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NONLOCAL IMPULSIVE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH TIME-VARYING GENERATING OPERATORS IN BANACH SPACES

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Abstract. In this paper, we study the existence and uniqueness of the PC-mild solution for a class of impulsive fractional differential equations with time-varying generating operators and nonlocal conditions. By means of the generalized Ascoli-Arzela Theorem given by us and the fixed point theorem, some existence and uniqueness results are obtained. Finally, an example is given to illustrate the theory.

Keywords: nonlocal conditions, impulsive equations, fractional differential equations, time-varying generating operators, PC-mild solution.

Mathematics Subject Classification: 34G10, 34G20.

1. INTRODUCTION

During the past decades, impulsive differential equations have attracted many authors since it is much richer than the corresponding theory of differential equations (see for instance [16–19, 38] and references therein). Recently, impulsive evolution equations and their optimal control problems in infinite dimensional spaces have been investigated by many authors including Ahmed, Benchohra, Ntouyas, Liu, Nieto and us (see for instance [3–5, 7–10, 21, 25, 26, 34–37] and references therein). Particularly, by constructing impulsive periodic evolution operators and generalized Gronwall inequalities, we studied the impulsive periodic evolution system in infinite dimensional spaces (see [29–33]).

On the other hand, the nonlocal condition has a better effect on the solution and is more precise for physical measurements than the classical condition $x(0) = x_0$ alone. For the importance of nonlocal conditions in different fields, we refer the reader to [12–15] and the references contained therein.

Very recently, Liang et al. in [23] studied the existence and uniqueness of the PC-mild solution for a nonlinear impulsive differential equation with nonlocal conditions

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in J = [0, b], \quad t \neq t_i, \\ x(0) + g(x) = x_0, \\ \Delta x(t_i) = I_i(x(t_i)), & i = 1, 2, \ldots, p, \quad 0 < t_1 < t_2 < \ldots < t_p < b, \end{cases}$$

under the differential assumptions on f, g and I_i such as Lipschitz conditions and compactness conditions.

Mophou in [24] studied the existence and uniqueness of the PC-mild solution to impulsive fractional differential equations

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t)), & 0 < \alpha < 1, \quad t \in J = [0, b], \quad t \neq t_i, \\ x(0) = x_0, \\ \Delta x(t_i) = I_i(x(t_i)), & i = 1, 2, \ldots, p, \quad 0 < t_1 < t_2 < \ldots < t_p < b, \end{cases}$$

where D_t^α is the Caputo fractional derivative.

Benchohra et al. in [1, 11] establish sufficient conditions for the existence of solutions for a class of initial value problems for impulsive fractional differential equations involving the Caputo farctional derivative of order $\alpha \in (0, 1]$ and $\alpha \in (1, 2]$. Ahmad et al. in [2] give some existence results for two-point boundary value problems involving nonlinear impulsive hybrid differential equations of fraction order $\alpha \in (1, 2]$.

To date differential equations with time-varying generating operators have not been covered in detail, impulsive conditions and nonlocal conditions. Here, motivated by [1, 11, 23, 24] and [31], we will combine these works and extend the study to the following semi-linear differential equation with time-varying generating operators which combines impulsive conditions and nonlocal conditions

$$\begin{cases} D_t^\alpha x(t) = A(t)x(t) + f(t, x(t)), & 0 < \alpha < 1, \quad t \in J = [0, b], \quad t \neq t_i, \\ x(0) = x_0 + g(x), \\ \Delta x(t_i) = I_i(x(t_i)), & i = 1, 2, \ldots, p, \quad 0 < t_1 < t_2 < \ldots < t_p < b, \end{cases} \tag{1.1}$$

where $\{A(t), t \in J\}$ is a family of closed densely defined linear unbounded operators on X. f: $J \times X \to X$ is a given continuous functions, g is a given function satisfying some assumptions and constitutes a nonlocal Cauchy problem, x_0 is an element of the Banach space X, I_i: $X \to X$, $0 = t_0 < t_1 < t_2 < \ldots < t_p < t_{p+1} = b$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $x(t_i^+) = \lim_{h \to 0^+} = x(t_i + h)$ and $x(t_i^-) = x(t_i)$ represent respectively the right and left limits of $x(t)$ at $t = t_i$.

By virtue of the generalized Ascoli-Arzela Theorem given by us and some fixed point theorems such as the Schaefer fixed point theorem and the Krasnoselskii fixed point theorem, we will derive some results concerning the PC-mild solution for the system (1.1) under the different assumptions on f, g, I_i and $U(\cdot, \cdot)$.

The paper is organized as follows. In Section 2, we introduce the PC-mild solution of system (1.1) and recall some Lemmas which are used in the sequel. In Section 3, we study the existence and uniqueness of PC-mild solutions of system (1.1) under some suitable conditions. An example to illustrate our results is given at last.

2. PRELIMINARIES

Let $B(X)$ be the Banach space of all linear and bounded operators on X. $C(J,X)$ be the Banach space of all X-valued continuous functions from $J=[0,T]$ into X endowed with the norm $\|x\|_C=\sup_{t\in J}\|x(t)\|$. We also introduce the set of functions $PC(J,X)\equiv\big\{x:J\to X\mid x$ is continuous at $t\in J\backslash\{t_1,t_2,\ldots,t_p\}$, and x is continuous from left and has right hand limits at $t\in\{t_1,t_2,\ldots,t_p\}\big\}$. Endowed with the norm

$$\|x\|_{PC}=\max\left\{\sup_{t\in J}\|x(t+0)\|,\sup_{t\in J}\|x(t-0)\|\right\},$$

$(PC(J,X),\|\cdot\|_{PC})$ is a Banach space.

[HA]: For $t\in[0,b]$ one has:

(A_1) The domain $D(A(t))=D$ is independent of t and is dense on X.

(A_2) For $t\geq 0$, the resolvent $R(\lambda,A(t))=(\lambda I-A(t))^{-1}$ exists for all λ with $Re\lambda\leq 0$, and there is a constant $\widetilde{M}$ independent of λ and t such that

$$\|R(\lambda,A(t))\|\leq\widetilde{M}(1+|\lambda|)^{-1}\qquad\text{for}\qquad Re\lambda\leq 0.$$

(A_3) There exist constants $L>0$ and $0<\alpha\leq 1$ such that

$$\big\|\big(A(t)-A(\theta)\big)A^{-1}(\tau)\big\|\leq L|t-\theta|^{\alpha}\qquad\text{for}\qquad t,\theta,\tau\in[0,b].$$

(A_4) The resolvent $R(\lambda,A(t))(t\geq 0)$ is compact.

