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# A CLASS OF NONLOCAL INTEGRODIFFERENTIAL EQUATIONS VIA FRACTIONAL DERIVATIVE AND ITS MILD SOLUTIONS

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**Abstract.** In this paper, we discuss a class of integrodifferential equations with nonlocal conditions via a fractional derivative of the type:

$$D_t^q x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + t^n f(t, x(t)), \quad t \in [0,T],\ n \in Z^+,$$

$$q \in (0,1],\ x(0) = g(x) + x_0.$$

Some sufficient conditions for the existence of mild solutions for the above system are given. The main tools are the resolvent operators and fixed point theorems due to Banach's fixed point theorem, Krasnoselskii's fixed point theorem and Schaefer's fixed point theorem. At last, an example is given for demonstration.

**Keywords:** integrodifferential equations, fractional derivative, nonlocal conditions, resolvent operator and their norm continuity, fixed point theorem, mild solutions.

**Mathematics Subject Classification:** 34G20, 45J05.

## 1. INTRODUCTION

The origin of fractional differential equations goes back to Newton and Leibniz in the seventieth century. They has recently been shown to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economy and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetics, etc. [9, 14–16, 25, 26]. In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Kilbas *et al.* [18], Miller *et al.* [24], Podlubny [29], Lakshmikantham *et al.* [19], and the papers [7, 10, 11, 17, 20, 21, 27, 28] and the references therein.

On the other hand, the study of initial value problems with nonlocal conditions arises to deal specially with some situations in physics. For the comments and motivations for a nonlocal Cauchy problem in different fields, we refer the reader to [1]–[6] and the references contained therein.

Most of the practical systems are integrodifferential equations in nature and hence the study of integrodifferential systems is very important. Recently, Liang *et al.* [22], use the Schaefer fixed point theorem and derive some new results about the existence of mild solutions for nonlocal Cauchy problems for the nonlinear integrodifferential equation which can be derived from the study of heat conduction in materials with memory and viscoelasticity such as

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f\left(t, x(t)\right), \quad t \in J = [0,T], \qquad (1.1)$$

$$x(0) = g(x) + x_0,$$

where $A$ is the generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ in a Banach space $X$, $\{B(t) \mid t \in J\}$ is a family of unbounded operators in $X$.

Very recently, Wang *et al.* [28] study a class of fractional integrodifferential equations of mixed type with time-varying generating operators and nonlocal conditions. By virtue of the contraction mapping principle and Krasnoselskii's fixed point theorem via Gronwall's inequailty, the existence and uniqueness of mild solutions are given. Meanwhile, the associated fractional optimal control is considered.

Motivated by the above works [22, 28], the purpose of this paper is to discuss the following fractional integrodifferential equations with nonlocal initial conditions:

$$D_t^q x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + t^n f\left(t, x(t)\right), \quad t \in J = [0,T], \quad n \in Z^+, \quad q \in (0,1], \qquad (1.2)$$

$$x(0) = g(x) + x_0 \qquad (1.3)$$

in a general Banach space $(X, \|\cdot\|)$, $x_0 \in X$, $f\colon J \times X \to X$ is a $X$-value function, and $g : C(J,X) \to X$ constitutes a nonlocal Cauchy problem. The derivative $D_t^q$ is understood here in the Riemann-Liouville sense.

We make the following assumption:

[HA]: $A$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ on $X$ with domain $D(A)$. Note that $D(A)$ endowed with the graph norm $|x| = \|x\| + \|Ax\|$ is a Banach space which will be denoted by $(Y, |\cdot|)$.

Following [8], we make the following assumption of $\{B(t) \mid t \in J\}$.

[HB]: $\{B(t) \mid t \in J\}$ is a family of continuous linear operators from $(Y, |\cdot|)$ into $(X, \|\cdot\|)$. Moreover, there is an integrable function $\widetilde{c} : J \to R^+$ such that for any $y \in Y$, the map $t \to B(t)y$ belongs to $W^{1,1}(J,X)$ and

$$\left\| \frac{dB(t)y}{dt} \right\| \leq \widetilde{c}(t)|y|, y \in Y, t \in J.$$

In fact, this assumption is satisfied in the study of heat conduction in materials with memory (see [12]) and viscoelasticity (see [13]), where $B(t) = K(t)A$ for a family of continuous operators $\{K(t) \mid t \in J\}$ on $X$ satisfying some additional conditions.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and introduce the mild solution of system (1.2)–(1.3). In Section 3, we study the existence and uniqueness of mild solutions for system (1.2)–(1.3) by virtue of the Banach contraction principle, Krasnoselskii's fixed point theorem and Schaefer's fixed point theorem respectively. At last, an example is also given to illustrate our theory.

