x^i)}{\Lambda_N} \sum_{j=1}^{l} \mu^j.$$

This yields

$$\tilde{\lambda}_1^N(m, x^1, \dots, x^l) \ge \int\limits_\Omega |\nabla u|^2\,dx + \frac{\Lambda_N}{\max_j m(x^j)} \left(1 - \int\limits_\Omega \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2} u^2\,dx\right) \ge$$

$$\ge \tilde{\lambda}_1^N(m, x^1, \dots, x^l) \int\limits_\Omega \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2} u^2\,dx +$$

$$+ \frac{\Lambda_N}{\max_j m(x^j)} \left(1 - \int\limits_\Omega \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2} u^2\,dx\right).$$

This equivalent to

$$\tilde{\lambda}_1^N - \frac{\Lambda_N}{\max_j m(x^j)} \ge \left(\tilde{\lambda}_1^N - \frac{\Lambda_N}{\max_j m(x^j)}\right) \int\limits_\Omega \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2} u^2\,dx$$

implying that

$$\int\limits_\Omega \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2} u^2\,dx \ge 1.$$

On the other hand since $\nu_{j_\circ} > 0$ we have

$$1 = \int\limits_\Omega \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2} u^2\,dx + \sum_{j=1}^{l} \nu_j(x^j) m(x^j) > \int_\Omega \sum_{j=1}^{l} \frac{m(x)}{|x-x^j|^2} u^2\,dx$$

and we have arrived at a contradiction. □

We point out that the principal eigenfunction from Theorem 5.1 satisfies around each singular point x^j an estimate of type (4.8).

We close this paper with the following remark concerning simplicity of the principal eigenvalues: all principal eigenvalues constructed in this article are simple. This can be proved using the arguments of Proposition 2.3 and 2.4 from [18].

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