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EXISTENCE AND STABILIZABILITY OF STEADY-STATE FOR SEMILINEAR PULSE-WIDTH SAMPLER CONTROLLED SYSTEM

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Abstract. In this paper, we study the steady-state of a semilinear pulse-width sampler controlled system on infinite dimensional spaces. Firstly, by virtue of Schauder's fixed point theorem, the existence of periodic solutions is given. Secondly, utilizing a generalized Gronwall inequality given by us and the Banach fixed point theorem, the existence and stabilizability of a steady-state for the semilinear control system with pulse-width sampler is also obtained. At last, an example is given for demonstration.

Keywords: pulse-width sampler system, compact semigroup, steady-state, existence, stabilizability.

Mathematics Subject Classification: 34G10, 34G20, 93C25.

1. INTRODUCTION

In the design of distributed parameter control systems, one of the important problems is to choose the controller and actuator. As the dimension of an industrial controller in actual applications is finite, it restricts us to consider the distributed parameter system with a finite dimensional output. In industrial process control systems on-off actuators have been preferred by engineer's because of their cheap price and the high reliability.

The interest in the pulse-width sampler control systems begun as early as the 1960s. It was motivated by applications to engineering problems and neural nets modeling. In modern times, the development of neurocomputers promises a rebirth of interest in this field. The theory of pulse-width sampler control systems is treated as a very important branch of engineering and mathematics. Nevertheless, it can supply a technical-minded mathematician with a number of new and interesting problems of a mathematical nature. There are some results such as steady-state control, stability analysis, robust control of pulse-width sampler systems [1–7], integral control by variable sampling based on steady-state data and adaptive sampled-data integral control [8–11].

In this paper, we will be concerned with control systems governed by a class of semilinear equations

$$\begin{cases} \dot{x}(t) = Ax(t) + f\,(t, x(t)) + Cu(t), \\ z(t) = K_1 x(t), \end{cases} \tag{1.1}$$

where the state variable $x(t)$ takes a value in a reflexive Banach space X, A is the infinitesimal generator of a compact semigroup $\{T\,(t)\,, t \geq 0\}$ on X. $f : [0, \infty) \times X \to X$ is continuous. Control variable $u(t)$ is a q dimensional vector, $u(t) \in \mathbb{R}^q$, C: $\mathbb{R}^q \to X$ is a bounded linear operator. K_1: $X \to \mathbb{R}^p$ is a bounded linear operator, $z(t)$ is a p dimensional output of the system (1.1).

Suppose that control signal $u(t)$ is the output of the q dimensional pulse-width sampler controller. $v(t)$ is the input of the q dimensional pulse-width sampler controller, which is the output of some dynamical controller

$$\dot{v}(t) = Jv(t) + K_2 z(t), \tag{1.2}$$

where J is a $q \times q$ matrix, K_2 is a $q \times p$ matrix. J is determined by the dynamic characteristics of the controller, K_2 is called the feedback matrix which will be chosen in the latter. The output signal $u(t) = (u_1(t), u_2(t), \ldots, u_q(t))^T$ and the input signal $v(t) = (v_1(t), v_2(t), \ldots, v_q(t))^T$ of the pulse-width sampler satisfy the following dynamic relation:

$$u_i(t) = \begin{cases} \text{sign } \alpha_{n_i}, & nT_0 \leq t < (n + |\alpha_{n_i}|)T_0, \quad i = 1, 2, \ldots, q; \\ 0, & (n + |\alpha_{n_i}|)T_0 \leq t < (n+1)T_0, \quad n = 0, 1, \ldots \end{cases} \tag{1.3}$$

and

$$\alpha_{n_i} = \begin{cases} v_i(nT_0), & |v_i(nT_0)| \leq 1, \quad i = 1, 2, \ldots, q; \\ \text{sign } v_i(nT_0), & |v_i(nT_0)| \geq 1, \quad n = 0, 1, \ldots, \end{cases} \tag{1.4}$$

where $T_0 > 0$ is the sampling period of the pulse-width sampler.

We end this introduction by giving some definitions.

Definition 1.1. The closed-loop system (1.1)–(1.4) is called a pulse-width sampling semilinear control system.