## 2. PRELIMINARIES

Let us recall the following known definitions. For more details see [29].

**Definition 2.1.** A real function $f(t)$ is said to be in the space $C_\alpha$, $\alpha \in R$ if there exists a real number $\kappa > \alpha$, such that $f(t) = t^\kappa g(t)$, where $g \in C[0,\infty)$ and it is said to be in the space $C_\alpha^m$ iff $f^{(m)} \in C_\alpha$, $m \in N$.

**Definition 2.2.** The Riemann-Liouville fractional integral operator of order $\gamma > 0$ of a function $f \in C_\alpha$, $\alpha \geq -1$ is defined as

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

**Definition 2.3.** If the function $f \in C_{-1}^m$, $m \in N$, the fractional derivative of order $\gamma > 0$ of a function $f(t)$ is the Caputo sense is given by

$$\frac{d^\gamma f(t)}{dt^\gamma} = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-s)^{m-\gamma-1} f^{(m)}(s) ds,\ m-1 < \gamma \leq m.$$

Now, we define the resolvent operator for Equation (1.1).

**Definition 2.4.** A family $\{R(t) \mid t \geq 0\}$ of continuous linear operators on $X$ is called a resolvent operator for Equation (1.1) if and only if:

(R1) $R(0) = I$, where $I$ is the identity operator on $X$,
(R2) the map $t \to R(t)x$ is continuous from $J$ to $X$, for all $x \in X$,
(R3) $R(t)$ is a continuous linear operator on $Y$, for all $t \in J$, and the map $t \to R(t)y$ belongs to $C(J,X) \cap C^1(J,X)$ and satisfies

$$\frac{dR(t)y}{dt} = AR(t)y + \int_0^t B(t-s)R(s)yds = R(t)Ay + \int_0^t R(t-s)B(s)yds.$$

The above definition follows [12], so we know that the existence of such a resolvent operator $R(\cdot)$ is well established from [12].

Accordingly, we can introduce the mild solution of system (1.2)–(1.3).

**Definition 2.5.** A mild solution of system (1.2)–(1.3) is a function in $C(J, X)$ such that

$$x(t) = R(t)(x_0 + g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n R(t-s) f(s, x(s)) ds.$$

The following results of the corresponding resolvent operator $R(\cdot)$ of Equation (1.1) are used throughout this paper.

**Theorem 2.6** (Theorem 2.1, [22])**.** *If the strongly continuous semigroup* $\{T(t), t \geq 0\}$ *generated by* $A$ *is compact, then the corresponding resolvent operator* $R(\cdot)$ *of Equation* (1.1) *exists and is also compact.*

**Theorem 2.7** (Theorem 2.2, [22])**.** *Then there exists a constant* $H = H(T)$ *such that*

$$\|R(t+h) - R(h)R(t)\|_{L_b(X)} \leq Hh, \text{ for } 0 \leq h \leq t \leq T,$$

*where* $L_b(X)$ *is the Banach space of all linear and bounded operators on* $X$*.*

**Theorem 2.8** (Theorem 2.3, [22])**.** *If the strongly continuous semigroup* $\{T(t), t \geq 0\}$ *generated by* $A$ *is compact, then the corresponding resolvent operator* $R(t)$ *of Equation* (1.1) *is an operator norm continuous for* $t > 0$.

**Lemma 2.9** (Krasnoselskii's fixed point theorem, [23])**.** *Let* $\mathfrak{B}$ *be a closed convex and nonempty subsets of a Banach space* $X$*. Suppose that* $\mathcal{L}$ *and* $\mathcal{N}$ *are in general nonlinear operators which map* $\mathfrak{B}$ *into* $X$ *such that:*

(1) $\mathcal{L}x + \mathcal{N}y \in \mathfrak{B}$ *whenever* $x, y \in \mathfrak{B}$,
(2) $\mathcal{L}$ *is a contraction mapping,*
(3) $\mathcal{N}$ *is compact and continuous.*

*Then there exists* $z \in \mathfrak{B}$ *such that* $z = \mathcal{L}z + \mathcal{N}z$.

