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POSTOPTIMAL ANALYSIS IN THE COEFFICIENTS MATRIX OF PIECEWISE LINEAR FRACTIONAL PROGRAMMING PROBLEMS WITH NON-DEGENERATE OPTIMAL SOLUTION

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Abstract. In this paper, we discuss how changes in the coefficients matrix of piecewise linear fractional programming problems affect the non-degenerate optimal solution. We consider separate cases when changes occur in the coefficients of the basic and non-basic variables and derive bounds for each perturbation, while the optimal solution is invariant. We explain that this analysis is a generalization of the sensitivity analysis for LP, LFP and PLP. Finally, the results are described by some numerical examples.

Keywords: piecewise linear fractional programming, degeneracy, optimal basis, fractional programming, piecewise linear programming, sensitivity analysis.

Mathematics Subject Classification: 90C31.

1. INTRODUCTION

We refer the reader to the collective work [8] for a review of postoptimal analysis in different areas of optimization problems. The work shows that the postoptimal topics lead to interesting questions and problems in various areas of optimization. The more papers designed on postoptimal analysis in linear fractional programming (see [1, 2]). These results have been extended to variations for both the numerator and the denominator of the objective function as well as with the right-hand-side of the constraints. Also some aspects concerning duality and sensitivity analysis in linear fractional program was discussed in [4]. The postoptimal analysis has been extended to variations for both the numerator and the denominator of the objective function of piecewise linear fractional program as well as with the right-hand-side of the constraints [9]. An alternative procedure studied for multi-parametric sensitivity analysis in linear programming by the concept of a maximum volume in the tolerance region, which is bounded by a symmetrically rectangular parallelepiped and can be solved by a maximization problem [15]. Kheirfam [10, 11] used the concept of maximal volume region to study the multiparametric sensitivity analysis of the objective function, right-hand-side vector and constraint matrix in a piecewise linear fractional programming problem. In this note, we consider the effect of changing the coefficients matrix in a piecewise linear fractional programming problem after we have obtained a non-degenerate optimal solution, and the problem is presented in the following way: *Is the given optimal solution still optimal after some change in the coefficients matrix of the initial problem?* We will consider separate cases when changes occur in the coefficients of the basic and non-basic variables. Since linear programming (LP) [5], piecewise linear programming problems (PLP) [7] and linear fractional programming problems (LFP) ([3, 13, 14]) are all special cases of the $PLFP$, therefore a unified framework of postoptimal analysis is presented which covers almost all approaches that have appeared in the literature.

2. PIECEWISE LINEAR FRACTIONAL PROGRAMMING PROBLEM

The piecewise linear fractional programming problem ($PLFP$) is defined as follows:

$$\min Z(x) = \frac{P(\mathbf{x})}{D(\mathbf{x})} = \frac{\alpha_0 + \sum_{j=1}^{n} f_j(x_j)}{\beta_0 + \sum_{j=1}^{n} g_j(x_j)} \qquad (PLFP)$$

$$s.t: \mathbf{Ax} = \mathbf{b}$$
$$0 \leq \mathbf{x} \leq \mathbf{u},$$

where $f_j(x_j)$ and $g_j(x_j)$, $j = 1, 2, \ldots, n$, are respectively continuous piecewise linear convex and concave functions such that $\beta_0 + \sum_{j=1}^{n} g_j(x_j) > 0$ for any feasible solution $\mathbf{x}$, $\mathbf{A}$ is an $m \times n$ matrix of full row rank, $\mathbf{b}$ is an m-vector and $\mathbf{u}$ is an n-vector.

Let $0 = \delta_0^j < \delta_1^j < \ldots < \delta_{\tau_j}^j < \delta_{\tau_j+1}^j = u_j$ be an ascending order of the breakpoints of both $f_j(x_j)$ and $g_j(x_j)$. Then within each subinterval $[\delta_i^j, \delta_{i+1}^j]$, $i = 0, 1, \ldots, \tau_j$, both $f_j(x_j)$ and $g_j(x_j)$ are linear functions. Therefore $f_j(x_j)$ and $g_j(x_j)$ can be stated as

$$f_j(x_j) = c_i^j x_j + \alpha_i^j, \quad \delta_i^j \leq x_j \leq \delta_{i+1}^j; \quad i = 0, 1, 2, \ldots, \tau_j, \tag{2.1}$$

and

$$g_j(x_j) = d_i^j x_j + \beta_i^j, \quad \delta_i^j \leq x_j \leq \delta_{i+1}^j; \quad i = 0, 1, 2, \ldots, \tau_j, \tag{2.2}$$

for some real numbers c_i^j, α_i^j, d_i^j and β_i^j, $i = 0, 1, \ldots, \tau_j$, $j = 1, 2, \ldots, n$.

The following lemmas determine the convexity and the concavity conditions for a continuous piecewise linear function [6].