Definition 1.2. In the closed-loop system (1.1)–(1.4), the q dimensional vector $\alpha_n = (\alpha_{n_1}, \alpha_{n_2}, \ldots, \alpha_{n_q})^T$ defined by (1.4) is called the duration ratio of the pulse-width sampler in the n-th sampling period, $n = 0, 1, \ldots$.

We defined a closed cube Ω in $\mathbb{R}^q$ as

$$\Omega = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q)^T \in \mathbb{R}^q \mid |\alpha_i| \leq 1, i = 1, 2, \ldots, q\},$$

then we have $\alpha_n \in \Omega$, for $n = 0, 1, \ldots$.

Definition 1.3. In the closed-loop system (1.1)–(1.4), if there exists a q dimensional vector

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q)^T \in \Omega,$$

and a corresponding periodicity rectangular-wave control signal $u(t) = u(t, \alpha)$ defined by

$$u_i(t) = u_i(t, \alpha) = \begin{cases} \text{sign } \alpha_i, & nT_0 \le t < (n + |\alpha_i|)T_0, \quad i = 1, 2, \ldots, q; \\ 0, & (n + |\alpha_i|)T_0 \le t < (n+1)T_0, \quad n = 0, 1, \ldots \end{cases} \tag{1.5}$$

such that the closed-loop system (1.1)–(1.4) has a corresponding T_0-periodic trajectory $x(\cdot) = x(\cdot, \alpha)$: $x(t+T_0, \alpha) = x(t, \alpha)$, $t \ge 0$, then the control signal (1.5) is called a steady-state control with respect to the disturbance f. The T_0-periodic trajectory $x(\cdot)$ is called a steady-state corresponding to the steady-state control $u(\cdot)$ and the constant vector $\alpha \in \Omega$ of the defining steady-state control (1.5) is called a steady-state duration ratio.

Definition 1.4. In the closed-loop system (1.1)–(1.4), if there exists some $\alpha \in \Omega$ such that

$$\lim_{n \to \infty} \alpha_n = \alpha, \quad \text{where} \quad \alpha_n = (\alpha_{n_1}, \alpha_{n_2}, \ldots, \alpha_{n_q})^T, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q)^T,$$

then system (1.1)–(1.4) is called **steady-state stable** with respect to the disturbance f.

System (1.1)–(1.4) is called **steady-state stabilizability** if we can choose a suitable $T_0 > 0$ and K_2 such that system (1.1)–(1.4) is steady-state stable with respect to the disturbance f.

2. PRELIMINARIES

Let $\pounds_b(\mathbb{R}^q, X)$ be the space of bounded linear operators from $\mathbb{R}^q$ to X, $\pounds_b(X, \mathbb{R}^p)$ be the space of bounded linear operators from X to $\mathbb{R}^p$. $\pounds_b(\mathbb{R}^q, X)$ and $\pounds_b(X, \mathbb{R}^p)$ are Banach spaces endowed with the usual operator norms, respectively. $C([0, T_0]; X)$ be the Banach space of continuous functions from $[0, T_0]$ to X with the usual operator norm.

In order to study system (1.1)–(1.4), we introduce the following assumptions:

- [H1] A is the infinitesimal generator of a compact semigroup $\{T(t), t \ge 0\}$ on X with domain $D(A)$.
- [H2] $f : [0, \infty) \times X \to X$ is continuous.
- [H3] $f(t, x)$ is T_0-periodic in t, i.e., $f(t + T_0, x) = f(t, x), t \ge 0$.
- [H4] Control signal $u(t)$ is a rectangular wave signal $u(t, \alpha)$ with a period T_0 defined by (1.5) for given $\alpha \in \Omega$.

By virtue of the compactness of $T(T_0)$ and Fredholm alterative theorem, one can complete the following results immediately.

Lemma 2.1. *Let assumption* [H1] *hold and the homogeneous periodic boundary problem*

$$\begin{cases} \dot{x}(t) = Ax(t), \\ x(0) = x(T_0). \end{cases} \tag{2.1}$$

has nontrivial T_0-periodic solution. Then operator $[I - T(T_0)]^{-1}$ exists and is bounded.