Lemma 2.1 (See [6], p.159, Lemma 2.2 in [32])**.** *Under the assumption* [HA]*, the Cauchy problem*

$$\dot{x}(t)+A(t)x(t)=0,\ t\in(0,b]\quad\textit{with}\quad x(0)=x_0 \tag{2.1}$$

has a unique evolution system $\{U(t,\theta)\mid 0\leq\theta\leq t\leq b\}$ *on* X *satisfying the following properties:*

1. $U(t,\theta)\in B(X)$ *for* $0\leq\theta\leq t\leq b$. *Denotes* $M=\sup_{t\in J}\|U(t,s)\|_{B(X)}$, *which is a finite number.*
2. $U(t,r)U(r,\theta)=U(t,\theta)$ *for* $0\leq\theta\leq r\leq t\leq b$.
3. $U(\cdot,\cdot)x\in C(\Lambda,X)$ *for* $x\in X$, $\Lambda=\{(t,\theta)\in J\times J\mid 0\leq\theta\leq t\leq b\}$.
4. *For* $0\leq\theta<t\leq b$, $U(t,\theta)$: $X\longrightarrow D$ *and* $t\longrightarrow U(t,\theta)$ *is strongly differentiable on* X. *The derivative* $\frac{\partial}{\partial t}U(t,\theta)\in B(X)$ *and it is strongly continuous on* $0\leq\theta<t\leq b$. *Moreover,*

$$\frac{\partial}{\partial t}U(t,\theta)=-A(t)U(t,\theta)\quad\textit{for}\quad 0\leq\theta<t\leq b,$$

$$\left\|\frac{\partial}{\partial t}U(t,\theta)\right\|_{B(X)}=\|A(t)U(t,\theta)\|_{B(X)}\leq\frac{C}{t-\theta},$$

$$\big\|A(t)U(t,\theta)A(\theta)^{-1}\big\|_{B(X)}\leq C\quad\textit{for}\quad 0\leq\theta\leq t\leq b.$$

(5) *For every* $v \in D$ *and* $t \in (0, b], U(t, \theta)v$ *is differentiable with respect to* θ *on* $0 \le \theta \le t \le b$

$$\frac{\partial}{\partial\theta}U(t,\theta)v = U(t,\theta)A(\theta)v.$$

(6) $U(t, \theta)$ *is a compact operator for* $0 \le \theta < t \le b$.

And, for each $x_0 \in X$*, the Cauchy problem* (2.1) *has a unique classical solution* $x \in C^1(J, X)$ *given by*

$$x(t) = U(t, 0)x_0, \quad t \in J.$$

Let us recall the following known definitions. For more details see [27].

Definition 2.2. A real function $f(t)$ is said to be in the space C_α, $\alpha \in R$ if there exists a real number $\kappa > \alpha$, such that $f(t) = t^\kappa g(t)$, where $g \in C[0, \infty)$ and it is said to be in the space C_α^m iff $f^{(m)} \in C_\alpha$, $m \in N$.

Definition 2.3. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\alpha$, $\alpha \ge -1$ is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds \tag{2.2}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.4. If the function $f \in C_{-1}^m$, $m \in N$, the fractional derivative of order $\alpha > 0$ of a function $f(t)$ in the Caputo sense is given by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds, \; m-1 < \alpha \le m.$$

Consider the following system

$$\begin{cases} D_t^\alpha x(t) = A(t)x(t) + f\left(t, x(t)\right), \quad 0 < \alpha < 1, \quad t \in J, \\ x(0) = x_0 + g(x), x_0 \in X. \end{cases} \tag{2.3}$$

Lemma 2.5. *The system* (2.3) *is equivalent to the nonlinear integral equation*

$$\begin{aligned} x(t) = x_0 + g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s)x(s)ds + \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s)\right) ds, \quad t \in J. \end{aligned} \tag{2.4}$$

In other words, every solution of the integral equation (2.4) *is also solution of the system* (2.3) *and vice versa.*

Proof. It can be proved by applying the integral operator (2.2) to both sides of the system (2.3), and using some classical results from fractional calculus to get (2.4). □

Definition 2.6. By a mild solution of system (2.3) we mean the function $x \in C(J, X)$ which satisfies

$$x(t) = U(t,0)[x_0 + g(x)] + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s, x(s))\, ds, \quad t \in J.$$

Definition 2.7. By a PC-mild solution of the system (1.1) we mean the function $x \in PC(J, X)$ which satisfies

$$\begin{aligned} x(t) = U(t,0)[x_0 + g(x)] + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} U(t,s) f(s, x(s))\, ds + \\ + \frac{1}{\Gamma(\alpha)} \int_{t_i}^t (t-s)^{\alpha-1} U(t,s) f(s, x(s))\, ds + \sum_{0<t_i<t} U(t,t_i) I_i(x(t_i)), \quad t \in J. \end{aligned} \tag{2.5}$$

Remark 2.8. Note that our definition is well defined. In fact, for $t \in [0, t_1]$, the function

$$x(t) = U(t,0)[x_0 + g(x)] + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s, x(s))\, ds$$

is the mild solution of

$$\begin{cases} D_t^{\alpha} x(t) = A(t)x(t) + f(t, x(t)), \quad 0 < \alpha < 1, \quad t \in [0, t_1], \\ x(0) = x_0 + g(x), \ x_0 \in X, \end{cases} \tag{2.6}$$

in the sense of Definition 2.6. Further, by the impulsive condition,

$$\begin{aligned} x(t_1^+) = x(t_1) + I_1(x(t_1)) = U(t_1, 0)[x_0 + g(x)] + \\ + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} U(t_1, s) f(s, x(s))\, ds + I_1(x(t_1)). \end{aligned}$$

Moreover, for $t \in (t_1, t_2]$, the function

$$\begin{aligned} x(t) &= U(t,t_1)x(t_1) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} U(t,s) f(s, x(s))\, ds = \\ &= U(t,0)[x_0 + g(x)] + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} U(t,s) f(s, x(s))\, ds = \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} U(t,s) f(s, x(s))\, ds + \sum_{0<t_1<t} U(t,t_1) I_1(x(t_1)) \end{aligned}$$

is the mild solution of

$$\begin{cases} D_t^\alpha x(t) = A(t)x(t) + f\left(t, x(t)\right), \quad 0 < \alpha < 1, \quad t \in (t_1, t_2], \\ x(t_1) = x(t_1^+). \end{cases} \tag{2.7}$$

By repeating the same procedure, for $t \in (t_{i-1}, t_i]$, $i = 1, 2, \ldots, p$, we can easily deduce that the expression (2.5) is just the mild solution of

$$\begin{cases} D_t^\alpha x(t) = A(t)x(t) + f\left(t, x(t)\right), \quad 0 < \alpha < 1, \quad t \in (t_{i-1}, t_i], \\ x(t_{i-1}) = x(t_{i-1}^+). \end{cases} \tag{2.8}$$

To end this section, we recall the following results which will be used in the sequel.