Lemma 2.1. *A continuous piecewise linear function is convex if and only if its slope is non-decreasing with respect to x_j; that is, $c_0^j \leq c_1^j \leq \ldots \leq c_{\tau_j}^j, \quad j = 1, 2, \ldots, n$.*

Lemma 2.2. *A continuous piecewise linear function is concave if and only if its slope is non-increasing with respect to x_j; that is, $d_0^j \geq d_1^j \geq \ldots \geq d_{\tau_j}^j, \quad j = 1, 2, \ldots, n$.*

Let $\mathbf{x}^0$ be an optimal solution to the $PLFP$. For each $j = 1, 2, \ldots, n$, choose an index j_i such that $\delta_{j_i}^j \leq x_j^0 \leq \delta_{j_i+1}^j$. Then any optimal solution to the LFP problem:

$$\min \frac{\alpha^* + \sum_{j=1}^{n} c_{j_i}^j x_j}{\beta^* + \sum_{j=1}^{n} d_{j_i}^j x_j} \qquad (LFP)$$

$$s.t: \mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\delta_{j_i}^j \leq x_j \leq \delta_{j_i+1}^j, \quad j = 1, 2, \ldots, n,$$

is also an optimal solution to the $PLFP$ where $\alpha^* = \alpha_0 + \sum_{j=1}^{n} \alpha_{j_i}^j, \beta^* = \beta_0 + \sum_{j=1}^{n} \beta_{j_i}^j$ [12]. The basic feasible solutions (BFS) for the $PLFP$ can be defined as follows:

Let $\mathbf{A} = [A_{.1}, \ldots, A_{.n}]$ be the coefficients matrix and $B = \{B_1, \ldots, B_m\} \subset \{1, \ldots, n\}$ be a subset of the indices of the columns of matrix $\mathbf{A}$, such that $\mathbf{B} = [A_{.B_1}, \ldots, A_{.B_m}]$ is a non-singular matrix with inverse $\mathbf{B}^{-1} = [\beta_{ij}]$. Let $N = \{1, 2, \ldots, n\} \setminus B$. The variables $x_{B_i}, \ i = 1, \ldots, m$, are called basic variables and $x_j, \ j \in N$, are referred to as non-basic variables. These vectors are denoted by $\mathbf{x}_B$ and $\mathbf{x}_N$, respectively. Consequently, the solution $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$, which

$$x_j = \delta_{\nu_j}^j, \quad j \in N, \qquad \nu_j \in \{0, 1, \ldots, \tau_j + 1\},$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in N} \mathbf{B}^{-1} A_{.j} x_j, \tag{2.3}$$

is called a basic solution. If, in addition $0 \leq \mathbf{x}_B \leq \mathbf{u}_B$, then $\mathbf{x}$ is a basic feasible solution (BFS). Moreover, if $x_{B_i} \in \{\delta_0^{B_i}, \delta_1^{B_i}, \ldots, \delta_{\tau_{B_i+1}}^{B_i}\}$ for some i, then $\mathbf{x}$ is a degenerate BFS. If $x_{B_i} \notin \{\delta_0^{B_i}, \delta_1^{B_i}, \ldots, \delta_{\tau_{B_i+1}}^{B_i}\}$ for any i, then it is a non-degenerate BFS.

It is shown [12] that there exists an optimal solution of the $PLFP$ which is a BFS. The optimality criterion given by Punnen and Pandy [12] for the $PLFP$ using the simplex algorithm is stated as follows:

Let $\mathbf{B}$ denote the optimal basis matrix and let $\mathbf{x}^* = (\mathbf{x}_B^*, \mathbf{x}_N^*)$ be the corresponding non-degenerate basic feasible solution for the $PLFP$. This solution will be optimal if

$$\eta_j^-(\mathbf{x}^*) = (c_{\nu_j-1}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - Z(\mathbf{x}^*)(d_{\nu_j-1}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \leq 0,$$

and

$$\eta_j^+(\mathbf{x}^*) = (c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - Z(\mathbf{x}^*)(d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \geq 0,$$

for $j = 1, 2, \ldots, n$, where $Z(\mathbf{x}^*)$ is the objective function value at the optimal solution $\mathbf{x}^*$, $\mathbf{c}_B$ and $\mathbf{d}_B$ are the sub-vectors of $\mathbf{c}$ and $\mathbf{d}$ such that their i-th coordinates corresponding to $\mathbf{B}$ are $c^{B_i}_{\mu(B_i)}$ and $d^{B_i}_{\mu(B_i)}$, respectively. If $\nu_j = \tau_j + 1$ then η^+_j is defined as 0. Similarly when $\nu_j = 0$ then η^-_j is defined as 0. Note that $\mu(B_i)$ denotes the index for which $\delta^{B_i}_{\mu(B_i)} \leq x^*_{B_i} \leq \delta^{B_i}_{\mu(B_i)+1}$.

3. CHANGES IN THE COEFFICIENTS OF A NON-BASIC VARIABLE

Let us replace entry A_{ik} by $A'_{ik} = A_{ik} + \delta$ in the vector $\mathbf{A}_{.k} = (A_{1k}, \ldots, A_{ik}, \ldots, A_{mk})^T$ and investigate how this change affects the optimal solution $\mathbf{x}^*$ and the optimal value of the objective function $Z(\mathbf{x})$. So from (2.3) we will have

$$\begin{aligned}\bar{\mathbf{x}}_B &= \mathbf{B}^{-1}\mathbf{b} - \sum_{\substack{j\in N\\ j\neq k}} \mathbf{B}^{-1}\mathbf{A}_{.j}\delta^j_{\nu_j} - \mathbf{B}^{-1}\mathbf{A}'_{.k}\delta^k_{\nu_k} = \\ &= \mathbf{B}^{-1}\mathbf{b} - \sum_{j\in N} \mathbf{B}^{-1}\mathbf{A}_{.j}\delta^j_{\nu_j} - \delta\beta_{.i}\delta^k_{\nu_k} = \mathbf{x}^*_B - \delta\beta_{.i}\delta^k_{\nu_k},\end{aligned}$$

where $\beta_{.i}$ is the i-th column $\mathbf{B}^{-1}$. Now the h-th component of $\bar{\mathbf{x}}_B$ is given by

$$\bar{x}_{B_h} = x^*_{B_h} - \delta\beta_{hi}\delta^k_{\nu_k}, \quad h = 1, \ldots, m.$$