**Lemma 2.10** (Schaefer's fixed point theorem, [30])**.** *Let* $S$ *be a convex subset of a normed linear space* $E$ *and assume* $0 \in S$*. Let* $\Psi : S \to S$ *be a continuous and compact map, and let the set* $\{x \in S : x = \lambda \Psi x \text{ for some } \lambda \in (0,1)\}$ *be bounded. Then* $\Psi$ *has at least one fixed point in* $S$*.*

## 3. MAIN RESULTS

In this section, we give the existence and uniqueness of the mild solutions for system (1.2)–(1.3).

We need the following assumptions.

[Hf](1): $f : J \times X \to X$ is continuous with respect to $t$ on $J$.

[Hf](2): There exists a function $m \in L^1_{Loc}(J, R^+)$ such that

$$\|f(t,x) - f(t,y)\| \leq m(t)\|x - y\|$$

for all $x, y \in X$ and $t \in J$.

[Hg]: $g : C(J,X) \to X$ satisfies the Lipschitz continuous condition, i.e., there exists a constant $l_g > 0$ such that

$$\|g(x) - g(y)\| \leq l_g \|x - y\|_C$$

for arbitrary $x, y \in C(J,X)$, where $\|\cdot\|_C$ denotes $\|\cdot\|_{C(J,X)}$.

[H$\Omega_n$]: The function $\Omega_n : J \to R^+$, $n \in Z^+$, defined by

$$\Omega_n = M\left[l_g + \frac{t^{n+1}T^{q-1}}{(n+1)\Gamma(q)}\|m\|_{L^1_{Loc}(J,R^+)}\right]$$

satisfies $0 < \Omega_n \leq \omega < 1$ for all $t \in J$.

### 3.1. EXISTENCE AND UNIQUENESS RESULT WITH THE BANACH CONTRACTION MAPPING PRINCIPLE

**Theorem 3.1.** *Assume that the conditions* [HA], [HB], [Hf], [Hg] *and* [H$\Omega_n$] *are satisfied. Then system* (1.2)–(1.3) *has a unique mild solution.*

*Proof.* We consider the operator $\Gamma : C(J,X) \to C(J,X)$ defined by

$$(\Gamma x)(t) = R(t)[x_0 + g(x)] + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^n R(t-s)f(s,x(s))ds, \tag{3.1}$$

for all $t \in J$. Note that $\Gamma$ is well defined on $C(J,X)$.

Now, take $t \in J$ and $x, y \in C(J,X)$. Then we have

$$\|\Gamma x(t) - \Gamma y(t)\| \leq \|R(t)[g(x) - g(y)]\| + \\ + \frac{1}{\Gamma(q)}\left\|\int_0^t (t-s)^{q-1}s^n R(t-s)[f(s,x(s)) - f(s,y(s))]ds\right\|.$$

Hence, we get

$$\|\Gamma x(t) - \Gamma y(t)\| \leq M\|g(x) - g(y)\| + \frac{MT^{q-1}}{\Gamma(q)}\int_0^t s^n\|f(s,x(s)) - f(s,y(s))\|ds,$$

where

$$M = \sup_{t \in J}\{\|R(t)\|_{L_b(X)}\}$$

and $L_b(X)$ be the Banach space of all linear and bounded operators on $X$.

According to [Hf](2) and [Hg], we obtain

$$\|\Gamma x(t) - \Gamma y(t)\| \leq Ml_g\|x-y\|_C + M\frac{T^{q-1}}{\Gamma(q)}\int_0^t s^n m(t)\|x(s)) - y(s)\|ds \leq$$

$$\leq Ml_g\|x-y\|_C + M\frac{T^{q-1}}{\Gamma(q)}\|x-y\|_C\int_0^t s^n m(t)ds.$$

Therefore, we can deduce that

$$\|\Gamma x(t) - \Gamma y(t)\| \leq M\left[l_g + \frac{t^{n+1}T^{q-1}}{(n+1)\Gamma(q)}\|m\|_{L^1_{Loc}(J,R^+)}\right]\|x-y\|_C \leq$$
$$\leq \Omega_n(t)\|x-y\|_C.$$

