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CONVERGENCE THEOREMS FOR STRICTLY ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we establish the weak and strong convergence theorems for a k-strictly asymptotically pseudo-contractive mapping in the framework of Hilbert spaces. Our result improve and extend the corresponding result of Acedo and Xu, Liu, Marino and Xu, Osilike and Akuchu, and some others.

Keywords: strictly asymptotically pseudo-contractive mapping, implicit iteration scheme, common fixed point, strong convergence, weak convergence, Hilbert space.

Mathematics Subject Classification: 47H09, 47H10.

1. INTRODUCTION AND PRELIMINARIES

Let H be a real Hilbert space with the scalar product and norm denoted by the symbols $\langle .,.\rangle$ and $\|\cdot\|$ respectively, and C be a closed convex subset of H. Let T be a (possibly) nonlinear mapping from C into C. We now consider the following classes:

(1) T is contractive, i.e., there exists a constant $k < 1$ such that

$$\|Tx - Ty\| \le k\,\|x - y\|\,, \tag{1.1}$$

for all $x, y \in C$.

(2) T is nonexpansive, i.e.,

$$\|Tx - Ty\| \le \|x - y\|\,, \tag{1.2}$$

for all $x, y \in C$.

(3) T is uniformly L-Lipschitzian, i.e., there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\,\|x - y\|\,, \tag{1.3}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

(4) T is pseudo-contractive, i.e.,

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \tag{1.4}$$

for all $x, y \in C$.

(5) T is k-strictly pseudo-contractive, i.e., if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\,\|(x - Tx) - (y - Ty)\|^2, \tag{1.5}$$

for all $x, y \in C$.

(6) T is asymptotically nonexpansive [3], i.e., if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + r_n)\,\|x - y\|, \tag{1.6}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

(7) T is k-strictly asymptotically pseudo-contractive [10], i.e., if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$\|T^n x - T^n y\|^2 \leq (1 + r_n)^2\,\|x - y\|^2 + k\,\|(x - T^n x) - (y - T^n y)\|^2 \tag{1.7}$$

for some $k \in [0, 1)$ for all $x, y \in C$ and $n \in \mathbb{N}$.

Remark 1.1 ([10])**.** If T is a k-strictly asymptotically pseudo-contractive mapping, then it is uniformly L-Lipschitzian, but the converse does not hold.

The class of strictly pseudo-contractive mappings have been studied by several authors (see, for example [2, 4, 8, 12] and references therein).

In the case of a contractive mapping, the Banach Contraction Principle guarantees not only the existence of a unique fixed point, but also obtain the fixed point by successive approximation (or Picard iteration). But outside the class of contractive mappings, the classical iteration scheme no longer applies. So some other iteration scheme is required.

Two iteration processes are often used to approximate the fixed point of nonexpansive and pseudo-contractive mappings. The first iteration process is known as Mann's iteration [9], where $\{x_n\}$ is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.8}$$

where the initial guess x_0 is taken in C arbitrary and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$.

The second iteration process is known as Ishikawa iteration process [5] which is defined by

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)Ty_n, \\ y_n &= \beta_n x_n + (1 - \beta_n)Tx_n; \quad n \geq 0, \end{aligned} \tag{1.9}$$

where the initial guess x_0 is taken in C arbitrary and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$.

Process (1.9) is indeed more general than the process (1.8). But research has been concentrated on the later, probably due to the reason that process (1.8) is simpler and that a convergence theorem for process (1.8) may possibly lead to a convergence theorem for process (1.9), provided that the sequence $\{\beta_n\}$ satisfy certain appropriate conditions.

If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's iteration process (1.8) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [14]).

Recently, Marino and Xu [8] extended the results of Reich [14] from nonexpansive mappings to strict pseudo-contractions and obtained a weak convergence theorem in Hilbert spaces. More precisely, they gave the following results.

Theorem 1.2 ([8]). *Let C be a closed convex subset of a Hilbert space H. Let $T\colon C \to C$ be a k-strict pseudo-contraction for some $0 \le k < 1$ and assume that T admits a fixed point in C. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Mann's algorithm* (1.8)*. Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $k < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

In 2001, Xu and Ori [15] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space H. Let C be a nonempty subset of H. Let $T_1, T_2, \dots, T_N$ be self-mappings of C and suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \dots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0,1)$, $x_0 \in C$:

$$\begin{aligned} x_1 &= t_1 x_0 + (1-t_1)T_1 x_1, \\ x_2 &= t_2 x_1 + (1-t_2)T_2 x_2, \\ &\ \vdots \\ x_N &= t_N x_{N-1} + (1-t_N)T_N x_N, \\ x_{N+1} &= t_{N+1} x_N + (1-t_{N+1})T_1 x_{N+1}, \\ &\ \vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1-t_n)T_n x_n, \quad n \ge 1, \tag{1.10}$$

where $T_k = T_{k \bmod N}$. (Here the mod N function takes values in $\{1, 2, \dots, N\}$). And they proved the weak convergence of the process (1.10).