This new basic solution $\bar{\mathbf{x}}_B$ will be feasible if

$$\delta^{B_h}_{\mu(B_h)} \leq x^*_{B_h} - \delta\beta_{hi}\delta^k_{\nu_k} \leq \delta^{B_h}_{\mu(B_h)+1}, \quad h = 1, \ldots, m.$$

Therefore, we obtain the following range for δ:

$$\begin{aligned}&\max\left\{ \max_{\substack{\beta_{hi}<0\\ 1\leq h\leq m}} \frac{x^*_{B_h} - \delta^{B_h}_{\mu(B_h)}}{\beta_{hi}\delta^k_{\nu_k}}, \max_{\substack{\beta_{hi}>0\\ 1\leq h\leq m}} \frac{x^*_{B_h} - \delta^{B_h}_{\mu(B_h)+1}}{\beta_{hi}\delta^k_{\nu_k}} \right\} \leq \delta \leq \\ &\leq \min\left\{ \min_{\substack{\beta_{hi}<0\\ 1\leq h\leq m}} \frac{x^*_{B_h} - \delta^{B_h}_{\mu(B_h)+1}}{\beta_{hi}\delta^k_{\nu_k}}, \min_{\substack{\beta_{hi}>0\\ 1\leq h\leq m}} \frac{x^*_{B_h} - \delta^{B_h}_{\mu(B_h)}}{\beta_{hi}\delta^k_{\nu_k}} \right\}.\end{aligned} \tag{3.1}$$

The new solution $\bar{\mathbf{x}}$ is an optimal solution for the perturbed $PLFP$ problem if

$$\eta^+_j(\bar{\mathbf{x}}) = (c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j}) - Z(\bar{\mathbf{x}})(d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j}) \geq 0, \qquad \forall j \in N, \tag{3.2}$$

and

$$\eta^-_j(\bar{\mathbf{x}}) = (c^j_{\nu_j-1} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j}) - Z(\bar{\mathbf{x}})(d^j_{\nu_j-1} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j}) \leq 0, \quad \forall j \in N. \tag{3.3}$$

It is obvious that reduced costs $c^j_{\nu_j-1} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j}$, $d^j_{\nu_j-1} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j}$, $c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j}$ and $d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j}$ are dependent directly on the coefficients matrix

$\mathbf{A}$ by (3.2) and (3.3). So, any change in $\mathbf{A}_{.k}$ may affects the value of objective function $Z(\mathbf{x})$. Hence, we have

$$\begin{aligned} Z(\bar{\mathbf{x}}) &= \frac{\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} + \sum_{\substack{j\in N\\ j\neq k}}(c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j})\delta^j_{\nu_j} + \alpha + (c^k_{\nu_k} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}'_{.k})\delta^k_{\nu_k}}{\mathbf{d}_B\mathbf{B}^{-1}\mathbf{b} + \sum_{\substack{j\in N\\ j\neq k}}(d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j})\delta^j_{\nu_j} + \beta + (d^k_{\nu_k} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}'_{.k})\delta^k_{\nu_k}} = \\ &= \frac{\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} + \sum_{j\in N}(c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j})\delta^j_{\nu_j} + \alpha - \delta\mathbf{c}_B\beta_{.i}\delta^k_{\nu_k}}{\mathbf{d}_B\mathbf{B}^{-1}\mathbf{b} + \sum_{j\in N}(d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j})\delta^j_{\nu_j} + \beta - \delta\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k}} = \\ &= \frac{P(\mathbf{x}^*) - \delta\mathbf{c}_B\beta_{.i}\delta^k_{\nu_k}}{D(\mathbf{x}^*) - \delta\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k}}. \end{aligned} \tag{3.4}$$

To preserve the strict positivity of the denominator $D(\mathbf{x})$, we need to have

$$D(\mathbf{x}^*) - \delta\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k} > 0, \tag{3.5}$$

which implies

$$\delta \begin{cases} < \dfrac{D(\mathbf{x}^*)}{\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k}}, & \text{if} \quad \mathbf{d}_B\beta_{.i} > 0, \\ > \dfrac{D(\mathbf{x}^*)}{\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k}}, & \text{if} \quad \mathbf{d}_B\beta_{.i} < 0. \end{cases} \tag{3.6}$$

Moreover, by using (3.4) and the change of the k-th column, we can re-write (3.2) in the following form

$$\begin{aligned} \eta^+_j(\bar{\mathbf{x}}) &= (c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j}) - \frac{P(\mathbf{x}^*) - \delta\mathbf{c}_B\beta_{.i}\delta^k_{\nu_k}}{D(\mathbf{x}^*) - \delta\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k}}(d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j}) = \\ &= \Delta'_j - \frac{P(\mathbf{x}^*) - \delta\mathbf{c}_B\beta_{.i}\delta^k_{\nu_k}}{D(\mathbf{x}^*) - \delta\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k}}(\Delta''_j) \geq 0, \quad \forall j \in N,\ j \neq k, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \eta^+_k(\bar{\mathbf{x}}) &= (c^k_{\nu_k} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}'_{.k}) - \frac{P(\mathbf{x}^*) - \delta\mathbf{c}_B\beta_{.i}\delta^k_{\nu_k}}{D(\mathbf{x}^*) - \delta\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k}}(d^k_{\nu_k} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}'_{.k}) = \\ &= (\Delta'_k - \mathbf{c}_B\beta_{.i}\delta) - \frac{P(\mathbf{x}^*) - \delta\mathbf{c}_B\beta_{.i}\delta^k_{\nu_k}}{D(\mathbf{x}^*) - \delta\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k}}(\Delta''_k - \mathbf{d}_B\beta_{.i}\delta) \geq 0, \end{aligned} \tag{3.8}$$

where $\Delta'_j = c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j}, \quad \Delta''_j = d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j}, \quad \forall j \in N.$

