Opuscula Mathematica • Vol. 30 • No. 3 • 2010

http://dx.doi.org/10.7494/OpMath.2010.30.3.311

ASYMPTOTIC BEHAVIOUR AND APPROXIMATION OF EIGENVALUES FOR UNBOUNDED BLOCK JACOBI MATRICES

Maria Malejki

Abstract. The research included in the paper concerns a class of symmetric block Jacobi matrices. The problem of the approximation of eigenvalues for a class of a self-adjoint unbounded operators is considered. We estimate the joint error of approximation for the eigenvalues, numbered from 1 to N, for a Jacobi matrix J by the eigenvalues of the finite submatrix J_n of order $pn \times pn$, where $N = \max\{k \in \mathbb{N} : k \leq rpn\}$ and $r \in (0, 1)$ is suitably chosen. We apply this result to obtain the asymptotics of the eigenvalues of J in the case $p = 3$.

Keywords: symmetric unbounded Jacobi matrix, block Jacobi matrix, tridiagonal matrix, point spectrum, eigenvalue, asymptotics.

Mathematics Subject Classification: 47A75, 47B25, 47B36, 15A18.

1. INTRODUCTION

Tridiagonal matrices are very useful in many problems in mathematics and in applications, and the theory and methods related to tridiagonal matrices are still developed and generalized (see [20]). In the context of advances and applications, block tridiagonal matrices are very interesting (see, e.g., [6] and [8]). This work is devoted to spectral properties of a class of block Jacobi matrices with discrete spectrum. The problem, when the linear operator defined by a Jacobi matrix has discrete spectrum, i.e., its spectrum consists of isolated eigenvalues of finite multiplicity, was already investigated and partially solved (see, e.g., [7, 10] and [12]). It is well known that sometimes it is possible to calculate exact formulas for eigenvalues of Jacobi matrices (see, e.g., [9,18] and [11]), but it is not possible in general. So, asymptotic and approximate approaches have to be applied (see, e.g., [3–5,12,13,17,21] and [22]). Projective methods, that use finite submatrices to investigate spectral properties of operators given by infinite Jacobi matrices are applied successfully (see [1, 2, 10, 15, 16, 21]). In this paper we continue the research related to the approximation of the discrete spectrum of selfadjoint operators in the Hilbert space $l^2(\mathbb{N})$ and generalize the results included in [16] and [15].

The paper is organized as follows. In Section 2 we introduce conditions that are needed to apply the projective method and obtain the result. The method, that is used in this paper, is based on the Volkmer's results ([21]). Section 3 includes a generalization of the lemmta, which come originally from [21], and other technical facts. In section 4 we formulate the main result of the article. There we estimate the joint error of approximation for the eigenvalues, numbered from 1 to N, of J by the eigenvalues of the finite submatrix J_n of order $pn \times pn$, where $N = \max\{k \in \mathbb{N} : k \leq rpn\}$ and $r \in (0, 1)$ is suitably chosen. Section 5 is devoted to an application of the main result to obtain asymptotic formulas for the eigenvalues of an operator that is defined by an infinite real symmetric 5-diagonal matrix and acts in the Hilbert space $l^2(\mathbb{N})$.

2. NOTATIONS AND PRELIMINARIES

The notations $(\cdot, \cdot)$ and $\|\cdot\|$ are used for an inner product and a norm, respectively, in the Euclidian space $\mathbb{C}^p$ as well as in any Hilbert spaces. Moreover, the notation $\|\cdot\|$ is also used for the operator norm.

Let $M_{k\times l}(\mathbb{C})$ be the set of complex matrices with k rows and l columns for any integers $k, l \geq 1$.

Next we introduce some concepts from abstract operator theory which we will need later. Let H be a Hilbert space and $T : D(T) \subset H \to H$ be a self-adjoint operator in H. Assume that T has a compact resolvent and is bounded from below in the sense that there exists $c \in \mathbb{R}$ such that $(Tf, f) \geq c\|f\|^2$ for $f \in D(T)$. Then the spectrum of T consists of the eigenvalues that can be ordered non-decreasingly: $\lambda_1(T) \leq \lambda_2(T) \leq \lambda_3(T) \leq \ldots$. By the minimum-maximum principle, for all $k \in \mathbb{N}$, there holds

$$\lambda_k(T) = \min_{E_k} \max\{(Tx, x) : x \in E_k, \|x\| = 1\}, \tag{2.1}$$

where the minimum is taken over all linear subspaces $E_k \subseteq D(T)$ of dimension k.

Denote by x_k the eigenvector of T associated with the eigenvalue $\lambda_k(T)$. We will assume that the system of eigenvectors $\{x_1, x_2, x_3, \ldots\}$ is orthonormal in H, so it forms an orthonormal basis of H.

Let E_N be a N-dimensional subspace of H. Assume that $E_N \subset D(T)$. Denote by P_N the orthogonal projection onto E_N and $Q_N = I - P_N$. Let us consider the following operator on E_N :

$$T_N : E_N \ni v \to P_N T v \in E_N.$$

Denote by μ_i, $1 \leq i \leq N$, the eigenvalues of T_N by assuming that $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_N$.

For any $k = 1, \ldots, N$, define

$$L^{(k)} = (L_{i,j})_{i,j=1,\ldots,k} \in M_{k\times k}(\mathbb{C}) \quad \text{with} \quad L_{i,j} = (Q_N x_i, x_j),$$

and

$$M^{(k)} = (M_{i,j})_{i,j=1,\dots,k} \in M_{k\times k}(\mathbb{C}) \quad \text{with} \quad M_{i,j} = ((P_N T P_N - T)x_i, x_j).$$

The following lemma is fundamental to obtain the results in this paper.

Lemma 2.1 (Volkmer [21]). *If* $\|L^{(k)}\| < 1$ *then*

$$0 \le \mu_k - \lambda_k(T) \le \frac{\|M^{(k)} + \lambda_k(T)L^{(k)}\|}{1 - \|L^{(k)}\|},$$

where $1 \le k \le n$.

Let $p \ge 1$ be an integer and also denote

$$l^2(\mathbb{N}, \mathbb{C}^p) = \left\{ \{f_n\}_{n=1}^{\infty} : f_n \in \mathbb{C}^p,\ n \ge 1, \text{ and } \sum_{k=1}^{\infty} \|f_k\|^2 < +\infty \right\}.$$

Consider a Jacobi operator J in the Hilbert space $l^2 = l^2(\mathbb{N}, \mathbb{C}^p)$ given by the symmetric block Jacobi matrix

$$J = \begin{pmatrix} D_1 & C_1^* & 0 & \cdots & \cdots \\ C_1 & D_2 & C_2^* & 0 & \ddots \\ 0 & C_2 & D_3 & C_3^* & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad D_n = D_n^*,\ C_n \in M_{p\times p}(\mathbb{C}),\ n \ge 1, \tag{2.2}$$

more exactly, J acts on the maximum domain

$$D(J) = \left\{ \{f_n\}_{n=1}^{\infty} \in l^2 : \{C_{n-1}f_{n-1} + D_n f_n + C_n^* f_{n+1}\}_{n=1}^{\infty} \in l^2 \right\}, \tag{2.3}$$

and it is defined by

$$Jf = \{C_{n-1}f_{n-1} + D_n f_n + C_n^* f_{n+1}\}_{n=1}^{\infty} \text{ for } f = \{f_n\}_{n=1}^{\infty} \in D(J),$$

where $f_n \in \mathbb{C}^p,\ n \ge 1$ and $C_0 := 0$.

