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ON STRONGLY MIDCONVEX FUNCTIONS

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Abstract. In this paper we collect some properties of strongly midconvex functions. First, counterparts of the classical theorems of Bernstein-Doetsch, Ostrowski and Sierpiński are presented. A version of Rodé support theorem for strongly midconvex functions and a Kuhn-type result on the relation between strongly midconvex functions and strongly t-convex functions are obtained. Finally, a connection between strong midconvexity and generalized convexity in the sense of Beckenbach is established.

Keywords: strongly convex functions, strongly midconvex functions, Bernstein-Doetsch-type theorem, Kuhn theorem, Rodé support theorem, Beckenbach convexity.

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1. INTRODUCTION

Let X be a normed space, D a convex subset of X and let $c > 0$. A function $f : D \to \mathbb{R}$ is called *strongly convex with modulus* c (see e.g. [4, 15]) if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) - c\lambda(1-\lambda)\|x-y\|^2 \tag{1.1}$$

for all $x, y \in D$ and $\lambda \in [0,1]$; f is said to be *strongly midconvex* (or *strongly Jensen convex*) *with modulus* c if (1.1) is assumed only for $\lambda = \frac{1}{2}$, that is

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2, \quad x, y \in D. \tag{1.2}$$

Recall also that the usual notions of convex and midconvex functions correspond to the case $c = 0$.

Strongly convex functions have been introduced by Polyak [14] and he used them for proving the convergence of a gradient type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance, [5, 9, 10, 13–15, 17]).

Condition (1.2) defining strongly midconvex functions appears in [15] and [17], but no properties are stated. (In [15, p. 268] there is a task: Show that (1.2) is equivalent to (1.1), but it is not true.) The aim of this note is to collect some results on strongly midconvex functions. Of course, condition (1.2) is much weaker than (1.1). In particular there exist discontinuous and non-measurable strongly midconvex functions defined on $\mathbb{R}$, whereas strongly convex functions defined on an open subset of $\mathbb{R}^n$ are continuous. On the other hand, condition (1.2) is much easier to verify than (1.1). Therefore, it can be interesting and important for possible applications that under weak regularity assumptions the classes of strongly midconvex and strongly convex functions coincide. As examples of such results we present, in Section 2, some versions of the classical theorems of Bernstein-Doetsch, Ostrowski and Sierpiński. In Section 3 we prove a Kuhn-type theorem stating that strongly t-convex functions are strongly midconvex. Section 4 contains a counterpart of the result of Rodé that characterizes midconvex functions via their supports. Jensen-type inequalities are obtained in Section 5. Finally, in Section 6, we discuss connections with the theory of generalized convex functions due to Beckenbach.

2. BERNSTEIN-DOETSCH-TYPE RESULTS

Obviously, every strongly convex function is strongly midconvex, but not conversely. For instance, if $a : \mathbb{R} \to \mathbb{R}$ is an additive discontinuous function and $f : \mathbb{R} \to \mathbb{R}$ is given as $f(x) := a(x) + x^2$, then f is strongly midconvex with modulus 1, but it is not strongly convex (with any modulus) because it is not continuous. In the class of continuous functions, strong midconvexity is equivalent to strong convexity in view of Corollary 2.2 below. In fact, strong convexity can be deduced from strong midconvexity under conditions formally much weaker than continuity. In this section we present a few results of such type. They are versions of the classical theorems of Bernstein-Doetsch, Ostrowski and Sierpiński (see [7], and [15]). We start with the following lemma.

Lemma 2.1. *Let D be a convex subset of a normed space and let $c > 0$. If $f : D \to \mathbb{R}$ is strongly midconvex with modulus c then*

$$f\left(\frac{k}{2^n}x+\left(1-\frac{k}{2^n}\right)y\right)\leq \frac{k}{2^n}f(x)+\left(1-\frac{k}{2^n}\right)f(y)-c\frac{k}{2^n}\left(1-\frac{k}{2^n}\right)\|x-y\|^2, \quad (2.1)$$

for all $x, y \in D$ and all $k, n \in \mathbb{N}$ such that $k < 2^n$.