We consider the integral equation on X given by

$$x(t,\alpha) = [I - T(T_0)]^{-1} \int_t^{t+T_0} T(t + T_0 - \theta)\big[f(\theta, x(\theta)) + Cu(\theta,\alpha)\big]d\theta. \tag{2.2}$$

Lemma 2.2. *$x(t,\alpha)$ is the T_0-periodic solution of the following periodic boundary problem*

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)) + Cu(t), \\ x(0) = x(T_0), \end{cases} \tag{2.3}$$

if and only if $x(t,\alpha)$ is the continuous T_0-periodic solution of the integral equation (2.2).

Proof. Suppose that $x(t,\alpha)$ is the T_0-periodic solution of (2.3), let $x_0 = x(0)$, then

$$x(t,\alpha) = T(t)x_0 + \int_0^t T(t-\theta)\big[f(\theta, x(\theta)) + Cu(\theta,\alpha)\big]d\theta. \tag{2.4}$$

Further,

$$x(t + T_0,\alpha) = T(t + T_0)x_0 + \int_0^{t+T_0} T(t + T_0 - \theta)\big[f(\theta, x(\theta)) + Cu(\theta,\alpha)\big]d\theta, \tag{2.5}$$

that is

$$x(t,\alpha) = T(t + T_0)x_0 + \int_0^{t+T_0} T(t + T_0 - \theta)\big[f(\theta, x(\theta)) + Cu(\theta,\alpha)\big]d\theta. \tag{2.6}$$

It comes from (2.6)−(2.4)×$T(T_0)$ that

$$[I - T(T_0)]x(t,\alpha) = \int_t^{t+T_0} T(t + T_0 - \theta)\big[f(\theta, x(\theta)) + Cu(\theta,\alpha)\big]d\theta.$$

By Lemma 2.1, we have

$$x(t,\alpha) = [I - T(T_0)]^{-1} \int_t^{t+T_0} T(t + T_0 - \theta)\big[f(\theta, x(\theta)) + Cu(\theta,\alpha)\big]d\theta.$$

On the other hand, let $x(t,\alpha)$ be the continuous T_0-periodic solution of (2.2), then

$$\begin{aligned}\frac{dx}{dt} &= [I-T(T_0)]^{-1}\int\limits_t^{t+T_0} AT(t+T_0-\theta)\big[f\left(\theta,x(\theta)\right)+Cu(\theta,\alpha)\big]d\theta+\\ &\quad+[I-T(T_0)]^{-1}\big[(f\left(t+T_0,x(t+T_0)\right)-T(T_0)f\left(t,x(t)\right))+\\ &\quad+(Cu(t+T_0,\alpha)-T(T_0)Cu(t,\alpha))\big]=\\ &= Ax(t)+f\left(t,x(t)\right).\end{aligned}$$

Thus, $x(t,\alpha)$ satisfies (2.3). This completes the proof. □

Set

$$E_{T_0}=\Big\{x\mid x\in C([0,T_0],X)\text{ and }x(t)=x(t+T_0)\\ \text{with the norm }\|\cdot\|_{E_{T_0}}=\sup_{t\in[0,T_0]}\|\cdot\|\le M\Big\},$$

where $M=L_0M_0\int_0^{T_0}\left(\|f\left(\theta,x(\theta)\right)\|+q\|C\|_{\pounds_b(\mathbb{R}^q,X)}\right)d\theta$, $M_0=\sup_{t\in[0,T_0]}\|T(t)\|$ and $L_0=\|[I-T(T_0)]^{-1}\|$. It is obvious that E_{T_0} is a Banach space. Meanwhile, it is also a closed, bounded and convex set.

Define operator P on E_{T_0} given by

$$P(x)(t,\alpha)=[I-T(T_0)]^{-1}\int\limits_t^{t+T_0} T(t+T_0-\theta)\big[f\left(\theta,x(\theta)\right)+Cu(\theta,\alpha)\big]d\theta,$$

where $x(t,\alpha)\in E_{T_0}$.