Denote

$$d_n^{min} = \inf\{(D_n f, f) :\ f \in \mathbb{C}^p,\ \|f\| = 1\}, \tag{2.4}$$

$$d_n^{max} = \sup\{(D_n f, f) :\ f \in \mathbb{C}^p,\ \|f\| = 1\}. \tag{2.5}$$

We assume the following conditions:

(C1) $D_n = D_n^*$ for $n \ge 1$ and there exist $\alpha > 0$, $\delta_1 \ge \delta_2 > 0$ and $\{\epsilon_n\}_{n=1}^{\infty} \subset [0, +\infty)$, $\lim_{n\to\infty} \epsilon_n = 0$, such that

$$\delta_2 n^{\alpha}(1 - \epsilon_n) \le d_n^{min} \le d_n^{max} \le \delta_1 n^{\alpha}(1 + \epsilon_n),\ n \ge 1;$$

(C2) there exist $\beta \in \mathbb{R}$ and $S > 0$ such that

$$\|C_n\| \leq S n^{\beta}, \ n \geq 1;$$

(C3) $\alpha > \beta$.

Proposition 2.2. *If* (C1)–(C3) *are satisfied then:*

1. $D(J) = \{\{f_n\}_{n=1}^{\infty} \in l^2 : \{D_n f_n\} \in l^2\}$,
2. *J is a selfadjoint operator in l^2,*
3. *J is bounded from below,*
4. *$(J-\lambda)^{-1}$ is compact for any λ belonging to the resolvent set of J.*

Proof. Let

$$c = \inf\{d_n^{min} - 2Sn^{\beta} : \ n \geq 1\},$$

then (C1)–(C3) yield $c \in \mathbb{R}$. Denote

$$A = \begin{pmatrix} D_1 & 0 & 0 & & \\ 0 & D_2 & 0 & 0 & \ddots \\ 0 & 0 & D_3 & 0 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} 0 & C_1^* & 0 & & \\ C_1 & 0 & C_2^* & 0 & \ddots \\ 0 & C_2 & 0 & C_3^* & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let $\lambda \in (-\infty, c)$. For $n \geq 1$, $D_n - \lambda$ is invertible and

$$\|(D_n - \lambda)^{-1}\| \leq (d_n^{min} - \lambda)^{-1},$$

because $\lambda < d_n^{min}$. Moreover, the operator given by the matrix $A - \lambda$ is also invertible, $(A-\lambda)^{-1}$ is a compact operator on l^2 and

$$\|(A-\lambda)^{-1}\| \leq \sup_{n \geq 1} (d_n^{min} - \lambda)^{-1} < +\infty,$$

because

$$\lim_{n\to\infty} (d_n^{min} - \lambda)^{-1} = 0.$$

Next calculate

$$B(A-\lambda)^{-1} = \begin{pmatrix} 0 & C_1^*(D_2-\lambda)^{-1} & 0 & \ddots & \\ C_1(D_1-\lambda)^{-1} & 0 & C_2^*(D_3-\lambda)^{-1} & 0 & \\ 0 & C_2(D_2-\lambda)^{-1} & 0 & C_3^*(D_4-\lambda)^{-1} & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The operator norm for the matrix $B(A-\lambda)^{-1}$ is estimated as follows

$$\|B(A-\lambda)^{-1}\| \leq 2 \sup\{Sn^{\beta}(d_n^{min} - \lambda)^{-1} : n \geq 1\},$$

because

$$\|C_n(D_n-\lambda)^{-1}\| \le Sn^\beta(d_n^{min}-\lambda)^{-1}, \quad n\ge 1,$$

and

$$\|C_{n-1}^*(D_n-\lambda)^{-1}\| \le Sn^\beta(d_n^{min}-\lambda)^{-1}, \quad n\ge 2.$$

Clearly, $\lim_{n\to\infty} n^\beta(d_n^{min}-\lambda)^{-1}=0$, so

$$2\sup_{n\ge 1}\{Sn^\beta(d_n^{min}-\lambda)^{-1}\} = 2Sn_0^\beta(d_{n_0}^{min}-\lambda)^{-1}$$

for a $n_0\in\mathbb{N}$. Notice that

$$2Sn_0^\beta(d_{n_0}^{min}-\lambda)^{-1}<1 \iff \lambda < d_{n_0}^{min}-2Sn_0^\beta.$$

The last inequality is satisfied because $\lambda<c$. Thus $\|B(A-\lambda)^{-1}\|<1$ and we observe the infinite matrix $I+B(A-\lambda)^{-1}$ acts as a bounded and boundedly invertible operator in l^2.

Notice that the matrices J, A and B satisfy the following formal identity:

$$J-\lambda = A-\lambda+B = (I+B(A-\lambda)^{-1})(A-\lambda). \tag{2.6}$$

Consequently

$$D(J)=D(A)=\{\{f_n\}\in l^2 : \{D_nf_n\}\in l^2\}; \tag{2.7}$$

moreover,

$$(J-\lambda)^{-1} = (A-\lambda)^{-1}(I+B(A-\lambda)^{-1})^{-1}.$$

Thus $(J-\lambda)^{-1}$ is compact for $\lambda<c$ and, therefore, for all λ from the resolvent set. In particular, due to the fact that J is symmetric, it follows that J, in fact, is a self-adjoint operator in l^2. Consequently, we had proved that J is bounded from below by a lower bound c and $(Jf,f)\ge c\|f\|^2$ for $f\in D(J)$ because $(-\infty,c)$ is included in the resolvent set of J. □

Let J be an operator given by (2.2) and assume (C1)–(C3). The spectrum of J consists of the sequence of the eigenvalues of finite multiplicities only:

$$\sigma(J)=\{\lambda_k(J) : k=1,2,3,\ldots\},$$

and we can assume

$$\lambda_1(J)\le\lambda_2(J)\le\lambda_3(J)\le\ldots.$$

Let $x_i\in l^2$ be an eigenvector of J, such that $Jx_i=\lambda_i(J)x_i$ $(i=1,2,3,\ldots)$. Moreover, we can assume $\{x_i : i=1,2,3,\ldots\}$ is an orthonormal basis in l^2. Let

$$x_i=\{x_{i,n}\}_{n=1}^\infty,$$

where

$$x_{i,n}=(w_{i,(n-1)p+1}, w_{i,(n-1)p+2},\ldots,w_{i,np})^\top\in\mathbb{C}^p.$$

Then

$$\|x_i\|^2 = \sum_{n=1}^{\infty} \|x_{i,n}\|^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{p} |w_{i,(n-1)p+k}|^2 = 1.$$

Denote $e_i = \{\Delta_{i,n}\}_{n=1}^{\infty}$ for $i = 1, 2, 3, \ldots$, where $\Delta_{i,n}$ is defined as follow. If $i = (n-1)p + k$, where $n \geq 1$ and $k \in \{1, 2, \ldots, p\}$, then

$$\Delta_{i,m} = (0, 0, \ldots, 0)^\top \in \mathbb{C}^p, \; for \; m \neq n,$$

and

$$\Delta_{i,n} = (\delta_{1,k}, \delta_{2,k}, \ldots, \delta_{p,k})^\top \in \mathbb{C}^p, \text{ where } \delta_{t,s} = \begin{cases} 0, & t \neq s, \\ 1, & t = s. \end{cases}$$

The system $\{e_i : i = 1, 2, 3, \ldots\}$ is the canonical orthonormal basis in $l^2 = l^2(\mathbb{N}, \mathbb{C}^p)$.