Thus,

$$\|\Gamma x - \Gamma y\|_C \leq \Omega_n(t)\|x-y\|_C.$$

Hence, assumption $[\mathrm{H}\Omega_n]$ allows us to conclude in view of the contraction mapping principle, that $\Gamma$ has a unique fixed point $x \in C(J, X)$, and

$$x(t) = R(t)[x_0 + g(x)] + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^n R(t-s)f(s, x(s))ds,$$

which is the unique mild solution of system (1.2)–(1.3). □

### 3.2. EXISTENCE RESULT WITH THE KRASNOSELSKII FIXED POINT THEOREM

We suppose that

[Hf](1'): $f : J \times X \to X$, for a.e. $t \in J$, the function $f(t, \cdot) : X \to X$ is continuous and for all $x \in X$, the function $f(\cdot, x) : J \to X$ is measurable.

[Hf](3): There exists a function $\rho \in L^1_{Loc}(J, R^+)$ such that

$$\|f(t, x)\| \leq \rho(t)$$

for all $x \in X$ and $t \in J$.

Now we are ready to state and prove a new existence result.

**Theorem 3.2.** *Assume that the conditions* [HA], [HB], [Hf](1'), [Hf](3), [Hg] *are satisfied. Then system* (1.2)–(1.3) *has at least one mild solution on* $J$ *provided that*

$$Ml_g < 1,$$

*where* $M = \sup_{t\in J}\{\|R(t)\|_{L_b(X)}\}$.

*Proof.* Let us choose

$$r = M(\|x_0\| + G) + M\frac{T^{n+q}}{(n+1)\Gamma(q)}\|\rho\|_{L^1_{Loc}(J,R^+)} + Mc_0 + c_2\frac{T^{n+q}}{n+1}$$

with

$$G = \sup_{x \in C(J,X)}\{\|g(x)\|\}, \tag{3.2}$$

$c_0$ and $c_2$ defined respectively by (3.3) and (3.4) below.

Consider the ball

$$B_r = \{x \in C(J,X) : \|x\|_C \leq r\}.$$

Define on $B_r$ the operators $\Gamma_1$ and $\Gamma_2$ by

$$(\Gamma_1 x)(t) = R(t)[x_0 + g(x)],$$

and

$$(\Gamma_2 x)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^n R(t-s)f(s,x(s))ds.$$

*Step 1.* Let us observe that if $x, y \in B_r$ then $\Gamma_1 x + \Gamma_2 y \in B_r$.

In fact, one can obtain

$$\|(\Gamma_1 x)(t) + (\Gamma_2 y)(t)\| \leq M\|x_0 + g(x)\| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^n\|f(s,y(s))\|ds \leq$$

$$\leq M(\|x_0\| + \|g(x)\|) + M\frac{T^{q-1}}{\Gamma(q)}\int_0^t s^n\|f(s,y(s))\|ds,$$

which according to (3.2), gives

$$\|(\Gamma_1 x)(t) + (\Gamma_2 y)(t)\| \leq M(\|x_0\| + G) + M\frac{T^{n+q}}{(n+1)\Gamma(q)}\|\rho\|_{L^1_{Loc}(J,R^+)} \leq r.$$

Hence, we can deduce that

$$\|\Gamma_1 x + \Gamma_2 x\|_C \leq r.$$

*Step 2.* We show that $\Gamma_1$ is a contraction mapping.

For any $t \in J$, $x, y \in C(J,X)$ we have

$$\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\| \leq M\|g(x) - g(y)\|,$$

which in view of [Hg], gives

$$\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\| \leq Ml_g\|x - y\|_C,$$

which implies that

$$\|\Gamma_1 x - \Gamma_1 y\| \leq Ml_g\|x - y\|_C.$$

Since $Ml_g < 1$, then $\Gamma_1$ is a contraction mapping.

*Step 3.* Let us prove that $\Gamma_2$ is continuous and compact.