From (3.5), the relation (3.7) is satisfied if

$$\Delta'_j(D(\mathbf{x}^*) - \delta\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k}) - \Delta''_j(P(\mathbf{x}^*) - \delta\mathbf{c}_B\beta_{.i}\delta^k_{\nu_k}) \geq 0$$

or

$$D(\mathbf{x}^*)\ \eta^+_j(\mathbf{x}^*) \geq \delta\delta^k_{\nu_k}(\mathbf{d}_B\Delta'_j - \mathbf{c}_B\Delta''_j)\beta_{.i},$$

which implies

$$\max_{\substack{j\in N\\ j\neq k}}\left\{\frac{D(\mathbf{x}^*)\ \eta_j^+(\mathbf{x}^*)}{\delta^k_{\nu_k}(\mathbf{d}_B\Delta^{'}_j-\mathbf{c}_B\Delta^{''}_j)\beta_{.i}}:(\mathbf{d}_B\Delta^{'}_j-\mathbf{c}_B\Delta^{''}_j)\beta_{.i}<0\right\}\leq\delta\leq$$
$$\leq\min_{\substack{j\in N\\ j\neq k}}\left\{\frac{D(\mathbf{x}^*)\ \eta_j^+(\mathbf{x}^*)}{\delta^k_{\nu_k}(\mathbf{d}_B\Delta^{'}_j-\mathbf{c}_B\Delta^{''}_j)\beta_{.i}}:\ (\mathbf{d}_B\Delta^{'}_j-\mathbf{c}_B\Delta^{''}_j)\beta_{.i}>0\right\}. \tag{3.9}$$

From (3.5), the relation (3.8) is satisfied if

$$(\Delta^{'}_k-\delta\mathbf{c}_B\beta_{.i})(D(\mathbf{x}^*)-\delta\mathbf{d}_B\beta_{.i}\delta^k_{\nu_k})-(\Delta^{''}_k-\delta\mathbf{d}_B\beta_{.i})(P(\mathbf{x}^*)-\delta\mathbf{c}_B\beta_{.i}\delta^k_{\nu_k})\geq 0,$$

or

$$D(\mathbf{x}^*)\ \eta_k^+(\mathbf{x}^*)+\delta\left[(P(\mathbf{x}^*)-\Delta^{'}_k\delta^k_{\nu_k})\mathbf{d}_B-(D(\mathbf{x}^*)-\Delta^{''}_k\delta^k_{\nu_k})\mathbf{c}_B\right]\beta_{.i}\geq 0,$$

which implies

$$\delta\begin{cases}\geq\dfrac{-D(\mathbf{x}^*)\eta_k^+(\mathbf{x}^*)}{H}, & \text{if}\quad H>0,\\ \leq\dfrac{-D(\mathbf{x}^*)\eta_k^+(\mathbf{x}^*)}{H}, & \text{if}\quad H<0,\end{cases} \tag{3.10}$$

where $H=\left[(P(\mathbf{x}^*)-\Delta^{'}_k\delta^k_{\nu_k})\mathbf{d}_B-(D(\mathbf{x}^*)-\Delta^{''}_k\delta^k_{\nu_k})\mathbf{c}_B\right]\beta_{.i}$.

Similarly, if $\eta_j^-(\bar{\mathbf{x}})\leq 0$ and $\eta_k^-(\bar{\mathbf{x}})\leq 0$ we obtain

$$\max_{\substack{j\in N\\ j\neq k}}\left\{\frac{D(\mathbf{x}^*)\ \eta_j^-(\mathbf{x}^*)}{\delta^k_{\nu_k}(\mathbf{d}_B\bar{\Delta}^{'}_j-\mathbf{c}_B\bar{\Delta}^{''}_j)\beta_{.i}}:(\mathbf{d}_B\bar{\Delta}^{'}_j-\mathbf{c}_B\bar{\Delta}^{''}_j)\beta_{.i}>0\right\}\leq\delta\leq$$
$$\leq\min_{\substack{j\in N\\ j\neq k}}\left\{\frac{D(\mathbf{x}^*)\ \eta_j^-(\mathbf{x}^*)}{\delta^k_{\nu_k}(\mathbf{d}_B\bar{\Delta}^{'}_j-\mathbf{c}_B\bar{\Delta}^{''}_j)\beta_{.i}}:\ (\mathbf{d}_B\bar{\Delta}^{'}_j-\mathbf{c}_B\bar{\Delta}^{''}_j)\beta_{.i}<0\right\}, \tag{3.11}$$

and

$$\delta\begin{cases}\leq\dfrac{-D(\mathbf{x}^*)\eta_k^-(\mathbf{x}^*)}{H^{'}}, & \text{if}\quad H^{'}>0,\\ \geq\dfrac{-D(\mathbf{x}^*)\eta_k^-(\mathbf{x}^*)}{H^{'}}, & \text{if}\quad H^{'}<0,\end{cases} \tag{3.12}$$

where $H^{'}=\left[(P(\mathbf{x}^*)-\bar{\Delta}^{'}_k\delta^k_{\nu_k})\mathbf{d}_B-(D(\mathbf{x}^*)-\bar{\Delta}^{''}_k\delta^k_{\nu_k})\mathbf{c}_B\right]\beta_{.i}$,

$\bar{\Delta}^{'}_j=c^j_{\nu_j-1}-\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j}$ and $\bar{\Delta}^{''}_j=d^j_{\nu_j-1}-\mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j},\quad\forall j\in N$.