Lemma 2.9 (The generalized Ascoli-Arzela theorem [34])**.** *Suppose $W \subset PC(J, X)$ be a subset. If the following conditions are satisfied:*

(1) *W is a uniformly bounded subset of $PC(J, X)$.*
(2) *W is equicontinuous in (t_i, t_{i+1}), $i = 0, 1, 2, \ldots, p$, where $t_0 = 0$, $t_{p+1} = b$.*
(3) *$W(t) \equiv \{x(t) \mid x \in W, t \in J \backslash \{t_1, \ldots, t_p\}\}$, $W(t_i + 0) \equiv \{x(t_i + 0) \mid x \in W\}$ and $W(t_i - 0) \equiv \{x(t_i - 0) \mid x \in W\}$ is a relatively compact subset of $PC(J, X)$.*

Then W is a relatively compact subset of $PC(J, X)$.

Lemma 2.10 (Schaefer's fixed point theorem [28])**.** *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a continuous and compact map, and let the set $\{x \in S : x = \lambda Fx$ for some $\lambda \in (0, 1)\}$ be bounded. Then F has at least one fixed point in S.*

Lemma 2.11 (Krasnoselskii's fixed point theorem [18])**.** *Let $\mathfrak{B}$ be a closed convex and nonempty subsets of Banach space X. Suppose that $\mathcal{L}$ and $\mathcal{N}$ are in general nonlinear operators which map $\mathfrak{B}$ into X such that:*

(1) *$\mathcal{L}x + \mathcal{N}y \in \mathfrak{B}$ whenever $x, y \in \mathfrak{B}$;*
(2) *$\mathcal{L}$ is a contraction mapping;*
(3) *$\mathcal{N}$ is compact and continuous.*

Then there exists $z \in \mathfrak{B}$ such that $z = \mathcal{L}z + \mathcal{N}z$.

3. MAIN RESULTS

In this section, we will derive some existence and uniqueness results concerning the PC-mild solution for the system (1.1) under the different assumptions on f, g, I_i and $U(\cdot, \cdot)$.

Case 1. f, g, I_i are uniformly Lipschitz, and $U(\cdot, \cdot)$ is not compact

Let us list the following hypotheses:

[Hf]: f: $J \times X \to X$ is continuous and there exists a function $L_f(t) \in L^\infty(J, R^+)$ such that

$$\|f(t, x) - f(t, y)\| \leq L_f(t)\|x - y\|, \quad t \in J, \ x, y \in X.$$

[Hg]: g: $PC(J,X) \to X$ and there exists a constant $L_g > 0$ such that

$$\|g(x) - g(y)\| \le L_g \|x - y\|_{PC}, \quad x, y \in PC(J,X).$$

[HI]: I_i: $X \to X$ and there exists a constant $h_i > 0$, $i = 1, 2, \ldots, p$, such that

$$\|I_i(x) - I_i(y)\| \le h_i \|x - y\|, \quad x, y \in X.$$

Theorem 3.1. *Let* [A$_1$]–[A$_3$], [Hf], [Hg] *and* [HI] *be satisfied. Then for every* $x_0 \in X$, *the system* (1.1) *has a unique PC-mild solution on* J *provided that*

$$0 < M\left[L_g + \frac{(p+1)b^\alpha M \|L_f\|_{L^\infty(J,R^+)}}{\Gamma(\alpha+1)} + \sum_{i=1}^{p} h_i\right] \le \gamma < 1. \tag{3.1}$$

Proof. Let $x_0 \in X$ be fixed. Define an operator Q on $PC(J,X)$ by

$$\begin{aligned}(Qx)(t) = U(t,0)[u_0 + g(x)] + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} U(t,s) f(s, x(s))\, ds + \\ + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-s)^{\alpha-1} U(t,s) f(s, x(s))\, ds + \\ + \sum_{0<t_i<t} U(t,t_i) I_i(x(t_i)), \quad t \in J.\end{aligned} \tag{3.2}$$

Then it is clear that Q: $PC(J,X) \to PC(J,X)$. In fact, for $0 \le \tau \le t \le t_1$,

$$\begin{aligned}\|(Qx)(t) - (Qx)(\tau)\| &\le \|U(t,0)[x_0 + g(x)] - U(\tau,0)[x_0 + g(x)]\| + \\ &\quad + \int_{\tau}^{t} \|U(t,s) f(s, x(s))\|\, ds + \\ &\quad + \int_{0}^{\tau} \|U(t,s) f(s, x(s)) - U(\tau,s) f(s, x(s))\|\, ds \le \\ &\le M\|U(t,\tau)x_0 - x_0\| + M\|U(t,\tau)g(x) - g(x)\| + \\ &\quad + \int_{0}^{\tau} \big\|U(t,\tau)\big[U(\tau,s) f(s, x(s))\big] - U(\tau,s) f(s, x(s))\big\|\, ds.\end{aligned}$$

By (3) of Lemma 2.1, we know that $U(\cdot,\cdot)x \in C(\Lambda, X)$ for $x \in X$, $\Lambda = \{(t,\tau) \in J \times J \mid 0 \le \tau \le t \le b\}$. Thus, we can deduce that $Qx \in C([0,t_1], X)$. Similarly we can also obtain that $Qx \in C((t_1,t_2], X)$, $Qx \in C((t_2,t_3], X)$, $\ldots$, $Qx \in C((t_p,b], X)$. That is, $Qx \in PC(J,X)$.