For this purpose, we assume that $x_n \to x$ in $C(J, X)$. Then by [Hf](1') we have that

$$f(s, x_n(s)) \to f(s, x(s)), \text{ as } n \to \infty, \ s \in J.$$

According to [Hf](3),

$$\|f(s, x_n(s)) - f(s, x(s))\| \leq 2\rho(s).$$

By the dominated convergence theorem,

$$\int_0^t \|f(s, x_n(s)) - f(s, x(s))\| ds \to 0, \text{ as } n \to \infty,$$

which implies that

$$\|(\Gamma_2 x_n)(t) - (\Gamma_2 x)(t)\| \leq \frac{MT^n}{(n+1)\Gamma(q)} \int_0^t \|f(s, x_n(s)) - f(s, x(s))\| ds \to 0, \text{ as } n \to \infty,$$

which implies that $\Gamma_2$ is continuous.

To prove that $\Gamma_2$ is a compact operator, we observe that $\Gamma_2$ is a composition of two operators, that is, $\Gamma_2 = U \circ V$ where

$$(Vx)(s) = R(t-s)f(s, x(s)), \, t \in J, \, 0 < s < t,$$

and

$$(Uy)(t) = \int_0^t (t-s)^{q-1} s^n y(s) ds, \, t \in J.$$

Since for the same reason as $\Gamma_1$, the operator $V$ is also continuous, it suffices to prove that $V$ is uniformly bounded and $U$ is compact to prove that $\Gamma_1$ is compact.

Let $x \in B_r$. In view of [Hf](1'), $f$ is bounded on the compact set $J \times B_r$. Therefore, we set

$$c_0 = \sup_{(t,x) \in J \times B_r} \|f(t, x(t))\| < \infty. \tag{3.3}$$

Then, using (3.3), we get

$$\|(Vx)(s)\| \leq \|R(t-s)\| \|f(s, x(s))\| \leq Mc_0 \leq r,$$

from which we deduce that

$$\|Vx\|_C \leq r.$$

This means that $V$ is uniformly bounded on $B_r$.

Since $y \in C(J, X)$, we set

$$c_2 = \sup_{t \in J} \|y(t)\| < \infty. \tag{3.4}$$

Then, on the other hand, we have

$$\|(Uy)(t)\| = \left\| \int_0^t (t-s)^{q-1} s^n y(s) ds \right\| \leq c_2 \int_0^t (t-s)^{q-1} s^n ds \leq c_2 \frac{T^{n+q}}{n+1} \leq r$$

and on the other hand, for $0 < s < t_2 < t_1 < T$,

$$\begin{aligned}
\|(Uy)(t_1) - (Uy)(t_2)\| &= \left\| \int_0^{t_1} (t_1-s)^{q-1} s^n y(s) ds - \int_0^{t_2} (t_2-s)^{q-1} s^n y(s) ds \right\| \leq \\
&\leq \left\| \int_0^{t_2} [(t_1-s)^{q-1} - (t_2-s)^{q-1}] s^n y(s) ds \right\| + \\
&\quad + \left\| \int_{t_1}^{t_2} (t_1-s)^{q-1} s^n y(s) ds \right\| \leq \\
&\leq \int_0^{t_2} |(t_1-s)^{q-1} - (t_2-s)^{q-1}| s^n \|y(s)\| ds + \\
&\quad + \int_{t_1}^{t_2} (t_1-s)^{q-1} s^n \|y(s)\| ds \leq \\
&\leq \frac{c_2 T^n}{q} |2(t_1-t_2)^q + t_2^q - t_1^q| \leq \\
&\leq \frac{c_2 T^n}{q} |t_1 - t_2|^q,
\end{aligned}$$

which does not depend on $y$. So $UB_r$ is relatively compact. By the Arzela-Ascoli Theorem, $U$ is compact. In short, we have proven that $\Gamma_2$ is continuous and compact, $\Gamma_1$ is a contraction mapping and $\Gamma_1 x + \Gamma_2 y \in B_r$ if $x, y \in B_r$. Hence, the Krasnoselskii theorem allows us to conclude that system (1.2)–(1.3) has at least one mild solution on $J$. □

### 3.3. EXISTENCE RESULT WITH THE SCHAEFER FIXED POINT THEOREM

We make the following assumptions:

[HA']: The strongly continuous semigroup $\{T(t), t \geq 0\}$ generated by $A$ is compact.