It is well known that the integral equation (2.2) has a periodic solution if and only if P has a fixed point on E_{T_0}. Thus, we only need to discuss the existence of the fixed point of P.

Lemma 2.3. *Operator P is a complete continuity on E_{T_0}.*

Proof. It is obvious that P maps E_{T_0} into itself. Since $f:[0,\infty)\times X\to X$ is continuous, we see that P is a continuous operator on E_{T_0}. Next, we verify that P is compact on E_{T_0}. Set $B=\{x\mid x\in E_{T_0},\|x\|_{E_{T_0}}\le\rho\}$. Let $t\in[0,T_0]$, $t+h\in[0,T_0]$, $h>0$, $x\in B$.

In fact,

$$\|P(x)(t+h,\alpha)-P(x)(t,\alpha)\| =$$
$$= \left\| [I-T(T_0)]^{-1}\Big[\int\limits_{t+h}^{t+h+T_0} T(t+h+T_0-\theta)\big[f(\theta,x(\theta))+Cu(\theta,\alpha)\big]d\theta - \right.$$
$$\left. - \int\limits_{t}^{t+T_0} T(t+T_0-\theta)\big[f(\theta,x(\theta))+Cu(\theta,\alpha)\big]d\theta\Big] \right\| \leq$$
$$\leq \|[I-T(T_0)]^{-1}\| \Big\{ \int\limits_{t+T_0}^{t+h+T_0} \|T(t+h+T_0-\theta)\big[f(\theta,x(\theta))+Cu(\theta,\alpha)\big]\|d\theta +$$
$$+ \int\limits_{t}^{t+h} \|T(t+h+T_0-\theta)\big[f(\theta,x(\theta))+Cu(\theta,\alpha)\big]\|d\theta +$$
$$+ \int\limits_{t}^{t+T_0} \|(T(h)-I)T(t+T_0-\theta)\big[f(\theta,x(\theta))+Cu(\theta,\alpha)\big]\|d\theta \Big\} \leq$$
$$\leq L_0 \Big\{ \int\limits_{t+T_0}^{t+h+T_0} M_0\big[\|f(\theta,x(\theta))\| + q\|C\|_{\pounds_b(\mathbb{R}^q,X)}\big]d\theta +$$
$$+ \int\limits_{t}^{t+h} M_0\big[\|f(\theta,x(\theta))\| + q\|C\|_{\pounds_b(\mathbb{R}^q,X)}\big]d\theta +$$
$$+ \int\limits_{t}^{t+T_0} \|T(h)-I\| M_0\big[\|f(\theta,x(\theta))\| + q\|C\|_{\pounds_b(\mathbb{R}^q,X)}\big]d\theta \Big\}$$

and $\lim_{h\to 0}\|T(h)-I\| = 0$, then P is equicontinuous in $[0,T_0]$. □

Moreover, it is obvious that PB is a bounded set. Denote $\Pi = PB$ and $\Pi(t) = \{(Px)(t) \mid x \in B\}$ for $t \in [0,T_0]$. Clearly, $\Pi(0) = \{x(0)\}$ is compact, and hence, it is only necessary to consider $t > 0$. For $0 < \varepsilon < t \leq T_0$, define

$$\Pi_\varepsilon(t) = (P_\varepsilon B)(t) =$$
$$= \{T(\varepsilon)(Px)(t-\varepsilon) \mid x \in B\} =$$
$$= \left\{ [I-T(T_0)]^{-1} \int\limits_{t-\varepsilon}^{t-\varepsilon+T_0} T(t+T_0-\theta)\big[f(\theta,x(\theta))+Cu(\theta,\alpha)\big]d\theta \mid x \in B \right\}.$$

By our hypothesis, $T(\cdot)$ is compact. It follows from the above expression that $\Pi_\varepsilon(t)$ is relatively compact for $t \in (\varepsilon, T_0]$.