Also, it comes from [Hf], [Hg] and [HI] that

$$\begin{aligned}
\|(Qx)(t)-(Qy)(t)\| \leq{}& \|U(t,0)[g(x)-g(y)]\| \times \\
&\times \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} \|U(t,s)\left(f(s,x(s))-f(s,y(s))\right)\| ds + \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-s)^{\alpha-1} \|U(t,s)\left(f(s,x(s))-f(s,y(s))\right)\| ds + \\
&+ \sum_{0<t_i<t} \|U(t,t_i)\| \|I_i(x(t_i)) - I_i(y(t_i))\| \leq \\
\leq{}& ML_g\|x-y\|_{PC} + \frac{ML_f\|x-y\|_{PC}}{\Gamma(\alpha)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} ds + \\
&+ \frac{ML_f\|x-y\|_{PC}}{\Gamma(\alpha)} \int_{t_i}^{t} (t-s)^{\alpha-1} ds + \\
&+ M\sum_{i=1}^{p} h_i \|x(t_i)-y(t_i)\| \leq \\
\leq{}& M\left[L_g + \frac{(p+1)b^\alpha M\|L_f\|_{L^\infty(J,R^+)}}{\Gamma(\alpha+1)} + \sum_{i=1}^{p} h_i\right] \|x-y\|_{PC} \equiv \\
\equiv{}& \mu\|x-y\|_{PC}, \quad x,y \in PC(J,X).
\end{aligned}$$

From (3.1), we can deduce

$$\mu \equiv M\left[L_g + \frac{(p+1)b^\alpha M\|L_f\|_{L^\infty(J,R^+)}}{\Gamma(\alpha+1)} + \sum_{i=1}^{p} h_i\right] < 1.$$

Thus, we find that Q is a contraction operator on $PC(J,X)$, thus Q has a unique fixed point, which gives rise to a unique PC-mild solution. This completes the proof. □

Case 2. f is not uniformly Lipschitz, g, I_i and $U(\cdot,\cdot)$ is compact

We make the following assumptions:

[C1]: f: $J \times X \to X$ is continuous and maps a bounded set into a bounded set.
[C2]: g: $PC(J,X) \to X$ and I_i: $X \to X$, $i = 1,2,\ldots,p$, are compact operators.
[C3]: For each $x_0 \in X$, there exists a constant $r > 0$ such that

$$M\left[\|x_0\| + \sup_{\phi\in Y_\Gamma} \|g(\phi)\| + \frac{(p+1)b^\alpha}{\Gamma(\alpha+1)} \sup_{s\in J,\phi\in Y_\Gamma} \|f(s,\phi(s))\| + \sup_{\phi\in Y_\Gamma} \sum_{i=1}^{p} \|I_i(\phi(t_i))\|\right] \leq r,$$

where

$$Y_\Gamma = \{\phi \in PC(J,X) \mid \|\phi\| \leq r \text{ for } t \in J\}.$$

Theorem 3.2. *Suppose that* [HA], [C1], [C2], [C3] *are satisfied. Then for every* $x_0 \in X$*, the system* (1.1) *has at least a PC-mild solution on* J.

Proof. Let $x_0 \in X$ be fixed. Define an operator Q on $PC(J, X)$ by

$$(Qv)(t) = (Q_0 v)(t) + (Q_1 v)(t) + (Q_2 v)(t)$$

where

$$\begin{aligned}
(Q_0 v)(t) &= U(t,0)[x_0 + g(v)], \quad t \in J, \\
(Q_1 v)(t) &= \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} U(t,s) f\left(s, v(s)\right) ds + \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-s)^{\alpha-1} U(t,s) f\left(s, v(s)\right) ds, \quad t, s \in J, \\
(Q_2 v)(t) &= \sum_{0<t_i<t} U(t,t_i) I_i(v(t_i)), \quad t \in J.
\end{aligned}$$

Step 1. We prove that Q is a continuous mapping from Y_Γ to Y_Γ. In order to derive the continuity of Q, we need only check that Q_0, Q_1 and Q_2 are all continuous.

For this purpose, we assume that $v_n \to v$ in Y_Γ. It comes from the continuity of f that

$$f\left(s, v_n(s)\right) \to f\left(s, v(s)\right), \text{ as } n \to \infty.$$

For every, $t \in J$, we have

$$\begin{aligned}
\|(Q_1 v_n)(t) - (Q_1 v)(t)\| \le \frac{M}{\Gamma(\alpha)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \|f\left(s, v_n(s)\right) - f\left(s, v(s)\right)\| ds + \\
+ \frac{M}{\Gamma(\alpha)} \int_{t_i}^{t} (t-s)^{\alpha-1} \|f\left(s, v_n(s)\right) - f\left(s, v(s)\right)\| ds \to \\
\to 0, \text{ as } n \to \infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(t_i - s)^{\alpha-1} \|f\left(s, v_n(s)\right) - f\left(s, v(s)\right)\| \le 2(t_i - s)^{\alpha-1} \sup_{s \in J, \phi \in Y_\Gamma} \|f\left(s, \phi(s)\right)\| \in L^1(J, R^+), \\
(t - s)^{\alpha-1} \|f\left(s, v_n(s)\right) - f\left(s, v(s)\right)\| \le 2(t - s)^{\alpha-1} \sup_{s \in J, \phi \in Y_\Gamma} \|f\left(s, \phi(s)\right)\| \in L^1(J, R^+).
\end{aligned}$$

By means of the Lebesgue dominated convergence theorem we obtain that

$$\int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \| f(s, v_n(s)) - f(s, v(s)) \| ds \to 0,$$

$$\int_{t_i}^{t} (t - s)^{\alpha-1} \| f(s, v_n(s)) - f(s, v(s)) \| ds \to 0.$$

Hence $Q_1 v_n \to Q_1 v$ in Y_Γ.

It comes from the compactness of g and I_i, $i = 1, 2, \ldots, p$ that

$$\|Q_0 v_n - Q_0 v\|_{Y_\Gamma} \leq M \|g(v_n) - g(v)\| \to 0, \text{ as } n \to \infty,$$

and

$$\|Q_2 v_n - Q_2 v\|_{Y_\Gamma} \leq M \sum_{0<t_i<t} \|I_i(v_n(t_i)) - I_i(v(t_i))\| \to 0, \text{ as } n \to \infty.$$

Thus, $Q_0 v_n \to Q_0 v$ in Y_Γ and $Q_2 v_n \to Q_2 v$ in Y_Γ. So, we can deduce that Q is a continuous mapping from Y_Γ to Y_Γ.

Step 2. We show that Q is a compact operator, or Q_0, Q_1 and Q_2 are all compact operators.