[Hf](2') There exist $a$, $a_1 \geq 0$ such that

$$\|f(t,x)\| \leq a\|x\| + a_1 \quad \text{for} \quad t \in J,\ x \in X.$$

[Hg'] $g : C(J,X) \to X$ is compact, and there exists $b, b_1 \geq 0$ such that

$$\|g(x)\| \leq b\|x\|_C + b_1 \quad \text{for} \quad x \in C(J,X).$$

**Theorem 3.3.** *Assume that the conditions* [HA'], [HB], [Hf](1), [Hf](2'), [Hg'] *are satisfied. Then system* (1.2)–(1.3) *has at least one mild solution on* $J$ *provided that*

$$M\left[b + a\frac{T^{n+q}}{(n+1)\Gamma(q)}\right] < 1, \tag{3.5}$$

*where* $M = \sup_{t\in J}\{\|R(t)\|_{L_b(X)}\}$.

*Proof.* For each $x \in C(J, X)$, define $\Gamma : C(J, X) \to C(J, X)$ by (3.1) in Theorem 3.1. Our goal is to prove that $\Gamma$ has a fixed point. To this end, we first prove that $\Gamma$ is a compact operator. Observe that for $x, y \in C(J, X)$,

$$\|\Gamma x - \Gamma y\|_C \le M\|g(x) - g(y)\| + \frac{MT^{q-1}}{\Gamma(q)}\int_0^t s^n\|f(s, x(s)) - f(s, y(s))\|ds,$$

so that $\Gamma$ is continuous by using assumption [Hf](2') and the dominated convergence theorem again.

Let $E \subset C(J, X)$ be bounded, we will prove that the set

$$\Gamma(E) = \{\Gamma x \mid x \in E\}$$

is precompact in $C(J, X)$ by using the Arzela-Ascoli theorem again. That is, we need to prove that for each $t \in J$, the set $\{(\Gamma x)(t) \mid x \in E\}$ is precompact in $X$ and the function in $\Gamma(E)$ are equicontinuous.

For $t = 0$, the set $\{(\Gamma x)(0) \mid x \in E\} = \{x_0 + g(x) \mid x \in E\}$ is precompact in $X$ because of [Hg'].

For $t \in (0, T]$,

$$(\Gamma x)(t) = R(t)[x_0 + g(x)] + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^n R(t-s)f(s, x(s))ds,$$

and the set $\{R(t)[x_0 + g(x)] \mid x \in E\}$ is precompact in $X$ because $R(t)$ is a compact operator for $t > 0$ based on Theorem 2.6.

Next, we need to prove the compactness of the set

$$Q = \left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^n R(t-s)f(s, x(s))ds \mid x \in E\right\}.$$

First, note that for any small $\varepsilon \in (0, t)$, the set

$$\left\{R(\varepsilon)\cdot\frac{1}{\Gamma(q)}\int_0^{t-\varepsilon} (t-s-\varepsilon)^{q-1}s^n R(t-s-\varepsilon)f(s, x(s))ds \mid x \in E\right\}$$

is precompact in $X$ because $R(\varepsilon)$ is a compact operator. Next, we have

$$\Bigg\| R(\varepsilon)\cdot\frac{1}{\Gamma(q)}\int_0^{t-\varepsilon}(t-s-\varepsilon)^{q-1}s^nR(t-s-\varepsilon)f(s,x(s))ds-$$
$$-\frac{1}{\Gamma(q)}\int_0^{t-\varepsilon}(t-s)^{q-1}s^nR(t-s)f(s,x(s))ds\Bigg\|\leq$$
$$\leq\frac{1}{\Gamma(q)}\int_0^{t-\varepsilon}\|R(\varepsilon)R(t-s-\varepsilon)-R(t-s)\|_{L_b(X)}|(t-s-\varepsilon)^{q-1}-(t-s)^{q-1}|s^n\|f(s,x(s))\|ds\leq$$
$$\leq\frac{1}{\Gamma(q)}\int_0^{t-\varepsilon}\|R(\varepsilon)R(t-s-\varepsilon)-R(t-s)\|_{L_b(X)}[(t-s-\varepsilon)^{q-1}+(t-s)^{q-1}]s^n\|f(s,x(s))\|ds+$$
$$+\frac{1}{\Gamma(q)}\int_{t-2\varepsilon}^{t-\varepsilon}\|R(\varepsilon)R(t-s-\varepsilon)-R(t-s)\|_{L_b(X)}[(t-s-\varepsilon)^{q-1}+(t-s)^{q-1}]s^n\|f(s,x(s))\|ds\leq$$
$$\leq\varepsilon H\frac{2T^{n+q}}{(n+1)\Gamma(q)}F+\varepsilon\frac{2T^{n+q}}{(n+1)\Gamma(q)}(M^2+M)F$$
$$\to 0 \text{ as } \varepsilon\to 0,$$

where $F=\max\{\|f(s,x(s))\|\mid x\in E, s\in J\}$ and Theorem 2.7 is used.