Put

$$E_n = \operatorname{span}\{e_1, e_2, \ldots, e_{np}\}, \tag{2.8}$$

then $dimE_n = np$. Let P_n be an orthogonal projection on E_n, and let

$$J_n : E_n \ni x \to P_n J x \in E_n. \tag{2.9}$$

Then J_n is represented, with respect the canonical basis of E_n, as the matrix

$$\begin{pmatrix} D_1 & C_1^* & 0 & & & \\ C_1 & D_2 & C_2^* & 0 & & \\ 0 & C_2 & D_3 & C_3^* & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 0 & C_{n-2} & D_{n-1} & C_{n-1}^* \\ & & & 0 & C_{n-1} & D_n \end{pmatrix}. \tag{2.10}$$

Denote by

$$\mu_{1,n} \leq \mu_{2,n} \leq \ldots \mu_{np-1,n} \leq \mu_{np,n}$$

the sequence of the eigenvalues of J_n.

From the min-max principle we derive

$$\lambda_k(J) \leq \mu_{k,n} \quad \text{and} \quad \lambda_k(J) \leq \|J_n\| \leq Cn^\alpha \quad \text{for} \quad k = 1, 2, \ldots, np.$$

3. AUXILIARY ESTIMATIONS

In this section we use the notations introduced in Section 2.

Denote

$$Q_n = I - P_n. \tag{3.1}$$

Let $k \in \{1, \ldots, np\}$ and define the following $k \times k$-matrices:

$$L^{(k,n)} = (L_{i,j}^{(n)})_{i,j=1,\ldots,k}, \quad \text{where} \quad L_{i,j}^{(n)} = (Q_n x_i, x_j),$$

and

$$M^{(k,n)} = (M^{(n)}_{i,j})_{i,j=1,\ldots,k}, \quad \text{where} \quad M^{(n)}_{i,j} = ((P_nJP_n - J)x_i, x_j).$$

Lemma 3.1. *If* $n \in \mathbb{N}$ *and* $k \in \{1, 2, \ldots, np\}$, *then*

$$\|L^{(k,n)}\| \leq \sum_{i=1}^{k} \|Q_n x_i\|^2;$$

$$\|M^{(k,n)} + \lambda_k(J)L^{(k,n)}\| \leq \|C_n\|(\sum_{i=1}^{k} \|x_{i,n+1}\|^2)^{1/2}(\sum_{j=1}^{k} \|x_{j,n}\|^2)^{1/2} + \\ + (\sum_{i=1}^{k} |\lambda_k(J) - \lambda_i(J)|^2 \|Q_n x_i\|^2)^{1/2}(\sum_{j=1}^{k} \|Q_n x_j\|^2)^{1/2}.$$

Proof. The proof follows the Volkmer's method (see [21]). At first notice that $|L^{(n)}_{i,j}| = |(Q_n x_i, x_j)| = |(Q_n x_i, Q_n x_j)| \leq \|Q_n x_i\|\|Q_n x_j\|$; therefore, the operator norm $\|L^{(k,n)}\|$ of the $k \times k$ matrix can be estimated as above.

Next notice that

$$JP_nx_j = \begin{pmatrix} D_1x_{i,1} + C_1^*x_{i,2} \\ C_1x_{i,1} + D_2x_{i,2} + C_1^*x_{i,3} \\ \vdots \\ C_{n-2}x_{i,n-2} + D_{n-1}x_{i,n-1} + C_n^*x_{i,n} \\ C_{n-1}x_{i,n-1} + D_nx_{i,n} \\ C_nx_{i,n} \\ 0 \\ \vdots \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_i(J)x_{i,1} \\ \lambda_i(J)x_{i,2} \\ \vdots \\ \lambda_i(J)x_{i,n-1} \\ \lambda_i(J)x_{i,n} - C_n^*x_{i,n+1} \\ C_nx_{i,n} \\ 0 \\ \vdots \end{pmatrix} = \lambda_i(J)P_nx_i + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -C_n^*x_{i,n+1} \\ C_nx_{i,n} \\ 0 \\ \vdots \end{pmatrix}$$

and

$$P_nJP_nx_i - Jx_i = \lambda_i(J)P_nx_i + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -C_n^*x_{i,n+1} \\ 0 \\ \vdots \end{pmatrix} - \lambda_i(J)x_i = \begin{pmatrix} 0 \\ \vdots \\ -C_n^*x_{i,n+1} \\ 0 \\ \vdots \end{pmatrix} - \lambda_i(J)Q_nx_i.$$

Then

$$M_{i,j}^{(n)} = (P_n J P_n x_i - J x_i, x_j) = -(C_n^* x_{i,n+1}, x_{j,n}) - \lambda_i(J)(Q_n x_i, x_j);$$

moreover,

$$\begin{aligned}|M_{i,j}^{(n)} + \lambda_k L_{i,j}^{(n)}| &= |(P_n J P_n x_i - J x_i, x_j) + \lambda_k (Q_n x_i, x_j)| = \\ &= | - (C_n^* x_{i,n+1}, x_{i,n}) - (\lambda_i(J) - \lambda_k(J))(Q_n x_i, x_j)| \leq \\ &\leq |(C_n^* x_{i,n+1}, x_{i,n})| + |\lambda_i(J) - \lambda_k(J)||(Q_n x_i, x_j)| \leq \\ &\leq \|C_n\| \|x_{i,n+1}\| \|x_{i,n}\| + |\lambda_i(J) - \lambda_k(J)| \|Q_n x_i\| \|Q_n x_j\|.\end{aligned}$$

Finally, from the above estimation we derive the second inequality of the lemma. □

Define

$$p_n = \max\{\epsilon_k k^\alpha : k \leq n\}, \quad q_n = \max\{S n^\beta, S\}, \quad n \geq 1. \tag{3.2}$$

Lemma 3.2. *Under assumptions* $(C1)$ *and* $(C2)$*, the sequences* $\{p_n\}$ *and* $\{q_n\}$ *are non-decreasing and*

$$\lim_{n\to\infty} \frac{p_n}{n^\alpha} = 0.$$

Proof. By definition $p_n = \epsilon_{k_n} k_n^\alpha$, for some $k_n \leq n$. Assume that $\{p_n\}$ is unbounded, then $\lim_{n\to\infty} k_n = +\infty$. So,

$$\left|\frac{p_n}{n^\alpha}\right| = \frac{\epsilon_{k_n} k_n^\alpha}{n^\alpha} \leq \epsilon_{k_n} \to 0, \quad n \to \infty,$$

because $\lim_{n\to\infty} \epsilon_n = 0$. □

The following estimates are satisfied for the eigenvalues of J.