Define

$$\Pi = QY_\Gamma = [Q_0 + Q_1 + Q_2] Y_\Gamma \quad \text{and} \quad \Pi(t) = \{(QY_\Gamma)(t) \mid x \in Y_\Gamma)\} \text{ for } t \in J.$$

Clearly, $\Pi(0) = \{x_0 + g(x)\}$ is a precompact since g is compact, hence, it is only necessary to check that $\Pi(t) = \{(QY_\Gamma)(t) \mid x \in Y_\Gamma)\}$ for $t \in (0, b]$ is also precompact.

For $0 < \varepsilon < t \leq b$, define

$$\Pi_\varepsilon(t) \equiv (Q_\varepsilon Y_\Gamma)(t) = \{U(t, t-\varepsilon)(QY_\Gamma)(t-\varepsilon) \mid x \in Y_\Gamma\}. \tag{3.3}$$

By our hypothesis, $\Pi_\varepsilon(t)$ is relatively compact for $t \in (\varepsilon, b]$ due to $U(\cdot, \cdot)$ is compact.

For interval $(0, t_1]$, (3.3) reduces to

$$\Pi_\varepsilon(t) \equiv (Q_\varepsilon Y_\Gamma)(t) = \{U(t, t-\varepsilon)([Q_0 + Q_1] Y_\Gamma)(t-\varepsilon) \mid x \in Y_\Gamma\}.$$

By elementary computation, we have

$$\sup_{x \in Y_\Gamma} \|(Qx)(t) - (Q_\varepsilon x)(t)\| \leq$$

$$\leq \sup \left\{ \frac{M}{\Gamma(\alpha)} \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} \sup_{s \in (0, t_1], \phi \in Y_\Gamma} \| f(s, \phi(s)) \| ds \right\} +$$

$$+ \sup \left\{ \frac{M}{\Gamma(\alpha)} \int_{0}^{t-\varepsilon} [(t-s)^{\alpha-1} - (t-\varepsilon-s)^{\alpha-1}] \sup_{s \in (0, t_1], \phi \in Y_\Gamma} \| f(s, \phi(s)) \| ds \right\} \leq$$

$$\leq [2\varepsilon^\alpha + |(t-\varepsilon)^\alpha - t^\alpha|] \frac{M}{\Gamma(\alpha+1)} \sup_{s \in (0, t_1], \phi \in Y_\Gamma} \| f(s, \phi(s)) \|.$$

It shows that the set $\Pi(t)$ can be approximated to an arbitrary degree of accuracy by a relatively compact set for $t \in (0, t_1]$. Hence, $\Pi(t)$ itself is a relatively compact set for $t \in (0, t_1]$.

For interval $(t_1, t_2]$, define

$$\Pi(t_1+0) \equiv \Pi(t_1-0)+I_1(\Pi(t_1-0)) = \Pi(t_1)+I_1(\Pi(t_1)) = \{(Qx)(t_1)+I_1(x(t_1)) \mid x \in Y_\Gamma\}.$$

By [C2], $I_1(\Pi(t_1)$ is relatively compact. So, $\Pi(t_1+0)$ is relatively compact. Then (3.3) reduces to

$$\Pi_\varepsilon(t) \equiv (Q_\varepsilon Y_\Gamma)(t) = \\ = \left\{ (Qx)(t_1+0) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t-\varepsilon} (t-\varepsilon-s)^{\alpha-1} U(t,s) f\left(s, v(s)\right) ds \mid x \in Y_\Gamma \right\}.$$

By elementary computation again, we have

$$\sup_{x\in Y_\Gamma} \|(Qx)(t) - (Q_\varepsilon x)(t)\| \le \\ \le \frac{M}{\Gamma(\alpha)} \int_{t_1}^{t-\varepsilon} [(t-\varepsilon-s)^{\alpha-1} - (t-s)^{\alpha-1}] \sup_{s\in(t_1,t_2],\phi\in Y_\Gamma} \|f\left(s, \phi(s)\right)\| ds + \\ + \frac{M}{\Gamma(\alpha)} \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} \sup_{s\in(t_1,t_2],\phi\in Y_\Gamma} \|f\left(s, \phi(s)\right)\| ds \le \\ \le [2\varepsilon^\alpha + |(t-t_1)^\alpha - (t-\varepsilon-t_1)^\alpha|] \frac{M}{\Gamma(\alpha+1)} \sup_{s\in(t_1,t_2],\phi\in Y_\Gamma} \|f\left(s, \phi(s)\right)\|.$$

Hence, $\Pi(t)$ itself is a relatively compact set for $t \in (t_1, t_2]$.

In general, for any given t_i, $i = 1, 2, \ldots, p$, we define that $x(t_i + 0) = x_i$, and

$$\Pi(t_i+0) \equiv \Pi(t_i-0) + I_i(\Pi(t_i-0)) = \Pi(t_i) + I_i(\Pi(t_i)) = \\ = \{(Qx)(t_i) + I_i(x(t_i)) \mid x \in Y_\Gamma\}, \quad i = 1, 2, \ldots, p.$$

By [C2] again, $I_i(\Pi(t_i)$ is relatively compact and the associated $\Pi_\varepsilon(t)$ over the interval $(t_i, t_{i+1}]$ is given by

$$\Pi_\varepsilon(t) \equiv (Q_\varepsilon Y_\Gamma)(t) = \left\{ (Qx)(t_i+0) + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t-\varepsilon} (t-\varepsilon-s)^{\alpha-1} U(t,s) f\left(s, v(s)\right) ds \mid x \in Y_\Gamma \right\}.$$

And

$$\sup_{x\in Y_\Gamma} \|(Qx)(t) - (Q_\varepsilon x)(t)\| \le \\ \le [2\varepsilon^\alpha + |(t-t_i)^\alpha - (t-\varepsilon-t_i)^\alpha|] \frac{M}{\Gamma(\alpha+1)} \sup_{s\in(t_i,t_{i+1}],\phi\in Y_\Gamma} \|f\left(s, \phi(s)\right)\|.$$

Now, we repeat the procedures until the time interval is expanded. Thus, we obtain that the set $\Pi(t)$ itself is relatively compact for $t \in J \setminus \{t_1, t_2, \dots, t_p\}$ and $\Pi(t_i + 0)$ is relatively compact for t_i, $i = 1, 2, \dots, p$.

Step 3. We show that Π is equicontinuous on the interval (t_i, t_{i+1}).