Furthermore,

$$\begin{aligned}
&\sup\{\|(Px)(t)-(P_\varepsilon x)(t)\| \mid x\in B\} \le \\
&\le \sup\left\{\int_{t-\varepsilon}^{t} M_0\big[\|f(\theta,x(\theta))\| + q\|C\|\big]d\theta\right\} + \\
&\quad + \sup\left\{\int_{t-\varepsilon+T_0}^{t+T_0} M_0\big[\|f(\theta,x(\theta))\| + q\|C\|\big]d\theta\right\} \le \\
&\le M_f\varepsilon. \quad (M_f \text{ do not depend on } x)
\end{aligned}$$

It is shown that the set $\Pi(t)$ can be approximated to an arbitrary degree of accuracy by a relatively compact for $t\in[0,T_0]$. Hence, $\Pi(t)$ itself is relatively compact for $t\in[0,T_0]$.

By the Ascoli-Arzela theorem, $P(x)$ is compact on E_{T_0}. As a result, $P(x)$ is a complete continuity operator on E_{T_0}.

We end this section by collecting a generalized Gronwall inequality which plays an essential role in the study of nonlinear problems on infinite dimensional spaces.

Lemma 2.4 ([13, Lemma 2])**.** *Let $a\ge 0$, $b\ge 0$, $c\ge 0$, $0\le\lambda_1\le 1$, $0\le\lambda_2<1$. If $x\in C([0,T_0];X)$ satisfies*

$$\|x(t)\| \le a + b\int_0^t \|x(\theta)\|^{\lambda_1}d\theta + c\int_0^{T_0}\|x(\theta)\|^{\lambda_2}d\theta, \qquad \text{for all} \quad t\in[0,T_0],$$

then there exists a constant $M^*=M^*(a,b,c,\lambda_2,T_0)>0$ such that

$$\|x(t)\|\le M^* \quad \text{for all} \quad t\in[0,T_0].$$

3. EXISTENCE AND STABILIZABILITY OF STEADY-STATE

Using Lemma 2.2, Lemma 2.3 and the Schauder fixed point theorem, P has a fixed point on E_{T_0}. Then we have the following result.

Lemma 3.1. *Assumptions* [H1]–[H4] *hold. For every $u(t,\alpha)$, system* (1.1) *has a T_0-periodic solution given by*

$$x(t,\alpha) = [I-T(T_0)]^{-1}\int_t^{t+T_0} T(t+T_0-\theta)\big[f(\theta,x(\theta)) + Cu(\theta,\alpha)\big]d\theta,$$

which is equivalent to

$$x(t,\alpha) = T(t)x_0 + \int_0^t T(t-\theta)\big[f(\theta,x(\theta)) + Cu(\theta,\alpha)\big]d\theta, \ \text{with } x_0 = x(T_0).$$

By Lemma 3.1, we can obtain the following result.

Theorem 3.2. *Under the assumptions of Lemma 3.1, if the sampler periodic T_0 has the following properties:*

$$i\omega_n \in \rho(J),\ \omega_n = \frac{2n\pi}{T_0},\ n = 0, \pm 1, \pm 2, \ldots, \tag{3.1}$$

where $\rho(J)$ is the resolvent set of the matrix J, i satisfies $i^2 = -1$, then the following open-loop control system

$$\begin{cases} \dot{x}(t,\alpha) = Ax(t,\alpha) + f(t, x(t)) + Cu(t,\alpha), \\ z(t) = K_1 x(t), \\ \dot{v}(t,\alpha) = Jv(t,\alpha) + K_2 z(t,\alpha), \end{cases} \tag{3.2}$$

has a unique T_0-periodic solution $v(t,\alpha)$ given by

$$v(t,\alpha) = e^{Jt}\left[(I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2 z(s,\alpha) ds\right] + \int_0^t e^{J(t-s)} K_2 z(s,\alpha) ds.$$

Proof. Using (3.1), we know that $e^{i\omega_n T_0} = e^{i2n\pi} = 1$, that is $1 \in \rho(e^{JT_0})$. Thus $(I - e^{JT_0})^{-1}$ exists and is bounded. It is not difficult to see that

$$v(t,\alpha) = e^{Jt} v_0 + \int_0^t e^{J(t-s)} K_2 z(s,\alpha) ds, \tag{3.3}$$

where $v_0 = v(0,\alpha)$.