Proof. The proof is by induction on n. For $n = 1$ (2.1) reduces to (1.2). Assuming (2.1) to hold for some $n \in \mathbb{N}$ and all $k < 2^n$, we will prove it for $n + 1$. Fix $x, y \in D$

and take $k < 2^{n+1}$. Without loss of generality we may assume that $k < 2^n$. Then, by (1.2) and the induction assumption, we get

$$f\left(\frac{k}{2^{n+1}}x+\left(1-\frac{k}{2^{n+1}}\right)y\right)= f\left(\frac{1}{2}\left(\frac{k}{2^n}x+\left(1-\frac{k}{2^n}\right)y\right)+\frac{1}{2}y\right)\leq$$
$$\leq \frac{1}{2}f\left(\frac{k}{2^n}x+\left(1-\frac{k}{2^n}\right)y\right)+\frac{1}{2}f(y)-\frac{c}{4}\left\|\frac{k}{2^n}x+\left(1-\frac{k}{2^n}\right)y-y\right\|^2\leq$$
$$\leq \frac{1}{2}\left(\frac{k}{2^n}f(x)+\left(1-\frac{k}{2^n}\right)f(y)-c\frac{k}{2^n}\left(1-\frac{k}{2^n}\right)\|x-y\|^2\right)+\frac{1}{2}f(y)-\frac{c}{4}\frac{k^2}{2^{2n}}\|x-y\|^2\leq$$
$$\leq \frac{k}{2^{n+1}}f(x)+\left(1-\frac{k}{2^{n+1}}\right)f(y)-c\frac{k}{2^{n+1}}\left(1-\frac{k}{2^{n+1}}\right)\|x-y\|^2,$$

which finishes the proof. □

Since the set of dyadic numbers from $[0,1]$ is dense in $[0,1]$, we get the following result as an immediate consequence of Lemma 2.1.

Corollary 2.2. *Let D be a convex subset of a normed space and $c > 0$. Assume that $f : D \to \mathbb{R}$ is continuous. Then f is strongly convex with modulus c if and only if it is strongly midconvex with modulus c.*

Theorem 2.3. *Let D be an open convex subset of a normed space and let $c > 0$. If $f : D \to \mathbb{R}$ is strongly midconvex with modulus c and bounded from above on a set with nonempty interior, then it is continuous and strongly convex with modulus c.*

Proof. Being strongly midconvex, f is also midconvex. Since f is bounded from above on a set with nonempty interior, it is continuous in view of the Bernstein-Doetsch theorem. Consequently, by Corollary 2.2, it is strongly convex with modulus c. □

Theorem 2.4. *Let D be an open convex subset of $\mathbb{R}^n$ and let $c > 0$. If $f : D \to \mathbb{R}$ is strongly midconvex with modulus c and bounded from above on a set $A \subset D$ with positive Lebesgue measure $(\lambda(A) > 0)$, then it is continuous and strongly convex with modulus c.*

Proof. Suppose that $f \leq M$ on A. Since f is strongly midconvex

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2}-\frac{c}{4}\|x-y\|^2\leq M$$

for all $x, y \in A$. This means that f is bounded from above on the set $\frac{A+A}{2}$. Since $\lambda(A) > 0$, it follows, by the classical theorem of Steinhaus (cf. [7]), that $int\left(\frac{A+A}{2}\right) \neq \emptyset$. This proves the theorem in view of Theorem 2.3. □

Theorem 2.5. *Let D be an open convex subset of $\mathbb{R}^n$ and let $c > 0$. If $f : D \to \mathbb{R}$ is Lebesgue measurable and strongly midconvex with modulus c, then it is continuous and strongly convex with modulus c.*

Proof. For each $m \in \mathbb{N}$, define the set $A_m := \{x \in D : f(x) \leq m\}$. Since $D = \bigcup A_m$, there exists $m_0 \in \mathbb{N}$ such that $\lambda(A_{m_0}) > 0$. Hence, f is bounded from above on a set of positive Lebesgue measure, which in view of Theorem 2.4 completes the proof. □

3. KUHN-TYPE RESULTS

Let t be a fixed number in $(0,1)$ and let $c>0$. We say that a function $f : D \to \mathbb{R}$ is *strongly t-convex with modulus c* if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)-ct(1-t)\|x-y\|^2 \tag{3.1}$$

for all $x, y \in D$. It is known (Kuhn's Theorem, [8]) that t-convex functions (i.e., those that satisfy (3.1) with $c=0$) are midconvex. In this section we present a counterpart of that theorem for strongly t-convex functions. In the proof we apply the idea used in [3].