For interval $(0, t_1)$, we note that for $t_1 > h > 0$,

$$\|(Qx)(h) - (Qx)(0)\| \leq \|U(h,0) - I\|[\|x_0\| + \sup_{\phi \in Y_\Gamma} \|g(\phi)\|] + \\ + \frac{Mh^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0,t_1), \phi \in Y_\Gamma} \|f(s, \phi(s))\|,$$

and, for $t_1 \geq t + h \geq t \geq \gamma \geq 0$, $\gamma < h$ and $x \in Y_\Gamma$,

$$\begin{aligned}(Qx)(t+h) - (Qx)(h) = {} & U(t+h,h)(U(h,0) - I)[x_0 + g(x)] + \\ & + \frac{1}{\Gamma(\alpha)} \int_t^{t+h} (t+h-s)^{\alpha-1} U(t+h,s) f(s, v(s))\, ds + \\ & + \frac{1}{\Gamma(\alpha)} \int_{t-\gamma}^{t} (t+h-s)^{\alpha-1} [U(t+h,s) - U(t,s)] f(s, v(s))\, ds + \\ & + \frac{1}{\Gamma(\alpha)} \int_{t-\gamma}^{t} [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] U(t,s) f(s, v(s))\, ds + \\ & + \frac{1}{\Gamma(\alpha)} \int_{0}^{t-\gamma} (t+h-s)^{\alpha-1} [U(t+h,s) - U(t,s)] f(s, v(s))\, ds + \\ & + \frac{1}{\Gamma(\alpha)} \int_{0}^{t-\gamma} [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] U(t,s) f(s, v(s))\, ds,\end{aligned}$$

hence,

$$\begin{aligned}& \|(Qx)(t+h) - (Qx)(h)\| \leq M\|U(h,0)x_0 - x_0\| + M\|U(h,0) - I\| \sup_{\phi \in Y_\Gamma} \|g(\phi)\|] + \\ & + M \frac{h^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0,t_1), \phi \in Y_\Gamma} \|f(s, \phi(s))\| + \\ & + \frac{|h^\alpha - (h+\gamma)^\alpha|}{\Gamma(\alpha+1)} \sup_{s \in (0,t_1), \phi \in Y_\Gamma} \|f(s, \phi(s))\| \int_0^{t-\gamma} \|U(t+h,s) - U(t,s)\| ds + \\ & + M \frac{|\gamma^\alpha - (h+\gamma)^\alpha| + h^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0,t_1), \phi \in Y_\Gamma} \|f(s, \phi(s))\| + \\ & + \frac{|(t+h)^\alpha - (\gamma+h)^\alpha|}{\Gamma(\alpha+1)} \sup_{s \in (0,t_1), \phi \in Y_\Gamma} \|f(s, \phi(s))\| \int_0^{t-\gamma} \|U(t+h,s) - U(t,s)\| ds + \\ & + M \frac{|(t+h)^\alpha - t^\alpha| + |(\gamma+h)^\alpha - h^\alpha|}{\Gamma(\alpha+1)} \sup_{s \in (0,t_1), \phi \in Y_\Gamma} \|f(s, \phi(s))\|.\end{aligned} \tag{3.4}$$

Since $\lim_{h\to 0}\|U(t+h,s)-U(t,s)\|=0$, for all $b\ge t\ge s\ge 0$, thus the right hand side of (3.4) can be made as small as desired by choosing h sufficiently small. Hence, $\Pi(t)$ is equicontinuous in interval $(0,t_1)$.

In general, for time interval (t_i,t_{i+1}), we similarly obtain the following inequalities

$$\begin{aligned}
&\|(Qx)(t+h)-(Qx)(h)\|\le\\
&\le M\|U(h,0)x_0-x_0\|+M\|U(h,0)-I\|\sup_{\phi\in Y_\Gamma}\|g(\phi)\|]+\\
&\quad+M\frac{h^\alpha}{\Gamma(\alpha+1)}\sup_{s\in(t_i,t_{i+1}),\phi\in Y_\Gamma}\|f(s,\phi(s))\|+\\
&\quad+\frac{|h^\alpha-(h+\gamma)^\alpha|}{\Gamma(\alpha+1)}\sup_{s\in(t_i,t_{i+1}),\phi\in Y_\Gamma}\|f(s,\phi(s))\|\int_0^{t-\gamma}\|U(t+h,s)-U(t,s)\|ds+\\
&\quad+M\frac{|\gamma^\alpha-(h+\gamma)^\alpha|+h^\alpha}{\Gamma(\alpha+1)}\sup_{s\in(t_i,t_{i+1}),\phi\in Y_\Gamma}\|f(s,\phi(s))\|+\\
&\quad+\frac{|(t+h)^\alpha-(\gamma+h)^\alpha|}{\Gamma(\alpha+1)}\sup_{s\in(t_i,t_{i+1}),\phi\in Y_\Gamma}\|f(s,\phi(s))\|\int_0^{t-\gamma}\|U(t+h,s)-U(t,s)\|ds+\\
&\quad+M\frac{|(t+h)^\alpha-t^\alpha|+|(\gamma+h)^\alpha-h^\alpha|}{\Gamma(\alpha+1)}\sup_{s\in(t_i,t_{i+1}),\phi\in Y_\Gamma}\|f(s,\phi(s))\|.
\end{aligned}$$

Using the same method, we know that Π is equicontinuous on the interval (t_i,t_{i+1}).

Step 4. We prove that Q has a fixed point in $PC(J,X)$.

By the generalized Ascoli-Arzela theorem, we know that QY_Γ is a relatively compact subset of $PC(J,X)$. Thus, Q is a compact operator. By [C3], we know that the set $\{x\in Y_\Gamma \mid x=\sigma Qx, \sigma\in[0,1]\}$ is bounded subset of $Y_\Gamma\subset PC(J,X)$. Thus, by the Schaefer fixed point theorem, we obtain that Q has a fixed point in $Y_\Gamma\subset PC(J,X)$. This completes the proof. □

Case 3. f is not uniformly Lipschitz, g is uniformly Lipschitz, I_i and $U(\cdot,\cdot)$ are compact

We introduce the following assumptions:

[D1]: f: $J\times X\to X$ is continuous and there exists a function $\rho_f\in L^\infty(J,R^+)$ such that

$$\|f(t,x)\|\le\rho_f(t),\ \text{for all } x\in X \text{ and } t\in J.$$

[D2]: I_i: $X\to X$, $i=1,2,\dots,p$, are compacts.