Very recently, Acedo and Xu [1] still in the framework of Hilbert spaces introduced the following cyclic algorithm.

Let C be a closed convex subset of a Hilbert space H and let $\{T_i\}_{i=0}^{N-1}$ be N k-strict pseudo-contractions on C such that $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}$ be a sequence in $(0,1)$. The cyclic algorithm generates a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$\begin{aligned} x_1 &= \alpha_0 x_0 + (1-\alpha_0)T_0 x_0, \\ x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_1 x_1, \\ &\vdots \\ x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_{N-1} x_{N-1}, \\ x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_0 x_N, \\ &\vdots \end{aligned}$$

In general, $\{x_{n+1}\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]} x_n, \tag{1.11}$$

where $T_{[n]} = T_i$ with $i = n \ (mod\ N)$, $0 \le i \le N-1$. They also proved a weak convergence theorem for k-strict pseudo-contractions in Hilbert spaces by cyclic algorithm (1.11). More precisely, they obtained the following theorem:

Theorem 1.3 ([1]). *Let C be a closed convex subset of a Hilbert space H. Let $N \ge 1$ be an integer. Let for each $0 \le i \le N-1$, $T_i\colon C \to C$ be a k_i-strict pseudo-contraction for some $0 \le k_i < 1$. Let $k = \max\{k_i : 1 \le i \le N\}$. Assume the common fixed point set $\bigcap_{i=0}^{N-1} F(T_i)$ of $\{T_i\}_{i=0}^{N-1}$ is nonempty. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the cyclic algorithm (1.11). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k+\epsilon < \alpha_n < 1-\epsilon$ for all n and for some $\epsilon \in (0,1)$. Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.*

Motivated by Xu and Ori [15], Acedo and Xu [1] and some others we introduce and study the following:

Let C be a closed convex subset of a Hilbert space H and let $\{T_i\}_{i=0}^{N-1}$ be N k-strictly asymptotically pseudo-contractions on C such that $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}$ be a sequence in $(0,1)$. The implicit iteration scheme generates a sequence $\{x_n\}_{n=0}^{\infty}$ in the following way:

$$\begin{aligned} x_1 &= \alpha_0 x_0 + (1-\alpha_0)T_0 x_0, \\ x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_1 x_1, \\ &\vdots \\ x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_{N-1} x_{N-1}, \\ x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_0^2 x_0, \\ &\vdots \\ x_{2N} &= \alpha_{2N-1} x_{2N-1} + (1-\alpha_{2N-1})T_{N-1}^2 x_{2N-1}, \\ x_{2N+1} &= \alpha_{2N} x_{2N} + (1-\alpha_{2N})T_0^3 x_0, \\ &\vdots \end{aligned}$$

In general, $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^s_{[n]} x_n, \tag{1.12}$$

where $T^s_{[n]} = T^s_{n\,(mod\ N)} = T^s_i$ with $n = sN + i$ and $i \in I = \{0, 1, \dots, N-1\}$.

The purpose of this paper is to establish weak and strong convergence theorems of the implicit iteration process (1.12) for finite family of k-strictly asymptotically pseudo-contraction mappings in Hilbert spaces. Our results extend the corresponding results of Reich [14], Marino and Xu [8], Acedo and Xu [1] and many others.

In the sequel, we will need the following lemmas.

Lemma 1.4. *Let H be a real Hilbert space. There hold the following identities:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \quad \forall\ x, y \in H$.
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, $\forall\ t \in [0,1]$, $\forall\ x, y \in H$.
(iii) *If $\{x_n\}$ is a sequence in H that weakly converges to z,*

then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2 \quad \forall y \in H.$$

We use following notation:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x : \exists\ x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Lemma 1.5 ([13])**.** *Let $\{a_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + r_n) a_n + \beta_n, \quad n \ge 1.$$