Proposition 3.3. *Assume that* (C1)–(C3) *are fulfilled. Let* $j \geq 1$, $l \in \{1, 2, \ldots, p\}$ *and* $i = (j-1)p + l$, *then*

$$\lambda_i(J) \leq \|J_j\| \leq \delta_1 j^\alpha + p_j + 2q_j.$$

Proof. Notice that $i \leq pj$. By applying the minimum-maximum principle (2.1) and using (C1)–(C3), we derive the following estimate

$$\lambda_i \leq \mu_{i,j} \leq \|J_j\| \leq \max_{1\leq k\leq j} d_k^{max} + 2 \max_{1\leq k\leq j-1} \|C_k\| \leq \delta_1 j^\alpha + p_j + 2q_j. \qquad \square$$

Let $0 < r < r' < (\delta_2/\delta_1)^{1/\alpha}$, $1 \leq j \leq r'k$ and $i = (j-1)p+l$, where $l \in \{1, 2, \ldots, p\}$. Next, follows Volkmer ([21]), we define

$$f_{i,k} = \frac{\|C_{k-1}\|}{d_k^{min} - \|J_j\| - \|C_k\|}, \quad k \geq n.$$

If $k \geq n$ then $j \leq r'k$, so from Lemma 3.2 and Proposition 3.3

$$\begin{aligned} f_{i,k} &\leq \frac{Sk^\beta}{\delta_2 k^\alpha(1-\epsilon_k) - \delta_1 j^\alpha + \delta_1 p_j - 2Sj^\beta - Sk^\beta} \leq \\ &\leq \frac{Sk^\beta}{\delta_2 k^\alpha - \delta_1 (r'k)^\alpha - \delta_2 k^\alpha \epsilon_k - \delta_1 p_k - 3Sk^\beta} = \frac{Sk^\beta}{\delta_1 k^\alpha(\delta_2/\delta_1 - r'^\alpha) + \tilde{\epsilon}_k}, \end{aligned}$$

where $\tilde{\epsilon}_k = o(k^\alpha)$, $k \to \infty$, i.e.,

$$\lim_{k\to\infty} \frac{\tilde{\epsilon}_k}{k^\alpha} = 0.$$

Therefore,

$$f_{i,k} \leq \frac{c}{k^{\alpha-\beta}} \leq \frac{1}{2} \text{ for } k \geq K_0, \tag{3.3}$$

where K_0 is large enough and $c > 0$ is a constant independent of i and k.

Lemma 3.4. *Assume* (C1)–(C3). *If* $n \geq K_0$, $1 \leq j \leq r'n$, $i = (j-1)p + l$, $1 \leq l \leq p$, *then*

$$\|x_{i,n}\| \leq f_{i,n} \|x_{i,n-1}\|.$$

Proof. If $\lambda_i(J)$ is an eigenvalue of J and x_i is a normalized eigenvector associated to $\lambda_i(J)$, then

$$C_{k-1} x_{i,k-1} + (D_k - \lambda_i(J)) x_{i,k} + C_k^* x_{i,k+1} = 0, \quad k \geq 2.$$

There exists $k \geq n$ such that $\|x_{i,k+1}\| \leq \|x_{i,k}\|$. Then

$$C_{k-1} x_{i,k-1} = -(D_k - \lambda_i(J)) x_{i,k} - C_k^* x_{i,k+1},$$

so

$$\begin{aligned} (C_{k-1} x_{i,k-1}, x_{i,k}) &= -((D_k - \lambda_i(J)) x_{i,k}, x_{i,k}) - (C_k^* x_{i,k+1}, x_{i,k}), \\ |(C_{k-1} x_{i,k-1}, x_{i,k})| &\geq ((D_k - \lambda_i(J)) x_{i,k}, x_{i,k}) - |(C_k^* x_{i,k+1}, x_{i,k})| \end{aligned}$$

and

$$\|C_{k-1}\| \|x_{i,k-1}\| \|x_{i,k}\| \geq d_k^{min} \|x_{i,k}\|^2 - \lambda_i(J) \|x_{i,k}\|^2 - \|C_k\| \|x_{i,k+1}\| \|x_{i,k}\|.$$

Assume $\|x_{i,k}\| \neq 0$. Then

$$\begin{aligned} \|C_{k-1}\| \|x_{i,k-1}\| &\geq (d_k^{min} - \lambda_i(J)) \|x_{i,k}\| - \|C_k\| \|x_{i,k+1}\| \geq \\ &\geq (d_k^{min} - \lambda_i(J)) \|x_{i,k}\| - \|C_k\| \|x_{i,k}\| = \\ &= (d_k^{min} - \|J_j\| - \|C_k\|) \|x_{i,k}\|. \end{aligned}$$

Obviously, $k \geq K_0$, so $d_k^{min} - \|J_j\| - \|C_k\| > 0$,

$$\|x_{i,k}\| \leq \frac{\|C_{k-1}\|}{d_k^{min} - \|J_j\| - \|C_k\|} \|x_{i,k-1}\| \leq \frac{1}{2} \|x_{i,k-1}\| \leq \|x_{i,k-1}\|$$

and

$$\|x_{i,k}\| \leq f_{i,k} \|x_{i,k-1}\|.$$

If $k > n$ then we can repeat this procedure to obtain $\|x_{i,n}\| \leq f_{i,n} \|x_{i,n-1}\|$. □

4. APPROXIMATION FOR EIGENVALUES OF UNBOUNDED SELF-ADJOINT JACOBI MATRICES WITH MATRIX ENTRIES BY THE USE OF FINITE SUBMATRICES

The main result of this article is formulated as the following theorem.

Theorem 4.1. *Let J be an operator in the Hilbert space l^2 defined by the infinite matrix* (2.2) *satisfying (C1)–(C3). Then for every $\gamma > 0$ and $r \in (0, (\delta_2/\delta_1)^{1/\alpha})$ there exists $C > 0$ such that*

$$\sup_{1\le k\le rnp} |\mu_{k,n} - \lambda_k(J)| \le Cn^{-\gamma} \text{ for } n > r^{-1},$$

where $\lambda_k(J)$ is the k-th eigenvalue of J and $\mu_{1,n} \le \mu_{2,n} \le \ldots \le \mu_{pn,n}$ are the eigenvalues of the matrix J_n given by (2.10).