Therefore, we have proved the following theorem:

Theorem 3.1. *If* δ *satisfies* (3.1), (3.6), (3.9), (3.10), (3.11) and (3.12) *then* $\bar{\mathbf{x}}$ *is an optimal solution of the perturbed* $PLFP$ *problem.*

Remark 3.2. Lower and upper bounds given in Theorem 3.1 are a generalization of the corresponding bounds for LP, PLP and LFP. Indeed,

1. If $\beta_0 = 1$ and $g_j(x_j) = 0, j = 1, 2, \dots, n$, then the $PLFP$ reduces to PLP and this means that $D(\mathbf{x}^*) = 1$, $\Delta_j^{''} = \bar{\Delta}_j^{''} = 0$, $\eta_j^+(\bar{\mathbf{x}}) = \Delta_j^{'}$, $\eta_j^-(\bar{\mathbf{x}}) = \bar{\Delta}_j^{'}, j \in N$, and $Z(\bar{\mathbf{x}}) = P(\mathbf{x}^*) - \delta \mathbf{c}_B \beta_{.i} \delta_{\nu_k}^k$. Thus, bounds (3.1) in the current form are valid for PLP too, on the other hand $\eta_j^+(\bar{\mathbf{x}}) \geq 0$ and $\eta_j^-(\bar{\mathbf{x}}) \leq 0$, thus (3.9) and (3.11) hold. Therefore, $\bar{\mathbf{x}}$ is an optimal solution for PLP if δ satisfies in (3.1) and

$$\delta \begin{cases} \leq \dfrac{\Delta_k^{'}}{\mathbf{c}_B \beta_{.i}}, & \text{if} \quad \mathbf{c}_B \beta_{.i} > 0, \\ \geq \dfrac{\Delta_k^{'}}{\mathbf{c}_B \beta_{.i}}, & \text{if} \quad \mathbf{c}_B \beta_{.i} < 0, \end{cases} \tag{3.13}$$

and

$$\delta \begin{cases} \geq \dfrac{\bar{\Delta}_k^{'}}{\mathbf{c}_B \beta_{.i}}, & \text{if} \quad \mathbf{c}_B \beta_{.i} > 0, \\ \leq \dfrac{\bar{\Delta}_k^{'}}{\mathbf{c}_B \beta_{.i}}, & \text{if} \quad \mathbf{c}_B \beta_{.i} < 0. \end{cases} \tag{3.14}$$

2. If $\beta_0 = 1$, $g_j(x_j) = 0$ and $f_j(x_j)$, $j = 1, 2, \dots, n$, are linear functions then the $PLFP$ reduces to LP with bounded variables. In this case, the feasibility condition (3.1) and the optimality conditions (3.9), (3.11), (3.10) and (3.12) are respectively as follows

$$\begin{aligned} \max & \left\{ \max_{\substack{\beta_{hi} < 0 \\ 1 \leq h \leq m}} \frac{x_{B_h}^*}{\beta_{hi} t}, \max_{\substack{\beta_{hi} > 0 \\ 1 \leq h \leq m}} \frac{x_{B_h}^* - u_{B_h}}{\beta_{hi} t} \right\} \leq \delta \leq \\ \leq \min & \left\{ \min_{\substack{\beta_{hi} < 0 \\ 1 \leq h \leq m}} \frac{x_{B_h}^* - u_{B_h}}{\beta_{hi} t}, \min_{\substack{\beta_{hi} > 0 \\ 1 \leq h \leq m}} \frac{x_{B_h}^*}{\beta_{hi} t} \right\}, \end{aligned} \tag{3.15}$$

$$\eta_j^+(\mathbf{x}^*) = c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j} = c_j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j} \geq 0, \quad \text{if } x_j = 0,$$

$$\eta_j^-(\mathbf{x}^*) = c_{\nu_j - 1}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j} = c_j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j} \leq 0, \quad \text{if } x_j = u_j,$$

$$\delta \begin{cases} \leq \dfrac{\eta_k^+(\mathbf{x}^*)}{\mathbf{c}_B \beta_{.i}}, & \text{if} \quad \mathbf{c}_B \beta_{.i} > 0, \\ \geq \dfrac{\eta_k^+(\mathbf{x}^*)}{\mathbf{c}_B \beta_{.i}}, & \text{if} \quad \mathbf{c}_B \beta_{.i} < 0, \end{cases}$$

and

$$\delta \begin{cases} \le \dfrac{\eta_k^-(\mathbf{x}^*)}{\mathbf{c}_B\beta_{.i}}, & \text{if} \quad \mathbf{c}_B\beta_{.i} < 0, \\ \ge \dfrac{\eta_k^-(\mathbf{x}^*)}{\mathbf{c}_B\beta_{.i}}, & \text{if} \quad \mathbf{c}_B\beta_{.i} > 0, \end{cases}$$

where t is the non-basic variable value x_k ($t = 0$ or u_k).