Therefore, we see that the set

$$\left\{\frac{1}{\Gamma(q)}\int_0^{t-\varepsilon}(t-s)^{q-1}s^nR(t-s)f(s,x(s))ds\mid x\in E\right\}$$

is also precompact in $X$ by using the total boundedness.

To prove the compactness of $Q$, we apply this idea again and observe that

$$\Bigg\|\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}s^nR(t-s)f(s,x(s))ds-\frac{1}{\Gamma(q)}\int_0^{t-\varepsilon}(t-s)^{q-1}s^nR(t-s)f(s,x(s))ds\Bigg\|\leq$$
$$\leq\frac{1}{\Gamma(q)}\Bigg\|\int_{t-\varepsilon}^t(t-s)^{q-1}s^nR(t-s)f(s,x(s))\Bigg\|\leq\varepsilon M\frac{T^{n+q-1}}{(n+1)\Gamma(q)}F\to 0 \text{ as } \varepsilon\to 0.$$

Thus, the set $Q$ is also precompact. Therefore, for each $t\in J$, the set $\{(\Gamma x)(t)\mid x\in E\}$ is precompact in $X$.

Now, we prove that the functions in $\Gamma(E)$ are equicontinuous.

First, since $g$ is a compact operator, the functions

$$\{R(t)[x_0+g(x)]\mid x\in E\}$$

can be shown to be equicontinuous by using the strong continuity of the resolvent operator $R(\cdot)$. Next, for the functions in

$$W = \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n R(t-s) f(s, x(s)) ds \mid x \in E \right\}$$

we let $0 \le t_1 < t_2 \le T$ and obtain

$$\Big\| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} s^n R(t_2-s) f(s,x(s)) ds -$$

$$- \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} s^n R(t_1-s) f(s,x(s)) ds \Big\| \le$$

$$\le \frac{1}{\Gamma(q)} \int_0^{t_1} |(t_2-s)^{q-1} s^n - (t_1-s)^{q-1} s^n| \cdot \|R(t_2-s) - R(t_1-s)\|_{L_b(X)} \|f(s,x(s))\| ds +$$

$$+ M \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} s^n \|f(s,x(s))\| ds.$$

If $t_1 = 0$, then the right-hand side can be made small when $t_2$ is small independently of $x \in E$.

If $t_1 > 0$, then the right-hand side can be estimated as

$$\frac{1}{\Gamma(q)} \int_0^{t_1} |(t_2-s)^{q-1} s^n - (t_1-s)^{q-1} s^n| \cdot \|R(t_2-s) - R(t_1-s)\|_{L_b(X)} \|f(s,x(s))\| ds +$$

$$+ M \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} s^n \|f(s,x(s))\| ds \le$$

$$\le \frac{1}{\Gamma(q)} \int_0^{t_1-\eta} |(t_2-s)^{q-1} s^n - (t_1-s)^{q-1} s^n| \cdot \|R(t_2-s) - R(t_1-s)\|_{L_b(X)} \|f(s,x(s))\| ds +$$

$$+ \int_{t_1-\eta}^{t_1} |(t_2-s)^{q-1} s^n - (t_1-s)^{q-1} s^n| \cdot \|R(t_2-s) - R(t_1-s)\|_{L_b(X)} \|f(s,x(s))\| ds +$$

$$+ M \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} s^n \|f(s,x(s))\| ds \le$$

$$\le \frac{2T^{n+q}}{(n+1)\Gamma(q)} F \|R(t_2-s) - R(t_1-s)\|_{L_b(X)} + \frac{4\eta M T^{n+q-1}}{(n+1)\Gamma(q)} F + (t_2-t_1) \frac{2T^{n+q-1}}{(n+1)\Gamma(q)} F,$$

where $0 < \eta < t_1$ is a small number. Note that in Theorem 2.8, the functions in $W$ are equicontinuous. Hence, the functions in $\Gamma(E)$ are equicontinuous. Therefore, $\Gamma$ is a compact operator by the Arzela-Ascoli theorem again.