Consider

$$y = (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2 z(s,\alpha) ds$$

which is the unique solution of the following equation

$$y = e^{Jt} y + \int_0^t e^{J(t-s)} K_2 z(s,\alpha) ds.$$

Let

$$v_0 = y = (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2 z(s,\alpha) ds,$$

it comes from Lemma 3.1 that

$$z(t + T_0, \alpha) = z(t,\alpha), t \geq 0.$$

It is easy to verify that

$$v(t,\alpha) = e^{Jt}\left[(I - e^{JT_0})^{-1}\int\limits_0^{T_0} e^{J(T_0-s)}K_2 z(s,\alpha)ds\right] + \int\limits_0^t e^{J(t-s)}K_2 z(s,\alpha)ds$$

is just the T_0-periodic solution $v(t,\alpha)$ of the open-loop control system (3.2). □

In order to discuss existence and stabilizability of the steady-state for system (1.1)–(1.4), we define a map G: $\Omega \in \mathbb{R}^q \to \mathbb{R}^q$ given by

$$G(\alpha) = (I - e^{JT_0})^{-1}\int\limits_0^{T_0} e^{J(T_0-s)}K_2 K_1 x(s,\alpha)ds, \alpha \in \Omega,$$

where $x(\cdot,\alpha)$ is the T_0-periodic solution of system (1.1) corresponding to $\alpha \in \Omega$.

In the sequel, we make [H2] a little stronger as following.

[H2′] For any x_1, x_2, $y_1, y_2 \in X$, there exists a positive constant $L_f > 0$ and $0 \le \lambda < 1$ such that

$$\|f(t,x_1) - f(t,x_2)\| \le L_f \|x_1 - x_2\|^\lambda.$$

Lemma 3.3. *Under the assumptions of Theorem 3.2* ([H2] *replaced by* [H2′])*, there exists a constant* $M_1 > 0$ *such that*

$$\|G(\alpha) - G(\bar\alpha)\| \le M_1\|K_2\|\|\alpha - \bar\alpha\|, \quad \alpha, \bar\alpha \in \Omega.$$

Proof. Let $x_1(t,\alpha)$ and $x_2(t,\bar\alpha)$ be the T_0-periodic solution of system (1.1) corresponding to α and $\bar\alpha \in \Omega$ respectively, then

$$\begin{aligned} x_1(0) - x_2(0) = x_1(T_0) - x_2(T_0) &= T(T_0)(x_1(0) - x_2(0)) + \\ &+ \int\limits_0^{T_0} T(T_0-\theta)(f(\theta,x_1(\theta)) - f(\theta,x_2(\theta))d\theta + \\ &+ \int\limits_0^{T_0} T(T_0-\theta)C(u(\theta,\alpha) - u(\theta,\bar\alpha))d\theta. \end{aligned}$$

Using assumption [H2′], we obtain

$$\begin{aligned}
&\|x_1(0)-x_2(0)\| \le \\
&\le \|[I-T(T_0)]^{-1}\|M_0L_f\int_0^{T_0}\|x_1(\theta)-x_2(\theta)\|^{\lambda}d\theta+ \\
&\quad +\|[I-T(T_0)]^{-1}\|M_0\|C\|_{\pounds_b(\mathbb{R}^q,X)}\int_0^{T_0}\|u(\theta,\alpha)-u(\theta,\bar{\alpha})\|_{\mathbb{R}^q}d\theta \le \\
&\le L_0L_fM_0\int_0^{T_0}\|x_1(\theta)-x_2(\theta)\|^{\lambda}d\theta+ \\
&\quad +L_0M_0C\|_{\pounds_b(\mathbb{R}^q,X)}\int_0^{T_0}\|u(\theta,\alpha)-u(\theta,\bar{\alpha})\|_{\mathbb{R}^q}d\theta.
\end{aligned}$$