Theorem 3.1. *Let D be a convex subset of a normed space X, and let $t \in (0,1)$ be a fixed number. If $f : D \to \mathbb{R}$ is strongly t-convex with modulus c, then it is strongly midconvex with modulus c.*

Proof. Fix $x, y \in D$ and put $z := \frac{x+y}{2}$.

Consider the points $u := tx + (1-t)z$ and $v := tz + (1-t)y$. Then, one can easily check that

$$z = (1-t)u + tv.$$

Applying three times condition (3.1) in the definition of strong t-convexity, we obtain

$$\begin{aligned} f(z) &= (1-t)f(u) + tf(v) - ct(1-t)\|u-v\|^2 \le \\ &\le (1-t)\big[tf(x)+(1-t)f(z)-ct(1-t)\|x-z\|^2\big]+ \\ &\quad + t\big[tf(z)+(1-t)f(y)-ct(1-t)\|z-y\|^2\big]- \\ &\quad - t(1-t)\|u-v\|^2 = \\ &= t(1-t)[f(x)+f(y)] + [(1-t)^2+t^2]f(z)- \\ &\quad - ct(1-t)\big[(1-t)\|x-z\|^2 + t\|z-y\|^2 + \|u-v\|^2\big], \end{aligned}$$

and from this last inequality, after regrouping and simplifying, we get

$$2f(z) \le f(x) + f(y) - c[(1-t)\|x-z\|^2 + t\|z-y\|^2 + \|u-v\|^2\big]. \tag{3.2}$$

Now, since$\|x-z\| = \|z-y\| = \|u-v\| = \frac{\|x-y\|}{2}$, we have

$$(1-t)\|x-z\|^2 + t\|z-y\|^2 + \|u-v\|^2 = \frac{\|x-y\|^2}{2}.$$

Consequently, inequality (3.2) can be written as

$$f\left(\frac{x+y}{2}\right) = f(z) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2,$$

which shows that f is strongly midconvex with modulus c. This finishes the proof. $\square$

Remark 3.2. From Theorem 3.1 and Corollary 2.2 we infer that if a function $f : D \to \mathbb{R}$ is continuous and strongly t-convex with modulus c (with arbitrarily fixed $t \in (0,1)$), then it is strongly convex with modulus c. Similarly we can reformulate Theorems 2.3, 2.4 and 2.5 for strongly t-convex functions.

4. SUPPORT THEOREM

It is well known that convex functions are characterized by having affine support at every point of their domains (see e.g. [15]). An analogous result for midconvex functions, stating that they have Jensen support (that is, an additive function plus a constant) is due to Rodé [16], (cf. also [6, 11] for simpler proofs). In this section we present a counterpart of that result for strongly midconvex functions. In the proof we will use the following characterization of strongly midconvex functions in inner product spaces ([12]).

Lemma 4.1. *Let X be an inner product space, let D be a convex subset of X and let $c > 0$. A function $f : D \to \mathbb{R}$ is strongly midconvex with modulus c if and only if there exists a midconvex function $g : D \to \mathbb{R}$ such that*

$$f(x) = g(x) + c\|x\|^2$$

for all $x \in D$.

Proof. Assume first that $f : D \to \mathbb{R}$ is strongly midconvex with modulus c. Define

$$g(x) := f(x) - c\|x\|^2.$$

Then, applying the Jordan-von Neumann parallelogram law, we obtain

$$\begin{aligned} g\Big(\frac{x+y}{2}\Big) &= f\Big(\frac{x+y}{2}\Big) - c\Big\|\frac{x+y}{2}\Big\|^2 \leq \\ &\leq \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2 - \frac{c}{4}\|x+y\|^2 = \\ &= \frac{f(x)+f(y)}{2} - \frac{c}{4}(2\|x\|^2 + 2\|y\|^2) = \\ &= \frac{g(x)+g(y)}{2}\ , \end{aligned}$$

which proves that g is midconvex.