Proof. Let $s \in \mathbb{N}$ be such that

$$2s(\alpha-\beta) - \alpha - 1 \ge \gamma, \tag{4.1}$$

and choose $r < r' < (\delta_2/\delta_1)^{1/\alpha}$ and $K_0 \in \mathbb{N}$ for which (3.3) is satisfied, and put

$$N_0 = \max\{K_0 + s, \frac{r's}{r'-r}\}.$$

For $n \ge N_0$, $1 \le j \le rn$, $i = (j-1)p + l$, where $l \in \{1, 2, \ldots, p\}$, and $m > n$, by using Lemma 3.4, we deduce

$$\|x_{i,m}\| \le f_{i,m}\|x_{i,m-1}\| \le f_{i,m}f_{i,m-1}\cdot\ldots\cdot f_{i,n+1}\|x_{i,n}\| \le \left(\frac{1}{2}\right)^{m-n}\|x_{i,n}\|.$$

If $j \le rn$ and $n \ge N_0$ then $j \le r'(n-s)$, and then

$$\|x_{i,n}\| \le f_{i,n}f_{i,n-1}\cdot\ldots\cdot f_{i,n-s}\|x_{i,n-s}\| \le \frac{c^s}{[n(n-1)\ldots(n-s+1)]^{(\alpha-\beta)}} \le \frac{M}{n^{s(\alpha-\beta)}},$$

where $M = M(s,\alpha,\beta)$ is a positive constant independent of i and n. Now, we use Lemma 3.1 to continue the proof. At first notice that

$$\|Q_n x_i\|^2 = \sum_{m=n+1}^{\infty}\|x_{i,m}\|^2 \le \|x_{i,n}\|^2 \sum_{m=n+1}^{\infty}\left(\frac{1}{4}\right)^{m-n} \le \|x_{i,n}\|^2 \le \frac{M^2}{n^{2s(\alpha-\beta)}}.$$

Let $k \le rnp$. Then

$$\|L^{(k,n)}\| \le \sum_{i=1}^{k}\|Q_n x_i\|^2 \le prn\frac{M^2}{n^{2s(\alpha-\beta)}} = \frac{C}{n^{2s(\alpha-\beta)-1}}.$$

Since the sequence $\{\lambda_m(J)\}$ is non-decreasing and since

$$\lim_{m\to\infty}\lambda_m(J) = +\infty,$$

it follows

$$\max\{|\lambda_m(J)| : \lambda_m(J) < 0\} = \mu < +\infty,$$

and then by using Proposition 3.3, we obtain

$$\lambda_k(J) - \lambda_i(J) \le \lambda_k(J) + \mu \le C_0 n^{\alpha} \quad \text{for} \quad i \le k.$$

Thus

$$\sum_{i=1}^{k} |\lambda_k(J) - \lambda_i(J)|^2 \|Q_n x_i\|^2 \le prnC_0^2 n^{2\alpha} \frac{M^2}{n^{2s(\alpha-\beta)}} = \frac{M'}{n^{2s(\alpha-\beta)-2\alpha-1}}.$$

Next,

$$\sum_{i=1}^{k} \|x_{i,n}\|^2 \le \frac{M^2 pr}{n^{2s(\alpha-\beta)-1}},$$

$$\sum_{i=1}^{k} \|x_{i,n+1}\|^2 \le \sum_{i=1}^{k} f_{i,n}^2 \|x_{i,n}\|^2 \le \frac{c^2}{n^{2(\alpha-\beta)}} \sum_{i=1}^{k} \|x_{i,n}\|^2 \le \frac{c^2 M^2 pr}{n^{2(s+1)(\alpha-\beta)-1}}$$

and, finally, from Lemma 3.1 we derive

$$\|M^{(k,n)} + \lambda_k(J) L^{(k,n)}\| \le \frac{cM^2 pr}{n^{2s(\alpha-\beta)+\alpha-\beta-1}} + \frac{CM'}{n^{2s(\alpha-\beta)-\alpha-1}} \le \frac{M''}{n^{\gamma}}.$$

Assume

$$\frac{prM^2}{n^{2s(\alpha-\beta)-1}} \le \frac{1}{2} \quad \text{for} \quad n \ge N_1,$$

where N_1 is large enough and $N_1 > N_0$. Then

$$\|L^{(k,n)}\| \le \frac{1}{2} < 1$$

and, by Lemma 2.1,

$$\mu_{k,n} - \lambda_k(J) \le \frac{2M''}{n^{\gamma}} \quad \text{for} \quad k \le rnp.$$

Finally,

$$\sup_{1 \le k \le rpn} |\mu_{k,n} - \lambda_k(J)| \le \frac{2M''}{n^{\gamma}}$$

for $n \ge N_1$, and the proof is complete. □

Theorem 4.1 generalizes the results included in [16] and [15].

5. ASYMPTOTICS

Theorem 4.1 can be applied to obtain an asymptotic behaviour of the discrete spectrum for a concrete class of operators acting in $l^2(\mathbb{N})$. Let us consider a 5-diagonal symmetric infinite matrix

$$J = \left(\begin{array}{ccc|ccc|c} \alpha_1 & \beta_1 & \gamma_1 & 0 & & & \ddots \\ \beta_1 & \alpha_2 & \beta_2 & \gamma_2 & 0 & & \ddots \\ \gamma_1 & \beta_2 & \alpha_3 & \beta_3 & \gamma_3 & 0 & \ddots \\ \hline 0 & \gamma_2 & \beta_3 & \alpha_4 & \beta_4 & \gamma_4 & \ddots \\ & 0 & \gamma_3 & \beta_4 & \alpha_5 & \beta_5 & \ddots \\ & & 0 & \gamma_4 & \beta_5 & \alpha_6 & \ddots \\ \hline & & & \ddots & \ddots & \ddots & \ddots \end{array}\right). \tag{5.1}$$

We identify this matrix with a block Jacobi matrix with 3×3-matrix entries

$$\begin{pmatrix} D_1 & C_1^* & 0 & \ddots & \ddots \\ C_1 & D_2 & C_2^* & 0 & \ddots \\ 0 & C_2 & D_3 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$D_n = \begin{pmatrix} \alpha_{3n-2} & \beta_{3n-2} & \gamma_{3n-2} \\ \beta_{3n-2} & \alpha_{3n-1} & \beta_{3n-1} \\ \gamma_{3n-2} & \beta_{3n-1} & \alpha_{3n} \end{pmatrix}, \quad C_n = \begin{pmatrix} 0 & \gamma_{3n-1} & \beta_{3n-2} \\ 0 & 0 & \gamma_{3n} \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.2}$$

We introduce the following conditions:

(A1) $\alpha_n \in \mathbb{R}$ and $\alpha_n = \delta n^\alpha(1 + \Delta_n),\ n \geq 1$, where $\alpha > 0$ and $\lim_{n\to\infty} \Delta_n = 0$;
(A2) $\beta_n, \gamma_n \in \mathbb{R}$ and there exist $\beta \in \mathbb{R}$ and $B > 0$ such that

$$|\beta_n|, |\gamma_n| \leq Bn^\beta,\ n \geq 1;$$

(A3) $\alpha > \beta$;
(A4) $\alpha > \beta + 1$, $\alpha \geq 1$ and $\Delta_n - \Delta_{n-1} = o(\frac{1}{n})$, $n \to \infty$.