3. If both $g_j(x_j)$ and $f_j(x_j), j = 1, 2, \ldots, n$, are linear functions then the $PLFP$ reduces to LFP and this means that $c^j_{\nu_j} = c_j,\ d^j_{\nu_j} = d_j,\ \Delta'_j = c_j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}$ and $\Delta''_j = d_j - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}$. Therefore, $\bar{\mathbf{x}}$ is an optimal solution (by $u_j = \infty,\ \forall j$) if

$$\delta \begin{cases} \ge \dfrac{-D(\mathbf{x}^*)\eta_k(\mathbf{x}^*)}{H}, & \text{if} \quad H > 0, \\ \le \dfrac{-D(\mathbf{x}^*)\eta_k(\mathbf{x}^*)}{H}, & \text{if} \quad H < 0, \end{cases} \tag{3.16}$$

where $H = \big(P(\mathbf{x}^*)\mathbf{d}_B - D(\mathbf{x}^*)\mathbf{c}_B\big)\beta_{.i}$.

Example 3.3. Consider the $PLFP$ problem:

$$\min Z(\mathbf{x}) = \frac{\sum_{j=1}^4 f_j(x_j)}{\sum_{j=1}^4 g_j(x_j)}$$

$$\begin{array}{rrrrrl} s.t: & 3x_1+ & 4x_2+ & x_3+ & 2x_4 = & 21, \\ & x_1+ & 3x_2+ & x_3+ & 3x_4 = & 13, \\ & 2x_1+ & x_2+ & 2x_3+ & 3x_4 = & 14, \end{array}$$

$$0 \le x_1 \le 5, \quad 0 \le x_2 \le 3, \quad 0 \le x_3 \le 5, \quad 0 \le x_4 \le 5,$$

where

$$f_1(x_1) = \begin{cases} 3x_1, & 0 \le x_1 \le 1, \\ 4x_1 - 1, & 1 \le x_1 \le 5, \end{cases} \qquad g_1(x_1) = \begin{cases} 4x_1 + 1, & 0 \le x_1 \le 1, \\ 3x_1 + 2, & 1 \le x_1 \le 5, \end{cases}$$

$$f_2(x_2) = \begin{cases} 2x_2 + 1, & 0 \le x_2 \le 1, \\ 3x_2, & 1 \le x_2 \le 3, \end{cases} \qquad g_2(x_2) = \begin{cases} 3x_2 + 1, & 0 \le x_2 \le 1, \\ 2x_2 + 2, & 1 \le x_2 \le 3, \end{cases}$$

$$f_3(x_3) = \begin{cases} x_3 + 3, & 0 \le x_3 \le 2, \\ 2x_3 + 1, & 2 \le x_3 \le 3, \\ 3x_3 - 2, & 3 \le x_3 \le 5, \end{cases} \qquad g_3(x_3) = \begin{cases} 3x_3 + 1, & 0 \le x_3 \le 2, \\ 2x_3 + 3, & 2 \le x_3 \le 3, \\ x_3 + 6, & 3 \le x_3 \le 5, \end{cases}$$

$$f_4(x_4) = \begin{cases} x_4 + 1, & 0 \le x_4 \le 1, \\ 2x_4, & 1 \le x_4 \le 3, \\ 3x_4 - 3, & 3 \le x_4 \le 5, \end{cases} \qquad g_4(x_4) = \begin{cases} 4x_4 + 1, & 0 \le x_4 \le 1, \\ 2x_4 + 3, & 1 \le x_4 \le 3, \\ x_4 + 6, & 3 \le x_4 \le 5. \end{cases}$$

The optimal solution is $x^* = \Big(\frac{32}{10}, \frac{21}{10}, 2, \frac{1}{2}, 0, 0, 0\Big)^T$, and $Z(\mathbf{x}^*) = \frac{123}{139}$. Here $B = \{B_1, B_2, B_3\} = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$ and $\mathbf{x}^*_B = (x^*_2, x^*_4, x^*_1)^T = \Big(\frac{21}{10}, \frac{1}{2}, \frac{32}{10}\Big)^T$.

Using formulas (3.1), (3.6), (3.9), (3.10), (3.11), (3.12) and inverse matrix

$$(\beta_{ij}) = \mathbf{B}^{-1} = \begin{pmatrix} 3/20 & 1/4 & -7/20 \\ -1/4 & 1/4 & 1/4 \\ 3/10 & -1/2 & 3/10 \end{pmatrix},$$

we obtain the following range for δ, when $A'_{23} = A_{23} + \delta = 1 + \delta$,

$$-1 \leq \delta \leq 1.$$

Interpretation is producing one unit of commodity 3 now required A'_{23} units of resource 2 instead A_{23}.