For $0\le t\le T_0$, we obtain

$$\begin{aligned}
&\|x_1(t,\alpha)-x_2(t,\bar{\alpha})\| \le \\
&\le \|x_1(0)-x_2(0)\|+L_fM_0\int_0^{t}\|x_1(\theta)-x_2(\theta)\|^{\lambda}d\theta+ \\
&\quad +M_0\|C\|_{\pounds_b(\mathbb{R}^q,X)}\int_0^{t}\|u(\theta,\alpha)-u(\theta,\bar{\alpha})\|_{\mathbb{R}^q}d\theta \le \\
&\le L_fM_0\|C\|_{\pounds_b(\mathbb{R}^q,X)}\big(L_0+1\big)\int_0^{T_0}\|u(\theta,\alpha)-u(\theta,\bar{\alpha})\|_{\mathbb{R}^q}d\theta+ \\
&\quad +L_fM_0\int_0^{t}\|x_1(\theta)-x_2(\theta)\|^{\lambda}d\theta+L_0L_fM_0\int_0^{T_0}\|x_1(\theta)-x_2(\theta)\|^{\lambda}d\theta.
\end{aligned}$$

By Lemma 2.4, there exists a $M^*>0$ such that

$$\|x_1(t,\alpha)-x_2(t,\bar{\alpha})\|\le M^*.$$

Furthermore, we can choose a sufficient large number $M^{**}>0$ such that

$$\begin{aligned}
\int_0^{T_0}\|x_1(t,\alpha)-x_2(t,\bar{\alpha})\|dt &\le M^{**}\int_0^{T_0}\|u(\theta,\alpha)-u(\theta,\bar{\alpha})\|_{\mathbb{R}^q}d\theta \le \\
&\le M^{**}\int_0^{T_0}\sum_{l=1}^{q}|u_l(\theta,\alpha_l)-u_l(\theta,\bar{\alpha}_l)|d\theta.
\end{aligned}$$

For $\alpha_l\bar{\alpha}_l > 0$, without loss of any generality, we suppose that $0 < \alpha_l < \bar{\alpha}_l$, then we have

$$\int_0^{T_0} |u_l(\theta,\alpha_l) - u_l(\theta,\bar{\alpha}_l)|d\theta \le \int_{\alpha_l T_0}^{\bar{\alpha}_l T_0} |u_l(\theta,\alpha_l) - u_l(\theta,\bar{\alpha}_l)|d\theta \le T_0|\alpha_l - \bar{\alpha}_l|.$$

For $\alpha_l\bar{\alpha}_l < 0$, for example, $\alpha_l < 0 < \bar{\alpha}_l$, $|\bar{\alpha}_l| > \alpha_l$, we have

$$\int_0^{T_0} |u_l(\theta,\alpha_l) - u_l(\theta,\bar{\alpha}_l)|d\theta \le \int_{\alpha_l T_0}^{|\bar{\alpha}_l| T_0} |u_l(\theta,\alpha_l) - u_l(\theta,\bar{\alpha}_l)|d\theta \le 2T_0|\alpha_l - \bar{\alpha}_l|.$$

By elemental computation,

$$\begin{aligned}
&\|G(\alpha) - G(\bar{\alpha})\| \le \\
&\le \|(I - e^{JT_0})^{-1}\|\|e^{JT_0}\|\|K_2\|\|K_1\|_{\mathcal{L}_b(X,\mathbb{R}^p)} \int_0^{T_0} \|x_1(s,\alpha) - x_2(s,\bar{\alpha})\|ds \le \\
&\le \|(I - e^{JT_0})^{-1}\|\|e^{JT_0}\|\|K_2\|\|K_1\|_{\mathcal{L}_b(X,\mathbb{R}^p)} M^{**} \int_0^{T_0} \|u(\theta,\alpha) - u(\theta,\bar{\alpha})\|_{\mathbb{R}^q} d\theta \le \\
&\le 2\|(I - e^{JT_0})^{-1}\|\|e^{JT_0}\|\|K_2\|\|K_1\|_{\mathcal{L}_b(X,\mathbb{R}^p)} M^{**} T_0\|\alpha - \bar{\alpha}\|.
\end{aligned}$$

Let

$$M_1 = 2\|(I - e^{JT_0})^{-1}\|\|e^{JT_0}\|\|K_1\|_{\mathcal{L}_b(X,\mathbb{R}^p)} M^{**} T_0 > 0,$$

then

$$\|G(\alpha) - G(\bar{\alpha})\| \le M_1\|K_2\|\|\alpha - \bar{\alpha}\|, \quad \alpha, \bar{\alpha} \in \Omega.$$

□

By Lemma 3.3, we can derive the following result.