The converse implication follows analogously. □

Remark 4.2. It is shown in [12] that the assumption that $(X, \|\cdot\|)$ is an inner product space is not redundant in Lemma 4.1. Moreover, the condition that for every $f : D \to \mathbb{R}$, f is strongly midconvex if and only if $f - \|\cdot\|^2$ is midconvex, characterizes inner product spaces among all normed spaces.

Now, recall that a function $h : D \to \mathbb{R}$ is said to be a support for the function $f : D \to \mathbb{R}$ at a point $x_0 \in D$, if $h(x_0) = f(x_0)$ and $h(x) \leq f(x)$ for all $x \in D$.

Theorem 4.3. *Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space, let D be an open convex subset of X and let $c > 0$. A function $f : D \to \mathbb{R}$ is strongly midconvex with modulus c if and only if, at every point $x_0 \in D$, f has support of the form*

$$h(x) = c\|x - x_0\|^2 + a(x - x_0) + f(x_0),$$

where $a : X \to \mathbb{R}$ is an additive function (depending on x_0).

Proof. Suppose, in the first place, that $f : D \to \mathbb{R}$ is strongly midconvex with modulus c and fix $x_0 \in D$. Then, by Lemma 4.1, there exists a midconvex function $g : D \to \mathbb{R}$ such that

$$f(x) = g(x) + c\|x\|^2$$

for all $x \in D$. By Rodé's Theorem, the function g has support at x_0 of the form

$$h_1(x) = a_1(x - x_0) + g(x_0), \quad x \in D,$$

where $a_1 : X \to \mathbb{R}$ is an additive function. Hence, the function $h : D \to \mathbb{R}$ defined by

$$h(x) := c\|x\|^2 + a_1(x - x_0) + g(x_0)$$

supports f at x_0. Now, since $g(x_0) = f(x_0) - c\|x_0\|^2$, we can express h as

$$\begin{aligned} h(x) &= c(\|x\|^2 - \|x_0\|^2) + a_1(x - x_0) + f(x_0) = \\ &= c\|x - x_0\|^2 + 2c\langle x_0, x - x_0\rangle + a_1(x - x_0) + f(x_0) = \\ &= c\|x - x_0\|^2 + a(x - x_0) + f(x_0), \end{aligned}$$

where $a := a_1 + 2c\langle x_0, \cdot\rangle$ is also an additive function.

To prove the converse, fix arbitrary $x, y \in D$, put $z_0 := \frac{x+y}{2}$ and take a support of f at z_0 of the form

$$h(z) = c\|z - z_0\|^2 + a(z - z_0) + f(z_0), \quad z \in D.$$

Then

$$f(x) \geq c(\|x - z_0\|^2) + a(x - z_0) + f(z_0)$$

and

$$f(y) \geq c(\|y - z_0\|^2) + a(y - z_0) + f(z_0).$$

Hence

$$\frac{f(x) + f(y)}{2} \geq \frac{c}{2}(\|x - z_0\|^2 + \|y - z_0\|^2) + \frac{1}{2}(a(x - z_0) + a(y - z_0)) + f(z_0).$$

Finally, since

$$\frac{c}{2}(\|x - z_0\|^2 + \|y - z_0\|^2) = \frac{c}{4}(\|x - y\|^2,$$

and the additivity of a implies that

$$a(x - z_0) + a(y - z_0) = 0,$$

we conclude that

$$f\left(\frac{x + y}{2}\right) = f(z_0) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2,$$

which proves that f is strongly midconvex with modulus c. $\square$

5. JENSEN-TYPE INEQUALITIES

In this section we present two versions of the classical Jensen inequality for strongly midconvex functions and next we show that strongly midconvex functions are strongly q-convex for all $q \in \mathbb{Q} \cap (0,1)$. Note first that that if $s = (x_1 + x_2)/2$, then

$$\frac{1}{4}\|x_1 - x_2\|^2 = \frac{1}{2}\big(\|x_1 - s\|^2 + \|x_2 - s\|^2\big).$$

Therefore condition (1.2) in the definition of strongly midconvex function can be written in the form

$$f\left(\frac{x_1 + x_2}{2}\right) \le \frac{f(x_1) + f(x_2)}{2} - \frac{c}{2}\big(\|x_1 - s\|^2 + \|x_2 - s\|^2\big), \quad x, y \in D.$$

Extending this relation to convex combination of n points we get the following Jensen-type inequality.