4. CHANGES IN THE COEFFICIENTS OF A BASIC VARIABLE

In this section, our goal is to determine the lower and upper bounds for δ which guarantee that the replacement $\mathbf{A}_{.k}$ by $\mathbf{A}'_{.k} = \mathbf{A}_{.k} + \mathbf{e}_i\delta, k \in B$, does not affect the optimal basis, and the original optimal solution $\mathbf{x}^*$ remains feasible and optimal. By taking this replacement, the optimal basis $\mathbf{B}$ will be replaced with $\overline{\mathbf{B}} = \mathbf{B} + \delta\mathbf{e}_i\mathbf{e}_k^T$ where $\mathbf{e}_j$ is a unit vector. The inverse matrix $\overline{\mathbf{B}}$ is

$$\overline{\mathbf{B}}^{-1} = \mathbf{B}^{-1} - \delta\frac{\beta_{.i}\beta_{k.}}{1 + \delta\beta_{ki}}, \qquad 1 + \delta\beta_{ki} \neq 0, \tag{4.1}$$

by the Sherman-Morrison formulas. This change of the basis matrix will affect the feasibility of vector $\mathbf{x}^*$. However, it may affect the optimal value of $Z(\mathbf{x})$ and hence, can change the reduced costs $\eta_j^+(\mathbf{x}^*)$ and $\eta_j^-(\mathbf{x}^*)$. So, by replacing A_{ik} with $A_{ik} + \delta$, from (2.3) we will have

$$\begin{aligned} \bar{\mathbf{x}}_B &= \overline{\mathbf{B}}^{-1}\mathbf{b} - \sum_{j \in N} \overline{\mathbf{B}}^{-1}\mathbf{A}_{.j}\delta^j_{\nu_j} = \\ &= (\mathbf{B}^{-1} - \delta\frac{\beta_{.i}\beta_{k.}}{1 + \delta\beta_{ki}})\mathbf{b} - \sum_{j \in N}(\mathbf{B}^{-1} - \delta\frac{\beta_{.i}\beta_{k.}}{1 + \delta\beta_{ki}})\mathbf{A}_{.j}\delta^j_{\nu_j} = \\ &= \mathbf{x}^*_B - \delta\frac{\beta_{.i}\beta_{k.}}{1 + \delta\beta_{ki}}(\mathbf{b} - \sum_{j \in N}\mathbf{A}_{.j}\delta^j_{\nu_j}). \end{aligned}$$

Now the h-th component of $\bar{\mathbf{x}}_B$ is given by

$$\bar{x}_{B_h} = x^*_{B_h} - \delta\frac{\beta_{hi}\beta_{k.}}{1 + \delta\beta_{ki}}(b - \textstyle\sum_{j \in N} A_{.j}\delta^j_{\nu_j}), \quad h = 1, \ldots, m.$$

This new basic solution $\bar{\mathbf{x}}_B$ will be feasible if

$$\delta^{B_h}_{\mu(B_h)} \leq x^*_{B_h} - \delta\frac{\beta_{hi}\beta_{k.}}{1 + \delta\beta_{ki}}(b - \sum_{j \in N} A_{.j}\delta^j_{\nu_j}) \leq \delta^{B_h}_{\mu(B_h)+1}, \quad h = 1, \ldots, m. \tag{4.2}$$

From (4.2), we obtain a range for δ.

Since the change in the basis matrix will be affected in the feasibility of vector $\mathbf{x}^*$, thus, it may affect the optimal value of $Z(\mathbf{x})$ and hence, can change the reduced costs $\eta_j^+(\mathbf{x}^*)$ and $\eta_j^-(\mathbf{x}^*)$. Hence, we will have

$$Z(\bar{\mathbf{x}}) = \frac{\mathbf{c}_B(\mathbf{B}^{-1} - \delta\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}})\mathbf{b} + \sum_{j\in N}\left(c^j_{\nu_j} - \mathbf{c}_B(\mathbf{B}^{-1} - \delta\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}})\mathbf{A}_{.j}\right)\delta^j_{\nu_j} + \alpha}{\mathbf{d}_B(\mathbf{B}^{-1} - \delta\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}})\mathbf{b} + \sum_{j\in N}\left(d^j_{\nu_j} - \mathbf{d}_B(\mathbf{B}^{-1} - \delta\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}})\mathbf{A}_{.j}\right)\delta^j_{\nu_j} + \beta} = \tag{4.3}$$

$$= \frac{P(\mathbf{x}^*) - \delta\mathbf{c}_B\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}}(\mathbf{b} - \sum_{j\in N}\mathbf{A}_{.j}\delta^j_{\nu_j})}{D(\mathbf{x}^*) - \delta\mathbf{d}_B\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}}(\mathbf{b} - \sum_{j\in N}\mathbf{A}_{.j}\delta^j_{\nu_j})}.$$

To preserve the strict positivity of the denominator $D(\mathbf{x})$, we need to have

$$D(\mathbf{x}^*) - \delta\mathbf{d}_B\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}}(\mathbf{b} - \sum_{j\in N}\mathbf{A}_{.j}\delta^j_{\nu_j}) > 0. \tag{4.4}$$

But since $1+\delta\beta_{ki} \neq 0$, we assume that $1+\delta\beta_{ki} > 0$ and will have

$$\delta\begin{cases} > \dfrac{-1}{\beta_{ki}}, & \text{if } \beta_{ki} > 0, \\ < \dfrac{-1}{\beta_{ki}}, & \text{if } \beta_{ki} < 0. \end{cases} \tag{4.5}$$

From (4.5), the relation (4.4) is satisfied if

$$D(\mathbf{x}^*) + \delta\beta_{ki}D(\mathbf{x}^*) - \delta\mathbf{d}_B\beta_{.i}\beta_{k.}(\mathbf{b} - \sum_{j\in N}\mathbf{A}_{.j}\delta^j_{\nu_j}) > 0,$$

which implies

$$\delta\begin{cases} > \dfrac{-D(\mathbf{x}^*)}{G}, & \text{if } G > 0, \\ < \dfrac{-D(\mathbf{x}^*)}{G}, & \text{if } G < 0, \end{cases} \tag{4.6}$$

where $G = \beta_{ki}D(\mathbf{x}^*) - \mathbf{d}_B\beta_{.i}\beta_{k.}(\mathbf{b} - \sum_{j\in N}\mathbf{A}_{.j}\delta^j_{\nu_j})$.