If $\sum_{n=1}^\infty r_n < \infty$ and $\sum_{n=1}^\infty \beta_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^\infty$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.6. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, and let $T_i\colon C \to C$ be a k_i-strictly asymptotically pseudocontractive mapping for $i = 0, 1, \dots, N-1$ with a sequence $\{r_{n_i}\} \subset [0, \infty)$ such that $\sum_{n=1}^\infty r_{n_i} < \infty$ and for some $0 \le k_i < 1$, then there exist constants $L > 0$ and $k \in [0, 1)$ and a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that for any $x, y \in C$ and for each $i = 0, 1, \dots, N-1$ and each $n \ge 1$, the following hold:*

$$\|T_i^n x - T_i^n y\| \le (1 + r_n)^2 \|x - y\|^2 + k\,\|(x - T_i^n x) - (y - T_i^n y)\|^2, \tag{1.13}$$

and

$$\|T_i^n x - T_i^n y\| \le L\,\|x - y\|. \tag{1.14}$$

Proof. Since for each $i = 0, 1, \dots, N-1$, T_i is k_i-strictly asymptotically pseudo-contractive, where $k_i \in [0, 1)$ and $\{r_{n_i}\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_{n_i} = 0$. By Remark 1.1, T_i is L_i-Lipschitzian. Taking $r_n = \max\{r_{n_i}, i = 0, 1, \dots, N-1\}$ and $k = \max\{k_i, i = 0, 1, \dots, N-1\}$, hence, for each $i = 0, 1, \dots, N-1$, we have

$$\begin{aligned} \|T_i^n x - T_i^n y\| &\le (1 + r_{n_i})^2 \|x - y\|^2 + k_i\,\|(x - T_i^n x) - (y - T_i^n y)\|^2 \le \\ &\le (1 + r_n)^2 \|x - y\|^2 + k\,\|(x - T_i^n x) - (y - T_i^n y)\|^2. \end{aligned} \tag{1.15}$$

The conclusion (1.13) is proved. Again taking $L = \max\{L_i : i = 0, 1, \ldots, N-1\}$ for any $x, y \in C$, we have

$$\|T_i^n x - T_i^n y\| \leq L_i \|x - y\| \leq L \|x - y\| . \tag{1.16}$$

This completes the proof of Lemma 1.6. □

2. MAIN RESULTS

Theorem 2.1. *Let C be a closed convex subset of a Hilbert space H. Let $N \geq 1$ be an integer. Let for each $0 \leq i \leq N-1$, $T_i \colon C \to C$ be N k_i-strictly asymptotically pseudo-contraction mappings for some $0 \leq k_i < 1$, $\sum_{n=1}^{\infty} r_n < \infty$ and $I - T_{[n]}$ is demiclosed at zero. Let $k = \max\{k_i : 0 \leq i \leq N-1\}$ and $r_n = \max\{r_{n_i} : 0 \leq i \leq N-1\}$. Assume that $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an implicit iteration scheme (1.12). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k + \epsilon < \alpha_n < 1 - \epsilon$ for all n and for some $\epsilon \in (0,1)$. Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.*

Proof. Let $p \in F = \bigcap_{i=0}^{N-1} F(T_i)$. It follows from (1.12) and Lemma 1.1 (ii) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|\alpha_n x_n + (1-\alpha_n) T_{[n]}^s x_n - p\right\|^2 = \\
&= \left\|\alpha_n (x_n - p) + (1-\alpha_n)(T_{[n]}^s x_n - p)\right\|^2 = \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \left\|T_{[n]}^s x_n - p\right\|^2 - \\
&\quad - \alpha_n (1-\alpha_n) \left\|x_n - T_{[n]}^s x_n\right\|^2 \leq \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \Big[(1+r_n)^2 \|x_n - p\|^2 + \\
&\quad + k \left\|x_n - T_{[n]}^s x_n\right\|^2 \Big] - \alpha_n(1-\alpha_n) \left\|x_n - T_{[n]}^s x_n\right\|^2 \leq \\
&\leq \Big[\alpha_n (1+r_n)^2 + (1-\alpha_n)(1+r_n)^2\Big] \|x_n - p\|^2 - \\
&\quad - (\alpha_n - k)(1-\alpha_n) \left\|x_n - T_{[n]}^s x_n\right\|^2 = \\
&= (1+r_n)^2 \|x_n - p\|^2 - (\alpha_n - k)(1-\alpha_n) \left\|x_n - T_{[n]}^s x_n\right\|^2 = \\
&= (1+d_n) \|x_n - p\|^2 - (\alpha_n - k)(1-\alpha_n) \left\|x_n - T_{[n]}^s x_n\right\|^2 ,
\end{aligned} \tag{2.1}$$

where $d_n = r_n^2 + 2r_n$. Since $k + \epsilon < \alpha_n < 1 - \epsilon$ for all n, from (2.1) we have

$$\|x_{n+1} - p\|^2 \leq (1+d_n) \|x_n - p\|^2 - \epsilon^2 \left\|x_n - T_{[n]}^s x_n\right\|^2 . \tag{2.2}$$