Theorem 5.1. *Let D be an open and convex subset of an inner product space X. If $f\colon D \to \mathbb{R}$ is strongly midconvex with modulus c, then for all $n \in \mathbb{N}$, and $x_1, x_2, \dots, x_n \in D$:*

$$f\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{c}{n}\sum_{i=1}^{n} \|x_i - s\|^2,$$

where $s = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Proof. Fix $x_1, x_2, \dots, x_n \in D$ and put $s := \frac{1}{n}\sum_{i=1}^{n} x_i$. By Theorem 4.3 there exists an additive function a such that f has at s support of the form

$$h(x) = c\|x - s\|^2 + a(x - s) + f(s).$$

Thus, for each $i = 1, 2, \dots, n$,

$$f(x_i) \ge h(x_i) = c\|x_i - s\|^2 + a(x_i - s) + f(s).$$

Summing up these n inequalities, and using the fact that

$$\sum_{i=1}^{n} a(x_i - s) = a\left(\sum_{i=1}^{n} x_i - ns\right) = 0,$$

we have:

$$\sum_{i=1}^{n} f(x_i) \geq c\sum_{i=1}^{n} \|x_i - s\|^2 + \sum_{i=1}^{n} a(x_i - s) + nf(s)$$

$$\Longrightarrow \frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq \frac{c}{n}\sum_{i=1}^{n} \|x_i - s\|^2 + \frac{1}{n}\sum_{i=1}^{n} a(x_i - s) + f(s)$$

$$\Longrightarrow \frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq \frac{c}{n}\sum_{i=1}^{n} \|x_i - s\|^2 + \frac{1}{n}a\left(\sum_{i=1}^{n} x_i - ns\right) + f(s)$$

$$\Longrightarrow f\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) = f(s) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{c}{n}\sum_{i=1}^{n} \|x_i - s\|^2,$$

which was to be proved. □

Now we extend the above result to convex combinations with arbitrary rational coefficients.

Theorem 5.2. *Let D be an open and convex subset of an inner product space X. If $f : D \to \mathbb{R}$ is strongly midconvex with modulus c, then*

$$f\left(\sum_{i=1}^{n} q_i x_i\right) \leq \sum_{i=1}^{n} q_i f(x_i) - c\sum_{i=1}^{n} q_i \|x_i - s\|^2,$$

for all $x_1, \ldots, x_n \in D$, $q_1, \ldots, q_n \in \mathbb{Q} \cap (0,1)$ with $q_1 + \ldots + q_n = 1$ and $s = \sum_{i=1}^{n} q_i x_i$.

Proof. Fix $x_1, \ldots, x_n \in D$ and $q_1 = k_1/l_1, \ldots, q_n = k_n/l_n \in \mathbb{Q} \cap (0,1)$ with $q_1 + \ldots + q_n = 1$. Without loss of generality we may assume that $l_1 = \ldots = l_n =: l$. Then $k_1 + \ldots + k_n = l$. Put $y_{11} = \ldots = y_{1k_1} =: x_1$, $y_{21} = \ldots = y_{2k_2} =: x_2$, $\ldots$, $y_{n1} = \ldots = y_{nk_n} =: x_n$. Then

$$s = \sum_{i=1}^{n} q_i x_i = \frac{1}{l}\sum_{i=1}^{n}\sum_{j=1}^{k_i} y_{ij}.$$

Hence, using Theorem 5.1, we obtain

$$f\left(\sum_{i=1}^{n} q_i x_i\right) = f\left(\frac{1}{l}\sum_{i=1}^{n}\sum_{j=1}^{k_i} y_{ij}\right) \leq \frac{1}{l}\sum_{i=1}^{n}\sum_{j=1}^{k_i} f(y_{ij}) - \frac{c}{l}\sum_{i=1}^{n}\sum_{j=1}^{k_i} \|y_{ij} - s\|^2 =$$

$$= \sum_{i=1}^{n} q_i f(x_i) - c\sum_{i=1}^{n} q_i \|x_i - s\|^2,$$

which was to be proved. □

Under the same assumptions on X and D we obtain the following corollary.