Theorem 3.4. *Under the assumptions of Lemma 3.3, we can choose a suitable* $\|K_2\| > 0$ *such that system (1.1)–(1.4) has a unique steady-state for any given* $f \in X$ *and the fixed point of* G *is just the steady-state duration ratio.*

Proof. Let $x(t,\alpha)$ be the T_0-periodic solution of system (1.1) corresponding to $\alpha \in \Omega$, then

$$x(0) = x(T_0) = T(T_0)x(0) + \int_0^{T_0} T(T_0 - \theta)\big(f(\theta, x(\theta)) + Cu(\theta,\alpha)\big)d\theta,$$

that is,

$$x(0) = [I - T(T_0)]^{-1} \int_0^{T_0} T(T_0 - \theta)\big(f(\theta, x(\theta)) + Cu(\theta,\alpha)\big)d\theta.$$

It is obvious that

$$\|x(0)\| \leq L_0 M_0 \int_0^{T_0} (\|f(\theta, x(\theta))\| + q\|C\|_{\mathcal{L}_b(\mathbb{R}^q, X)}) d\theta \equiv M_2.$$

It comes from

$$G(\alpha) = (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2 K_1 T(t) x(0) ds +$$
$$+ (I - e^{JT_0})^{-1} \int_0^{T_0} e^{J(T_0-s)} K_2 K_1 \left(\int_0^t T(t-\theta)\big(f(\theta, x(\theta)) + Cu(\theta, \alpha)\big) d\theta \right) ds,$$

that

$$\|G(\alpha)\| \leq M_3 \|K_2\|,$$

where

$$M_3 = \|(I - e^{JT_0})^{-1}\| \|e^{JT_0}\| \|K_1\|_{\mathcal{L}_b(X, \mathbb{R}^p)} T_0 \left(M_0 M_2 T_0 + \frac{M_2}{L_0} \right).$$

It is not difficult to see that $G : \Omega \to \Omega$ is a contraction map when

$$0 < \|K_2\| < \frac{1}{\max(M_1, M_3)}.$$

By the Banach fixed point theorem, G has a unique fixed point $\alpha^* \in \Omega$. Obviously, the T_0-periodic solution of system (1.1) corresponding to α^* is just the unique steady-state. □

At last, an example is given to illustrate our theory. Consider the following system

$$\begin{cases} \frac{\partial}{\partial t} x(t, y) = \Delta x(t, y) + x^{\frac{1}{3}}(t, y) + \sin(t, y) + Cu(t), \quad y \in \Omega = (0, l), t \in (0, 2\pi], \\ x(t, y) \mid_{y \in \partial\Omega} = 0, \quad t > 0, \\ x(0, y) = x(2\pi, y), \\ z(t) = \int_\Omega x(t, y) dy, \end{cases} \tag{3.4}$$

and the output $v(t)$ satisfies (1.2).

Define $X = L^2(\Omega)$, $D(A) = H^2(\Omega) \bigcap H_0^1(\Omega)$, and $Ax = -\frac{\partial^2 x}{\partial y^2}$ for $x \in D(A)$. Define $x(\cdot)(y) = x(\cdot, y)$, $\sin(\cdot)(y) = \sin(\cdot, y)$, $f(\cdot, x(\cdot))(y) = x^{\frac{1}{3}}(\cdot, y) + \sin(\cdot, y)$ and $K_1 x(t)(y) = \int_\Omega x(t, y) dy$.

Thus system (3.4) can be rewritten as

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x) + Cu(t), \quad t \in (0, 2\pi], \\ x(0) = x(2\pi), \\ z(t) = K_1 x(t). \end{cases} \tag{3.5}$$

It satisfies all the assumptions given in Theorem 3.4, by choosing a suitable matrix K_2, system (3.4), (1.2)–(1.4) has a unique steady-state. Thus, system (3.4), (1.2)–(1.4) is steady-state stabilizability.

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