In this section we use the standard notations $o(a_n)$ and $O(a_n)$, as $n \to \infty$.
Apply formulas (2.4) and (2.5) to (5.2) and notice that

$$d_n^{min} \geq \min\{\alpha_{3n-2}, \alpha_{3n-1}, \alpha_{3n}\} - 6B(3n)^\beta \geq$$

$$\delta(3n-2)^\alpha - \delta(3n-2)^\alpha \max\{|\Delta_{3n}|, |\Delta_{3n-1}|, |\Delta_{3n-2}|\} - 6B3^\beta n^\beta =$$

$$= 3^\alpha \delta n^\alpha (1+\epsilon'_n), \quad \text{where} \quad \epsilon'_n = o(1),$$

and

$$d_n^{max} \le \max\{\alpha_{3n-2}, \alpha_{3n-1}, \alpha_{3n}\} + 6B(3n)^\beta =$$
$$= 3^\alpha \delta n^\alpha (1+\epsilon''_n), \quad \text{where} \quad \epsilon''_n = o(1).$$

Thus, $\epsilon_n = \max\{|\epsilon'_n|, |\epsilon''_n|\} = o(1)$, $n \to \infty$, and

$$3^\alpha \delta n^\alpha (1-\epsilon_n) \le d_n^{min} \le d_n^{max} \le 3^\alpha \delta n^\alpha (1+\epsilon_n), \ n \ge 1.$$

It is easy to verify that

$$\|C_n\| \le 3^{1+\beta} B^\beta n^\beta, \quad n \ge 1.$$

Therefore, (A1)–(A3) yield (C1)–(C3), where $\delta_1 = \delta_2 = 3^\alpha \delta$. Then J defines an operator in $l^2(\mathbb{N}, \mathbb{C})$ which is identified with a Jacobi operator in $l^2(\mathbb{N}, \mathbb{C}^3)$. Moreover, we can apply Theorem 4.1 with any $r \in (0,1)$ and $\gamma > 0$ to the operator given by the matrix J.

Define the Gerschgorin radius (see [19])

$$R_n = |\beta_n| + |\gamma_n| + |\beta_{n-1}| + |\gamma_{n-2}| \tag{5.3}$$

and let

$$K_n = \{x \in \mathbb{R} : |\alpha_n - x| \le R_n\}. \tag{5.4}$$

Lemma 5.1. *If* (A1)–(A4) *are satisfied then*

1.
$$\alpha_{n+1} - \alpha_n - R_{n+1} - R_n = \delta n^{\alpha-1} + o(n^{\alpha-1}), \ n \to \infty;$$

2. *there exists* $n_0 > 1$ *such that* $K_n \cap \left(\bigcup_{m\neq n} K_m\right) = \emptyset$ *for* $n \ge n_0$.

Denote

$$J_l^k = \begin{pmatrix} \alpha_k & \beta_k & \gamma_k & 0 & \\ \beta_k & \alpha_{k+1} & \beta_{k+1} & \gamma_{k+1} & \ddots \\ \gamma_k & \beta_{k+1} & \alpha_{k+2} & \beta_{k+2} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \gamma_{l-2} & \beta_{l-1} & \alpha_l \end{pmatrix}, \quad k \le l,$$

and

$$J_n = J_{3n}^1. \tag{5.5}$$

Let $\mu_{1,n}, \mu_{2,n} \dots, \mu_{3n,n}$ be the non-decreasingly arranged sequence of the eigenvalues of the matrix J_n.

Lemma 5.2. *Let* $\gamma > 0$. *If* (A1)–(A3) *are satisfied then*

$$\lambda_n(J) = \mu_{n,n} + O(n^{-\gamma}), \ n \to \infty.$$

Proof. Notice that $p = 3$. Let $r = \frac{1}{3}$, then $rnp = n$. From Theorem 4.1 we have

$$\sup_{1\le i\le n} |\mu_{i,n} - \lambda_i(J)| \le Cn^{-\gamma},$$

where C is independent of n and i. Thus

$$|\mu_{n,n} - \lambda_n(J)| \le Cn^{-\gamma}. \qquad \square$$

Remark 5.3. We apply the Gerschgorin theorem (see [19]) and the generalized Gerschorin theorem, which is given in the book of Saad (see Theorem 3.12, [19]), to the symmetric matrix J_n, and we observe that if $n_0 < i \le 3n$ then $\mu_{i,n} \in K_i$, where n_0 is given in Lemma 5.1 and K_i is defined by (5.4). Moreover, from Theorem 4.1 we derive

$$\lambda_i(J) = \lim_{n\to\infty} \mu_{i,n} \in K_i,\ i > n_0.$$

Lemma 5.4 (Lütkepohl [14]). *Let $A \in M_{k\times k}$, $D \in M_{l\times l}$, $B, C^\top \in M_{l\times k}$. Then*

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \begin{cases} \det A \det(D - CA^{-1}B), & \text{if } A \text{ is invertible,} \\ \det D \det(A - BD^{-1}C), & \text{if } D \text{ is invertible.}\end{cases}$$

Theorem 5.5. *Let J be an operator defined in the Hilbert space $l^2(\mathbb{N})$ by the matrix* (5.1)*. Under* (A1)–(A4) *the following asymptotic formula for the discrete spectrum of J is satisfied:*

$$\lambda_n(J) = \alpha_n - \frac{\beta_{n-1}^2}{\alpha_{n-1}-\alpha_n} - \frac{\beta_n^2}{\alpha_{n+1}-\alpha_n} + \frac{\gamma_{n-2}^2}{\alpha_{n-2}-\alpha_n} + \frac{\gamma_n^2}{\alpha_{n+2}-\alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right),$$

as $n \to \infty$.

Proof. Let $n > n_0 + 2$, where n_0 is given in Lemma 5.1, $N = 3n$ and $\lambda = \mu_{n,n}$ be the n-th eigenvalue of $J_n = J_N^1 \in M_{N\times N}$. Then

$$J_N^1 - \lambda = \begin{pmatrix} J_{n-2}^1 - \lambda & E_n^* \\ E_n & J_N^{n-1} - \lambda\end{pmatrix},$$

where

$$E_n = \begin{pmatrix} 0 & \cdots & \gamma_{n-3} & \beta_{n-2} \\ 0 & \cdots & 0 & \gamma_{n-2} \\ \cdots & & & \\ 0 & \cdots & 0 & 0\end{pmatrix} \in M_{(N-n+2)\times(n-2)}.$$