Corollary 5.3. *$f : D \to \mathbb{R}$ is strongly midconvex with modulus c if and only if*

$$f(qx + (1-q)y) \;\le\; qf(x) + (1-q)f(y) \;-\; cq(1-q)\|x-y\|^2,$$

for all $q \in \mathbb{Q} \cap (0,1)$ and $x, y \in D$:

Proof. Fix $x, y \in D$, $q \in \mathbb{Q} \cap (0,1)$ and put $s := qx + (1-q)y$.
Then, by Theorem 5.2, we get

$$\begin{aligned} f(qx + (1-q)y) &\le qf(x) + (1-q)f(y) - c\big(q\|x-s\|^2 + (1-q)\|y-s\|^2\big) = \\ &= qf(x) + (1-q)f(y) - cq(1-q)\|x-y\|^2. \end{aligned}$$

The converse is, of course, immediate. □

6. CONNECTIONS WITH GENERALIZED CONVEXITY

The geometric idea of convexity of a function is the following:
A function f is convex iff for any two distinct points on the graph of f, the line segment joining these points lies above the corresponding part of the graph of f.

In [1] E. F. Beckenbach generalized this idea replacing the line segments by graphs of continuous functions belonging to a two-parameter family $\mathcal{F}$ of functions. In this section we will show that strong midconvexity is equivalent to generalized convexity with respect to a certain two-parameter family.

Let $\mathcal{F}$ be a family of continuous real functions defined on an interval $I \subset \mathbb{R}$. A class of functions $\mathcal{F} \subset \mathbb{R}^I$ is said to be a *two-parameter family* if for any two points $(x_1, y_1),\ (x_2, y_2) \in I \times \mathbb{R}$ with $x_1 \neq x_2$ there exists exactly one $\varphi \in \mathcal{F}$ such that

$$\varphi(x_i) = y_i \quad \text{for} \quad i = 1, 2.$$

The unique function $\varphi \in \mathcal{F}$ determined by the points $(x_1, y_1), (x_2, y_2)$ will be denoted by $\varphi_{(x_1,y_1),(x_2,y_2)}$.

Following [2] (see also [15]) we say that a function $f : I \to \mathbb{R}$ is *midconvex with respect to $\mathcal{F}$* (shortly, *$\mathcal{F}$–midconvex*) if for any $x_1, x_2 \in I$, $x_1 < x_2$,

$$f\left(\frac{x_1+x_2}{2}\right) \le \varphi_{(x_1,f(x_1)),(x_2,f(x_2))}\left(\frac{x_1+x_2}{2}\right).$$

Theorem 6.1. *Let $I \subset \mathbb{R}$ be an interval and let c be a positive number. Consider the two-parameter family $\mathcal{F}_c := \{cx^2 + ax + b : a, b \in \mathbb{R}\} \subset \mathbb{R}^I$. A function $f : I \to \mathbb{R}$ is strongly midconvex with modulus c if and only if it is $\mathcal{F}_c$*–midconvex.

Proof. Let $x_1, x_2 \in I$. If $\varphi = \varphi_{(x_1,f(x_1)),\ (x_2,f(x_2))} \in \mathcal{F}_c$, then $\varphi(x) = cx^2 + ax + b$, where the coefficients a, b are uniquely determined by the conditions $\varphi(x_i) = f(x_i)$, $i = 1, 2$. Therefore

$$\begin{aligned}\varphi\left(\frac{x_1+x_2}{2}\right) &= c\left(\frac{x_1+x_2}{2}\right)^2 + a\left(\frac{x_1+x_2}{2}\right) + b = \\ &= c\left(\frac{x_1^2+2x_1x_2+x_2^2}{4}\right) + a\left(\frac{x_1+x_2}{2}\right) + b = \\ &= \frac{1}{2}\big(cx_1^2+ax_1+b\big) + \frac{1}{2}\big(cx_2^2+ax_2+b\big) - \frac{c}{4}\big(x_1^2-2x_1x_2+x_2^2\big) = \\ &= \frac{\varphi(x_1)+\varphi(x_2)}{2} - \frac{c}{4}\big(x_1-x_2\big)^2 = \\ &= \frac{f(x_1)+f(x_2)}{2} - \frac{c}{4}\big(x_1-x_2\big)^2.\end{aligned}$$

Consequently, f is strongly midconvex with modulus c if and only if it is $\mathcal{F}_c$–*midconvex.* □

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