[D3]: g: $PC(J,X)\to X$ and there exists a constant $L_g>0$ such that

$$\|g(x)-g(y)\|\le L_g\|x-y\|_{PC},\ \text{for all } t\in J,\ x,y\in PC(J,X).$$

Theorem 3.3. *Suppose that* [HA], [D1], [D2], [D3] *are satisfied. Then system* (1.1) *has at least a PC-mild solution on J provided that*

$$ML_g<\frac{1}{2}.$$

Proof. Choose

$$\varrho \geq 2M\left[\|x_0\| + \|g(0)\| + \frac{(p+1)b^\alpha}{\Gamma(\alpha+1)}\|\rho_f\|_{L^\infty(J,R^+)} + \sup_{\phi\in Y_\Gamma}\sum_{i=1}^{p}\|I_i(\phi(t_i))\|\right].$$

and consider

$$B_\varrho = \{x \in PC(J,X) \mid \|x\| \leq \varrho\}.$$

Define on B_ϱ the operators L and N by

$$(\mathcal{L}x)(t) = U(t,0)[x_0 + g(x)],$$

and

$$(\mathcal{N}x)(t) = \frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}U(t,s)f\left(s, x(s)\right)ds +$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-s)^{\alpha-1}U(t,s)f\left(s,x(s)\right)ds + \sum_{0<t_i<t}U(t,t_i)I_i(x(t_i)).$$

Similarly to prove Theorem 3.1, it is suffices to proceed exactly as in Step 1 to Step 4 of the proof while replacing Y_Γ by B_ϱ to obtain that $\mathcal{N}$ is continuous and compact. Thus, to complete the rest of the proof, it suffices to show that $\mathcal{L}$ is a contraction mapping and that if $x, y \in B_\varrho$ then $\mathcal{L}x + \mathcal{N}x \in B_\varrho$. Indeed, for any $x \in B_\varrho$, we have

$$\|(\mathcal{L}x)(t) + (\mathcal{N}x)(t)\| \leq$$
$$\leq \|U(t,0)(x_0 + g(0))\| + \|U(t,0)(g(x) - g(0))\| +$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}\|U(t,s)\|\|f\left(s, x(s)\|\right)ds +$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-s)^{\alpha-1}\|U(t,s)\|\|f\left(s,v(s)\right)\|ds + \sum_{0<t_i<t}\|U(t,t_i)\|\|I_i(x(t_i))\| \leq$$
$$\leq M\left[\|x_0\| + \|g(0)\| + \|g(x) - g(0)\| + \frac{(p+1)b^\alpha}{\Gamma(\alpha+1)}\|\rho_f\|_{L^\infty(J,R^+)} + \sup_{\phi\in Y_\Gamma}\sum_{i=1}^{p}\|I_i(\phi(t_i))\|\right].$$

Since $ML_g < \frac{1}{2}$, we can deduce that

$$\|(\mathcal{L}x)(t) + (\mathcal{N}x)(t)\| \leq M\left[\|x_0\| + \|g(0)\| + L_g\varrho + \frac{(p+1)b^\alpha}{\Gamma(\alpha+1)}\|\rho_f\|_{L^\infty(J,R^+)} +\right.$$
$$\left. + \sup_{\phi\in Y_\Gamma}\sum_{i=1}^{p}\|I_i(\phi(t_i))\|\right] \leq \varrho.$$

Hence, we can deduce that

$$\|(\mathcal{L}x) + (\mathcal{N}x)\|_{PC} \le \varrho.$$

Next, for any $t \in J$, $x, y \in PC(J, X)$, we have

$$\|(\mathcal{L}x)(t) - (\mathcal{L}y)(t)\| \le ML_g \|x - y\|_{PC}.$$

Therefore, we deduce that

$$\|(\mathcal{L}x) - (\mathcal{L}y)\|_{PC} \le ML_g \|x - y\|_{PC}.$$

And since $ML_g < \frac{1}{2}$, then $\mathcal{L}$ is contraction mapping.

As a result, by the Krasnoselskii fixed point theorem, we can deduce that system (1.1) has at least one PC-mild solution on J. □

Case 4. f **is uniformly Lipschitz,** g **is not Lipschitz and not compact,** I_i **and** $U(\cdot,\cdot)$ **are compact**

We need the following assumptions:

[P1]: $f\colon J \times X \to X$ is continuous and there exists a constant $L_f > 0$ such that

$$\|f(t,x) - f(t,y)\| \le L_f \|x - y\|, \quad t \in J,\ x, y \in X.$$

[P2]: $I_i\colon X \to X$, $i = 1, 2, \ldots, p$, are compact operators.

[P3]: For each $x_0 \in X$, there exists a constant $r > 0$ such that

$$M\left[\|x_0\| + \sup_{\phi \in Y_\Gamma} \|g(\phi)\| + \frac{(p+1)b^\alpha}{\Gamma(\alpha+1)} \sup_{s \in J, \phi \in Y_\Gamma} \|f(s, \phi(s))\| + \sup_{\phi \in Y_\Gamma} \sum_{i=1}^{p} \|I_i(\phi(t_i))\|\right] \le r,$$

where

$$Y_\Gamma = \{\phi \in PC(J, X) \mid \|\phi\| \le r \text{ for } t \in J\}.$$

[P4]: $g\colon PC(J, X) \to X$ is continuous and maps Y_Γ into a bounded set, and there is a $\delta = \delta(r) \in (0, t_1)$ such that $g(\phi) = g(\psi)$ for any $\phi, \psi \in Y_\Gamma$ with $\phi(s) = \psi(s)$, $s \in [\delta, b]$.

Theorem 3.4. *Suppose that* [HA], [P1], [P2], [P3], [P4] *are satisfied. Then system* (1.1) *has at least a PC-mild solution on J provided that*

$$\frac{ML_f(p+1)b^\alpha}{\Gamma(\alpha+1)} < 1. \tag{3.5}$$

Proof. For $\delta = \delta(r) \in (0, t_1)$, set

$$Y(\delta) = PC([\delta, b], X) = \text{ restriction of functions in } PC(J, X) \text{ on } [\delta, b],$$

and

$$Y_r(\delta) = \{\phi \in Y(\delta) \mid \|\phi\| \le r \text{ for } t \in [\delta, b]\}.$$

For $v \in Y_r(\delta)$ fixed, we define a mapping $\mathcal{F}_v$ on $Y_r(\delta)$ by

$$(\mathcal{F}_v\phi)(t) = U(t,0)[u_0 + g(\widetilde{v})] + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} U(t,s) f(s, \phi(s))\, ds +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-s)^{\alpha-1} U(t,s) f(s,\phi(s))\, ds + \sum_{0<t_i<t} U(t,t_i) I_i(v(t_i)), \quad t \in J,$$

where

$$\widetilde{v}(t) = \begin{cases} v(t), \text{ if } t \in [\delta, b], \\ v(\delta), \text{ if } t \in [0,\delta]. \end{cases} \tag{3.6}$$