J_{n-2}^1 is a real symmetric matrix, so

$$\|J_{n-2}^1\| = \max\{\mu \in \mathbb{R} : \mu \text{ is an eigenvalue of } J_{n-2}^1\} \in K_{n-2},$$

$$\|J_{n-2}^1\| \le \alpha_{n-2} + R_{n-2} < \alpha_n - R_n \le \mu_{n,n} = \lambda;$$

therefore, $J_{n-2}^1 - \lambda$ is invertible and, from Lemma 5.4, we derive

$$\det(J_N^1 - \lambda) = \det(J_{n-2}^1 - \lambda)\det(J_N^{n-1} - \lambda - E_n(J_{n-2}^1 - \lambda)^{-1}E_n^*). \qquad (5.6)$$

Denote

$$(J^1_{n-2} - \lambda)^{-1} = (m_{i,j}(\lambda))_{i,j=1}^{n-2}\,. \tag{5.7}$$

Then

$$E_n(J^1_{n-2} - \lambda)^{-1}E_n^* = \begin{pmatrix} a(\lambda) & b(\lambda) & 0 & \cdots \\ b(\lambda) & d(\lambda) & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & & & 0 \end{pmatrix} \in M_{(N-n+2)\times(N-n+2)},$$

where

$$a(\lambda) = \gamma_{n-3}^2 m_{n-3,n-3}(\lambda) + \beta_{n-2}^2 m_{n-2,n-2}(\lambda) + 2\gamma_{n-3}\beta_{n-3}m_{n-2,n-3}(\lambda), \tag{5.8}$$

$$b(\lambda) = \gamma_{n-2}\gamma_{n-3}m_{n-3,n-2}(\lambda) + \gamma_{n-2}\beta_{n-2}m_{n-2,n-2}(\lambda), \tag{5.9}$$

$$d(\lambda) = \gamma_{n-2}^2 m_{n-2,n-2}(\lambda). \tag{5.10}$$

Applying Lemma 5.4, we deduce

$$\begin{aligned} &\det(J_N^{n-1} - \lambda - E_n(J^1_{n-2} - \lambda)^{-1}E_n^*) = \\ &= \det\begin{pmatrix} J^{n-1}_{n+1} - \lambda - E(\lambda) & E'^*_{n+1} \\ E'_{n+1} & J_N^{n+2} - \lambda \end{pmatrix} = \\ &= \det(J_N^{n+2} - \lambda)\det\left(J^{n-1}_{n+1} - \lambda - E(\lambda) - E'^*_{n+1}(J_N^{n+2} - \lambda)^{-1}E'_{n+1}\right), \end{aligned} \tag{5.11}$$

where

$$E(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) & 0 \\ b(\lambda) & d(\lambda) & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_{3\times3}$$

and

$$E'_{n+1} = \begin{pmatrix} 0 & \gamma_n & \beta_{n+1} \\ 0 & 0 & \gamma_{n+1} \\ \cdots & & \\ 0 & 0 & 0 \end{pmatrix} \in M_{(N-n-1)\times3}.$$

Notice that $J_N^{n+2} - \lambda$ is invertible because from (5.4) and Remark 5.3

$$\lambda = \mu_{n,n} \le \alpha_n + R_n < \alpha_{n+2} - R_{n+2} \le \min\{\mu : \mu \text{ is an eigenvalue of } J_N^{n+2}\}.$$

Let

$$(J_N^{n+2} - \lambda)^{-1} = (s_{i,j}(\lambda))_{i,j=1}^{N-n-1}\,. \tag{5.12}$$

Thus

$$E'^*_{n+1}(J_N^{n+2} - \lambda)^{-1}E'_{n+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d'(\lambda) & b'(\lambda) \\ 0 & b'(\lambda) & c'(\lambda) \end{pmatrix} \in M_{3\times3},$$

where

$$d'(\lambda) = \gamma_n^2 s_{1,1}(\lambda), \tag{5.13}$$

$$c'(\lambda) = \beta_{n+1}^2 s_{1,1}(\lambda) + \gamma_{n+1}^2 s_{2,2}(\lambda) + 2\beta_{n+1}\gamma_{n+1}s_{1,2}(\lambda), \tag{5.14}$$

$$b'(\lambda) = \gamma_n(\beta_{n+1}s_{1,1}(\lambda) + \gamma_{n+1}s_{2,1}(\lambda)). \tag{5.15}$$

From (5.6) and (5.11) we deduce

$$\det(J_N^1 - \lambda) = \det(J_{n-2}^1 - \lambda)\det(J_N^{n+2} - \lambda)\det A_n(\lambda),$$

where

$$A_n(\lambda) = \begin{pmatrix} \alpha_{n-1} - \lambda - a(\lambda) & \beta_{n-1} - b(\lambda) & \gamma_{n-1} \\ \beta_{n-1} - b(\lambda) & \alpha_n - \lambda - d(\lambda) - d'(\lambda) & \beta_n - b'(\lambda) \\ \gamma_{n-1} & \beta_n - b'(\lambda) & \alpha_{n+1} - \lambda - c'(\lambda) \end{pmatrix}.$$

The matrices $J_{n-2}^1 - \lambda$ and $J_N^{n+2} - \lambda$ are invertible and $\lambda = \mu_{n,n} \in K_n$ is an eigenvalue of J_N^1, so $\det(J_N^1 - \lambda) = 0$, or, equvalently, $\det A_n(\lambda) = 0$, or also

$$\lambda = \alpha_n - d(\lambda) - d'(\lambda) + F_n(\lambda) + G_n(\lambda), \tag{5.16}$$

where

$$F_n(\lambda) = \tag{5.17}$$

$$= \frac{-(\beta_{n-1} - b(\lambda))^2}{\alpha_{n-1} - \lambda - a(\lambda)} + \frac{-(\beta_n - b'(\lambda))^2}{\alpha_{n+1} - \lambda - c'(\lambda)} + \frac{2\gamma_{n-1}(\beta_{n-1} - b(\lambda))(\beta_n - b'(\lambda))}{(\alpha_{n-1} - \lambda - a(\lambda))(\alpha_{n+1} - \lambda - c'(\lambda))}$$

and

$$G_n(\lambda) = \frac{\gamma_{n-1}^2 F_n(\lambda)}{(\alpha_{n-1} - \lambda - a(\lambda))(\alpha_{n+1} - \lambda - c'(\lambda)) - \gamma_{n-1}^2}. \tag{5.18}$$