At last, we will use Schaefer's fixed point theorem to derive fixed points for the operator $\Gamma$. To do so, let's define

$$\widehat{K} = \{x \in C(J, X) \mid \lambda\Gamma x = x \text{ for some } \lambda \in [0, 1]\}.$$

For $x \in \widehat{K}$, we have

$$\begin{aligned}
&\|x\|_C \le \|\Gamma x\|_C \le \\
&\quad \le M(\|x_0\| + \|g(x)\|) + M\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^n(a\|x(s)\| + a_1)ds \le \\
&\quad \le M(\|x_0\| + b\|x\|_C + b_1) + M\frac{T^{n+q}}{(n+1)\Gamma(q)}(a\|x\|_C + a_1) = \\
&\quad = M\left[b + a\frac{T^{n+q}}{(n+1)\Gamma(q)}\right]\|x\|_C + M(\|x_0\| + b_1 + a_1).
\end{aligned}$$

Now using (3.5), we obtain

$$\|x\|_C \le \frac{M(\|x_0\| + b_1 + a_1)}{1 - M\left[b + a\frac{T^{n+q}}{(n+1)\Gamma(q)}\right]},$$

which implies that $\widehat{K}$ is bounded. Therefore, Schafer's fixed point theorem yield that $\Gamma$ has a fixed point, which gives rise to a mild solution. This completes the proof. □

**Remark 3.4.** If $n = 0$, $q = 1$, we obtain the nonlocal problems discussed in Theorem 3.1 in Liang *et al.* [22]. Our results are new and extend some recent results due to Liang *et al.* [22].

To illustrate our results, we present the following example.

Let $\Omega$ be a bounded domain in $R^n$ with smooth boundary, and $X = L^2(\Omega)$. Consider the following nonlinear integrodifferential equation in $X$

$$\begin{aligned}
\frac{\partial^q}{\partial t^q}x(t, y) = \triangle x(t, y) + \int_0^t b(t-s, \xi)\triangle x(s, \xi)ds + \\
+ \sigma t^2 x(t, \xi)|\sin x(t, \xi)|^{\frac{1}{2}}, \quad q \in (0, 1], \quad t \in J, \quad \xi \in \Omega
\end{aligned} \tag{3.6}$$

with nonlocal condition

$$x(0) = x_0(\xi) + \int_\Omega \int_0^T h(t, \xi)\log(1 + |x(t, s)|^{\frac{1}{2}})dtds, \ \xi \in \Omega, \tag{3.7}$$

where $b(t,\xi) \in C^1(J \times \overline{\Omega})$, $\sigma \in R$, and $h(t,y) \in C(J \times \bar{\Omega})$.
Set $A = \triangle$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$,

$$(B(t)x)(\xi) = b(t,\xi)\triangle x(s,\xi), t \in J, x \in D(A), \xi \in \Omega,$$

$$f(t,x)(\xi) = \sigma x(\xi)|\sin x(\xi)|^{\frac{1}{2}}, t \in J, x \in X, \xi \in \Omega,$$

$$g(x)(\xi) = x_0(\xi) + \int_\Omega \int_0^T h(t,\xi)\log(1+|x(t,s)|^{\frac{1}{2}})dtds, \xi \in \overline{\Omega}, x \in C(J,X).$$

Then, $A$ generates a compact $C_0$-compact semigroup in $X$, and

$$f\,(t,x)) \le |\sigma|\|x\|, t \in J, x \in X,$$

$$\|g(x)\| \le T(mes(\Omega)) \max_{t\in J,\xi\in\bar{\Omega}} |h(t,\xi)|[\|x\| + (mes(\Omega))^{\frac{1}{2}}], x \in C(J,X),$$

and $g$ is compact (see [22]). Furthermore, $\sigma$, $T$ and $h(t,\xi)$ can be chosen such that

$$M\left[T(mes(\Omega)) \max_{t\in J,\xi\in\bar{\Omega}} |h(t,\xi)| + |\sigma|\frac{T^{2+q}}{3\Gamma(q)}\right] < 1$$

is also satisfied. Obviously, it satisfies all the assumptions given in Theorem 3.3, the problem has at least one mild solution in $C(J, L^2(\Omega))$.

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