From our assumptions, $\mathcal{F}$ is a continuous mapping from Y_r to Y_r. Moreover, we can see

$$\|(\mathcal{F}_v\phi)(t) - (\mathcal{F}_v\psi)(t)\| \le \frac{ML_f(p+1)b^\alpha}{\Gamma(\alpha+1)} \sup_{s\in[0,t]} \|\phi(s) - \psi(s)\|, t \in J, \phi, \psi \in Y_r.$$

Thus,

$$\|(\mathcal{F}_v\phi) - (\mathcal{F}_v\psi)\|_{PC} \le \frac{ML_f(p+1)b^\alpha}{\Gamma(\alpha+1)} \|\phi - \psi\|_{PC}, t \in J, \phi, \psi \in Y_r.$$

It comes from (3.4) that $\mathcal{F}_v$ is a contraction operator on Y_r. This implies that $\mathcal{F}_v$ has a unique fixed point $\phi_v \in Y_r$ given by

$$\phi_v(t) = U(t,0)[u_0 + g(\widetilde{v})] + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} U(t,s) f(s, \phi_v(s))\, ds +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-s)^{\alpha-1} U(t,s) f(s,\phi_v(s))\, ds + \sum_{0<t_i<t} U(t,t_i) I_i(v(t_i)), \quad t \in J.$$

Based on this fact, we define a mapping $\mathcal{G}$ from $Y_r(\delta)$ to $Y_r(\delta)$ by

$$(\mathcal{G}v)(t) = \phi_v(t) \mid_{[\delta,b]} =$$

$$= U(t,0)[u_0 + g(\widetilde{v})] + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} U(t,s) f(s, \phi_v(s))\, ds +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-s)^{\alpha-1} U(t,s) f(s,\phi_v(s))\, ds + \sum_{0<t_i<t} U(t,t_i) I_i(v(t_i)), \ \ t \in [\delta, b].$$

Similarly to the above proof of Theorem 3.1, we can verify that $\mathcal{G}$ is a compact operator. Therefore, we can use the Schaefer fixed point theorem to conclude that $\mathcal{G}$ has a fixed point $v_* \in Y_r(\delta)$.

Put $u = \phi_{v_*}$. Then we have

$$\begin{aligned} u(t) = U(t,0)[u_0 + g(\widetilde{v_*})] + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} U(t,s) f(s, u(s))\, ds + \\ + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-s)^{\alpha-1} U(t,s) f(s,u(s))\, ds + \sum_{0<t_i<t} U(t,t_i) I_i(v_*(t_i)), \quad t \in J. \end{aligned} \tag{3.7}$$

But

$$g(\widetilde{v_*}) = g(v) \quad \text{and} \quad v_*(t_i) = u(t_i),$$

since

$$v_*(t) = \mathcal{G}(v_*)(t) = \phi_{v_*}(t) = u(t), \quad t \in [\delta, b],$$

by the definition of $\mathcal{G}$. This concludes, together with (3.7), that u is just a PC-mild solution of the system (1.1). This completes the proof. □

4. AN APPLICATION

In this section, an example is given to illustrate our theory. Consider the following fractional differential equation with impulse

$$\begin{cases} D_t^\alpha x(t,y) = (t+1)\frac{\partial^2}{\partial y^2} x(t,y) + \frac{e^{-t}}{e^t + e^{-t}} \cdot \frac{|x(t,y)|}{1+|x(t,y)|} + e^{-t}, \\ \qquad\qquad \alpha \in (0,1), y \in (0,\pi), t \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1], \\ \Delta x(\frac{1}{2}, y) = \int_0^\pi \rho_1(\frac{1}{2}, y) \cos^2(x(\frac{1}{2}, y)) dy, \ t_1 = \frac{1}{2}, y \in (0,\pi), \rho_1 \in C([0,\pi] \times [0,\pi], R), \\ x(0,y) = \sum_{j=1}^{2} \lambda_j x(t,y) + x(1,y), \quad 0 < \lambda_1 < \lambda_2, \lambda_1, \lambda_2, t \in [0,1], y \in (0,\pi). \end{cases} \tag{4.1}$$

Let $X = L^2([0,\pi])$. Define $D = H^2([0,\pi]) \bigcap H_0^1([0,\pi])$, and $A(t)x = (t+1)\frac{\partial^2}{\partial y^2} x$ for $x \in D$ which can determined a strongly continuous evolutionary process $\{U(\cdot,\cdot)\}$ in $L^2([0,\pi])$ and it is also compact and there exists a $M > 0$ such that $\|U(\cdot,\cdot)\| \le M$.

Define $x(\cdot)(y) = x(\cdot, y)$,

$$f(\cdot, x(\cdot))(y) = \frac{e^{-\cdot}}{e^{\cdot} + e^{-\cdot}} \cdot \frac{|x(\cdot,y)|}{1 + |x(\cdot,y)|} + e^{-\cdot},$$

$$I_1(x(\cdot,y)) = \int_0^\pi \rho_1(\cdot,y) \cos^2(x(\cdot,y)) dy, \quad g(x(\cdot))(y) = \sum_{j=1}^{2} \lambda_j x(\cdot,y).$$

Thus, problem (4.1) can be rewritten as

$$\begin{cases} D_t^\alpha x(t) = A(t)x(t) + f(t,x(t)), \quad \alpha \in (0,1), t \in [0,1] \setminus \{t_1\}, \\ \Delta x(t_1) = I_1(x(t_1)), \quad t_1 = \frac{1}{2}, \\ x(0) = g(x) + x(1). \end{cases} \tag{4.2}$$

Clearly, $f : [0,1] \times R \to R$ and

$$|f(t,x)| \le \frac{e^{-t}}{e^t + e^{-t}} + e^{-t} \equiv \rho_f(t).$$

$I_1 : R \to R$ is compact, g: $PC([0,1], L^2([0,\pi])) \to L^2([0,\pi])$ and

$$|g(x_1) - g(x_2)| \le L_g|x_1 - x_2|, \quad L_g = \sum_{j=1}^{2} \lambda_j.$$

By choosing λ_j $(j = 1,2)$ small enough such that $M\sum_{j=1}^{2} \lambda_j < \frac{1}{2}$. Then all the assumptions in Theorem 3.3 are satisfied. Therefore, the problem (4.1) has at least one PC-mild solution.

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