Now (2.2) implies that

$$\|x_{n+1} - p\|^2 \leq (1 + d_n) \|x_n - p\|^2 . \tag{2.3}$$

Since $\sum_{n=1}^{\infty} r_n < \infty$ thus $\sum_{n=1}^{\infty} d_n < \infty$, it follows by Lemma 1.2, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists and so $\{x_n\}$ is bounded. Consider (2.2) again yields that

$$\left\|x_n - T_{[n]}^s x_n\right\|^2 \leq \frac{1}{\epsilon^2}[\|x_n - p\|^2 - \|x_{n+1} - p\|^2] + \frac{d_n}{\epsilon^2} \|x_n - p\|^2 . \tag{2.4}$$

Since $\{x_n\}$ is bounded and $d_n \to 0$ as $n \to \infty$. So, we get

$$\left\|x_n - T_{[n]}^s x_n\right\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.5}$$

From the definition of $\{x_n\}$, we have

$$\|x_{n+1} - x_n\| = (1 - \alpha_n) \left\|x_n - T_{[n]}^s x_n\right\| \to 0, \quad \text{as} \quad n \to \infty. \tag{2.6}$$

So, $\|x_n - x_{n+l}\| \to 0$ as $n \to \infty$ and for all $l < N$. Now for $n \geq N$, and since T is uniformly Lipschitzian (by Remark 1.1) with Lipschitz constant $L > 0$, so we have

$$\begin{aligned} \left\|x_n - T_{[n]}x_n\right\| &\leq \left\|x_n - T_{[n]}^s x_n\right\| + \left\|T_{[n]}^s x_n - T_{[n]}x_n\right\| \leq \\ &\leq \left\|x_n - T_{[n]}^s x_n\right\| + L\left\|T_{[n]}^{s-1} x_n - x_n\right\| \leq \\ &\leq \left\|x_n - T_{[n]}^s x_n\right\| + L\Big[\left\|T_{[n]}^{s-1} x_n - T_{[n-N]}^{s-1} x_{n-N}\right\| + \\ &\quad + \left\|T_{[n-N]}^{s-1} x_{n-N} - x_{n-N}\right\| + \|x_{n-N} - x_n\|\Big]. \end{aligned} \tag{2.7}$$

Since for each $n \geq N$, $n \equiv (n - N)$ $(mod\ N)$. Thus $T_{[n]} = T_{[n-N]}$, therefore from (2.7), we have

$$\begin{aligned} \left\|x_n - T_{[n]}x_n\right\| &\leq \left\|x_n - T_{[n]}^s x_n\right\| + L^2 \|x_n - x_{n-N}\| + \\ &\quad + L\left\|T_{[n-N]}^{s-1} x_{n-N} - x_{n-N}\right\| + L\|x_{n-N} - x_n\| . \end{aligned} \tag{2.8}$$

From (2.5) and (2.8), we obtain

$$\left\|x_n - T_{[n]}x_n\right\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.9}$$

Consequently, for any $l \in I = \{0, 1, \ldots, N - 1\}$,

$$\begin{aligned} \left\|x_n - T_{[n+l]}x_n\right\| &\leq \|x_n - x_{n+l}\| + \left\|x_{n+l} - T_{[n+l]}x_{n+l}\right\| + \left\|T_{[n+l]}x_{n+l} - T_{[n+l]}x_n\right\| \leq \\ &\leq (1 + L) \|x_n - x_{n+l}\| + \left\|x_{n+l} - T_{[n+l]}x_{n+l}\right\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.10}$$

This implies that

$$\lim_{n\to\infty} \left\|x_n - T_{[l]}x_n\right\| = 0, \quad \forall\ l \in I = \{0, 1, \ldots, N - 1\}. \tag{2.11}$$

Since $I-T_{[n]}$ is demiclosed at zero, (2.10) implies that $x_n \rightharpoonup x$, where x is a weak limit of $\{x_n\}$ and hence $\omega_w(x_n) \subset F = \bigcap_{i=0}^{N-1} F(T_i)$. Now we show that $\{x_n\}$ is weakly convergent. Let $p_1, p_2 \in \omega_w(x_n)$ and $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ which converges weakly to some p_1 and p_2 respectively.