Observe that, under conditions (A1)–(A4), if $1 \le k \le n-2$, $|\lambda - \alpha_n| \le R_n$ and $x \in \mathbb{R}^k$, then

$$\|(J_k^1 - \lambda)x\| \ge \lambda\|x\| - \|J_k^1 x\| \ge \lambda\|x\| - \|J_k^1\|\|x\| \ge (\alpha_n - R_n - \alpha_k - R_k)\|x\| \ge cn^{\alpha-1}\|x\|,$$

for a constant $c > 0$; therefore,

$$\|(J_k^1 - \lambda)^{-1}\| \le (cn^{\alpha-1})^{-1}.$$

Then, by Lemma 5.4, from (5.7) we derive

$$m_{n-3,n-3}(\lambda) = \frac{1}{\alpha_{n-3} - \alpha_n} + O\left(\frac{n^\beta}{n^{2(\alpha-1)}}\right),$$

$$m_{n-2,n-2}(\lambda) = \frac{1}{\alpha_{n-2} - \alpha_n} + O\left(\frac{n^\beta}{n^{2(\alpha-1)}}\right),$$

$$m_{n-2,n-3}(\lambda) = m_{n-3,n-2}(\lambda) = O\left(\frac{n^\beta}{n^{2(\alpha-1)}}\right),$$

if $|\lambda - \alpha_n| = O(n^\beta)$. Then we calculate the following asymptotic equalities

$$a(\lambda) = \frac{\gamma_{n-3}^2}{\alpha_{n-3}-\alpha_n} + \frac{\beta_{n-2}^2}{\alpha_{n-2}-\alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right), \tag{5.19}$$

$$b(\lambda) = \frac{\beta_{n-2}\gamma_{n-2}}{\alpha_{n-2}-\alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right), \tag{5.20}$$

$$d(\lambda) = \frac{\gamma_{n-2}^2}{\alpha_{n-2}-\alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right), \tag{5.21}$$

for $\lambda = \mu_{n,n} \in K_n$, where K_n is given by (5.4).

From (A1)–(A4) we derive also $\|(J_N^k - \lambda)^{-1}\| \le (cn^{\alpha-1})^{-1}$ for $k \ge n+2$ and $\lambda = \mu_{n,n}$. Then from Lemma 5.4 and equation (5.12) we deduce

$$s_{1,1}(\lambda) = \frac{1}{\alpha_{n+2}-\alpha_n} + O\left(\frac{n^{\beta}}{n^{2(\alpha-1)}}\right),$$

$$s_{2,2}(\lambda) = \frac{1}{\alpha_{n+3}-\alpha_n} + O\left(\frac{n^{\beta}}{n^{2(\alpha-1)}}\right)$$

and

$$s_{1,2}(\lambda) = s_{2,1}(\lambda) = O\left(\frac{n^{\beta}}{n^{2(\alpha-1)}}\right).$$

Then

$$d'(\lambda) = \frac{\gamma_n^2}{\alpha_{n+2}-\alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right), \tag{5.22}$$

$$c'(\lambda) = \frac{\beta_{n+1}^2}{\alpha_{n+2}-\alpha_n} + \frac{\gamma_{n+2}^2}{\alpha_{n+3}-\alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right) \tag{5.23}$$

and

$$b'(\lambda) = \frac{\gamma_n\beta_{n+1}}{\alpha_{n+2}-\alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right). \tag{5.24}$$

Notice that if $\lambda = \mu_{n,n}$ then $\lambda = \alpha_n + O(n^\beta)$ and using (5.19)–(5.21), (5.22)–(5.24), (5.17) and (5.18) we have

$$\begin{aligned} F_n(\lambda) = {}& \frac{-(\beta_{n-1} - \gamma_{n-2}\beta_{n-2}/(\alpha_{n-1}-\alpha_n))^2}{\alpha_{n-1}-\alpha_n} + \\ &+ \frac{-(\beta_n - \gamma_n\beta_{n+1}/(\alpha_{n+1}-\alpha_n))^2}{\alpha_{n+1}-\alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right) = \\ ={}& \frac{-\beta_{n-1}^2}{\alpha_{n-1}-\alpha_n} + \frac{-\beta_n^2}{\alpha_{n+1}-\alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right) \end{aligned}$$

and

$$G_n(\lambda) = O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right).$$

Thus

$$\lambda = \mu_{n,n} = \alpha_n - \frac{\beta_{n-1}^2}{\alpha_{n-1} - \alpha_n} - \frac{\beta_n^2}{\alpha_{n+1} - \alpha_n} + \frac{\gamma_{n-2}^2}{\alpha_{n-2} - \alpha_n} + \frac{\gamma_n^2}{\alpha_{n+2} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right). \tag{5.25}$$

Notice that the above estimate is satisfied under conditions (A1)–(A4).

Finally we apply Lemma 5.5 with a constant $\gamma > \max\{0, 2(\alpha-1) - 3\beta\}$ to obtain the asymptotic equality for the eigenvalues of the operator J:

$$\lambda_n(J) = \alpha_n - \frac{\beta_{n-1}^2}{\alpha_{n-1} - \alpha_n} - \frac{\beta_n^2}{\alpha_{n+1} - \alpha_n} + \frac{\gamma_{n-2}^2}{\alpha_{n-2} - \alpha_n} + \frac{\gamma_n^2}{\alpha_{n+2} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right),$$

as $n \to \infty$. □

Remark 5.6. The asymptotic formula for $\lambda_n(J)$ from Theorem 5.5 and formula (5.25) for $\mu_{n,n}$ are more precise then the estimates mentioned in Remark 5.3 even if we do not assume additional conditions on the sign of the expression $2(\alpha-1) - 3\beta$.

Example 5.7. Let consider a non-symmetric tridiagonal operator T on $l^2(\mathbb{N})$

$$\begin{pmatrix} 1 & a_1 & 0 & \ddots & \ddots \\ b_1 & 4 & a_2 & 0 & \ddots \\ 0 & b_2 & 9 & a_3 & \ddots \\ 0 & 0 & b_3 & 16 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{5.26}$$

where $\{a_n\}$ and $\{b_n\}$ are bounded real sequences. If $J = T^*T$ then J is symmetric 5-diagonal operator and the infinite matrix, associated with J, has the entries determined by the sequences

$$\alpha_n = n^4 + a_{n-1}^2 + b_n^2, \quad \text{for} \quad n \geq 2, \quad \alpha_1 = 1 + b_1^2,$$
$$\beta_n = n^2 a_n + (n+1)^2 b_n, \quad \gamma_n = a_n b_n, \quad \text{for} \quad n \geq 1.$$

The above sequences satisfy (A1)–(A4) with $\alpha = 4$ and $\beta = 2$, so we apply Theorem 5.5 to obtain

$$\begin{aligned} \lambda_n(T^*T) = \lambda_n(J) &= n^4 + \frac{\beta_{n-1}^2}{n^4 - (n-1)^4 - \rho_n} - \frac{\beta_n^2}{(n+1)^4 - n^4 + \rho_{n+1}} + O(1) = \\ &= n^4 + \frac{n((a_{n-1} + b_{n-1})^2 - (a_n + b_n)^2)}{4} + O(1), \quad n \to \infty, \end{aligned}$$

$(\rho_n = a_{n-1}^2 + b_n^2 - a_{n-2}^2 - b_{n-1}^2)$.

From the above result we deduce easily the asymptotic formula for the singular numbers of T as follow

$$s_n(T) = (\lambda_n(T^*T))^{\frac{1}{2}} = n^2 + \frac{(a_{n-1} + b_{n-1})^2 - (a_n + b_n)^2}{8n} + O\Big(\frac{1}{n^2}\Big), \quad n \to \infty.$$

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Maria Malejki
malejki@uci.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Cracow, Poland

Received: February 19, 2010.
Revised: April 26, 2010.
Accepted: April 30, 2010.