Since $\lim_{n\to\infty} \|x_n - z\|$ exists for every $z \in F$ and since $p_1, p_2 \in F$, we have

$$\lim_{n\to\infty} \|x_n - p_1\|^2 = \lim_{j\to\infty} \left\|x_{m_j} - p_1\right\|^2 = \lim_{j\to\infty} \left\|x_{m_j} - p_2\right\|^2 + \|p_2 - p_1\|^2 =$$
$$= \lim_{i\to\infty} \|x_{n_i} - p_1\|^2 + 2\|p_2 - p_1\|^2 = \lim_{n\to\infty} \|x_n - p_1\|^2 + 2\|p_2 - p_1\|^2 .$$

Hence $p_1 = p_2$. Thus $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$. This completes the proof. □

Theorem 2.2. *Let C be a closed convex compact subset of a Hilbert space H. Let $N \geq 1$ be an integer. Let for each $0 \leq i \leq N-1$, $T_i\colon C \to C$ be N k_i-strictly asymptotically pseudo-contraction mappings for some $0 \leq k_i < 1$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let $k = \max\{k_i : 0 \leq i \leq N-1\}$ and $r_n = \max\{r_{n_i} : 0 \leq i \leq N-1\}$. Assume that $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an implicit iteration scheme (1.12). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k + \epsilon < \alpha_n < 1 - \epsilon$ for all n and for some $\epsilon \in (0, 1)$. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.*

Proof. We only conclude the difference. By compactness of C this immediately implies that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to a common fixed point of $\{T_i\}_{i=0}^{N-1}$, say, p. Combining (2.3) with Lemma 1.5, we have $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof. □

For our next result, we shall need the following definition:

Definition 2.3. A mapping $T\colon C \longrightarrow C$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = x \in C$.

Theorem 2.4. *Let C be a closed convex subset of a Hilbert space H. Let $N \geq 1$ be an integer. For each $0 \leq i \leq N-1$, let $T_i\colon C \to C$ be N k_i-strictly asymptotically pseudo-contraction mappings for some $0 \leq k_i < 1$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let $k = \max\{k_i : 0 \leq i \leq N-1\}$ and $r_n = \max\{r_{n_i} : 0 \leq i \leq N-1\}$. Assume that $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an implicit iteration scheme (1.12). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k + \epsilon < \alpha_n < 1 - \epsilon$ for all n and for some $\epsilon \in (0, 1)$. Assume that one member of the family $\{T_i\}_{i=0}^{N-1}$ be semi-compact. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.*

Proof. Without loss of generality, we can assume that T_1 is semi-compact. It follows from (2.11) that

$$\lim_{n\to\infty} \left\|x_n - T_{[1]}x_n\right\| = 0. \tag{2.12}$$

By the semi-compactness of T_1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to u \in C$ strongly. Since C is closed, $u \in C$, and furthermore,

$$\lim_{n_k \to \infty} \left\| x_{n_k} - T_{[l]} x_{n_k} \right\| = \left\| u - T_{[l]} u \right\| = 0, \tag{2.13}$$

for all $l \in I = \{0, 1, \ldots, N-1\}$. Thus $u \in F$. Since $\{x_{n_k}\}$ converges strongly to u and $\lim_{n\to\infty} \|x_n - u\|$ exists, it follows from Lemma 1.5 that $\{x_n\}$ converges strongly to u. This completes the proof. □

Remark 2.5. Theorem 2.1 extends and improves the corresponding result of Reich [14] and Marino and Xu [8] from nonexpansive and strict pseudo-contraction mappings to the more general class of a finite family of k-strictly asymptotically pseudo-contraction mappings and implicit iteration schemes considered in this paper.

Remark 2.6. Theorem 2.1 also extends and improves the corresponding result of Acedo and Xu [1] from k-strictly pseudo-contraction mappings to the more general class of k-strictly asymptotically pseudo-contraction mappings.

Remark 2.7. Theorem 2.1 also extends and improves the corresponding result of Xu and Ori [15] from nonexpansive mappings to more the general class of k-strictly asymptotically pseudo-contraction mappings.

Remark 2.8. Theorem 2.2 extends and improves the corresponding result of Liu [7] in the following respects:

(i) We removed the uniformly L-Lipschitzian condition.
(ii) The modified Mann iteration process is replaced by implicit iteration process for a finite family of mappings.

Remark 2.9. Theorem 2.4 extends and improves the corresponding result of Kim and Xu [6].

Remark 2.10. Theorem 2.4 also extends and improves Theorem 1.6 of Osilike and Akuchu [11] from asymptotically pseudocontractive mappings to strictly asymptotically pseudocontractive mappings.

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