Now, the optimal solution $\mathbf{x}^*$ of the original $PLFP$ problem remains optimal for the perturbed $PLFP$ problem if

$$\eta_j^+(\bar{\mathbf{x}}) = \Delta_j^{'} + \delta\mathbf{c}_B\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}}\mathbf{A}_{.j} -$$

$$- \frac{P(\mathbf{x}^*) - \delta\mathbf{c}_B\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}}(\mathbf{b} - \sum_{j\in N}\mathbf{A}_{.j}\delta^j_{\nu_j})}{D(\mathbf{x}^*) - \delta\mathbf{d}_B\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}}(\mathbf{b} - \sum_{j\in N}\mathbf{A}_{.j}\delta^j_{\nu_j})}\left(\Delta_j^{''} + \delta\mathbf{d}_B\frac{\beta_{.i}\beta_{k.}}{1+\delta\beta_{ki}}\mathbf{A}_{.j}\right) \geq 0. \tag{4.7}$$

From (4.4), the last relation is satisfied if

$$\max_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\eta_j^+(\mathbf{x}^*)}{W}:W>0\right\}\le\delta\le\min_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\eta_j^+(\mathbf{x}^*)}{W}:\ W<0\right\},\qquad(4.8)$$

and similarly, for $\eta_j^-(\bar{\mathbf{x}})\le 0$ we will have

$$\max_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\eta_j^-(\mathbf{x}^*)}{W'}:W^{'}<0\right\}\le\delta\le\min_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\eta_j^-(\mathbf{x}^*)}{W'}:\ W^{'}>0\right\},\qquad(4.9)$$

where

$$W=\Big(D(\mathbf{x}^*)\mathbf{c}_B-P(\mathbf{x}^*)\mathbf{d}_B\Big)\beta_{.i}\beta_{k.}\mathbf{A}_{.j}+$$

$$+\Big(\mathbf{c}_B\Delta_j^{''}-\mathbf{d}_B\Delta_j^{'}\Big)\beta_{.i}\beta_{k.}(\mathbf{b}-\sum_{j\in N}\mathbf{A}_{.j}\delta_{\nu_j}^j)+\beta_{ki}D(\mathbf{x}^*)\eta_j^+(\mathbf{x}^*),$$

and

$$W^{'}=\Big(D(\mathbf{x}^*)\mathbf{c}_B-P(\mathbf{x}^*)\mathbf{d}_B\Big)\beta_{.i}\beta_{k.}\mathbf{A}_{.j}+$$

$$+\Big(\mathbf{c}_B\bar{\Delta}_j^{''}-\mathbf{d}_B\bar{\Delta}_j^{'}\Big)\beta_{.i}\beta_{k.}(\mathbf{b}-\sum_{j\in N}\mathbf{A}_{.j}\delta_{\nu_j}^j)+\beta_{ki}D(\mathbf{x}^*)\eta_j^-(\mathbf{x}^*).$$

Therefore, we have proved the following theorem:

Theorem 4.1. *If δ satisfies (4.2), (4.5), (4.6), (4.8) and (4.9) then $\bar{\mathbf{x}}$ is an optimal solution of the perturbed $PLFP$ problem.*

Remark 4.2. Lower and upper bounds given in Theorem 4.1 are a generalization of the corresponding bounds for LP, PLP and LFP. Indeed,

1. If $\beta_0=1$ and $g_j(x_j)=0, j=1,2,\ldots,n$, then the $PLFP$ reduces to PLP and this means that $D(\mathbf{x}^*)=1,\ \Delta_j^{''}=\bar{\Delta}_j^{''}=0,\ \eta_j^+(\mathbf{x}^*)=\Delta_j^{'}$ and $\eta_j^-(\mathbf{x}^*)=\bar{\Delta}_j^{'}, j\in N$. In this case, the relation (4.5) and bounds (4.2) are hold too. The relations (4.8) and (4.9) become respectively as follows

$$\max_{j\in N}\left\{\frac{-\Delta_j^{'}}{W}:W>0\right\}\le\delta\le\min_{j\in N}\left\{\frac{-\Delta_j^{'}}{W}:\ W<0\right\},$$

$$\max_{j\in N}\left\{\frac{-\bar{\Delta}_j^{'}}{W'}:W^{'}<0\right\}\le\delta\le\min_{j\in N}\left\{\frac{-\bar{\Delta}_j^{'}}{W'}:\ W^{'}>0\right\},$$

where

$$W=\big(\mathbf{c}_B\beta_{.i}\beta_{k.}\mathbf{A}_{.j}+\beta_{ki}\Delta_j^{'}\big),$$

and

$$W^{'}=\big(\mathbf{c}_B\beta_{.i}\beta_{k.}\mathbf{A}_{.j}+\beta_{ki}\bar{\Delta}_j^{'}\big).$$

2. If $\beta_0 = 1$, $g_j(x_j) = 0$ and $f_j(x_j), j = 1, 2, \ldots, n$, are linear functions then the $PLFP$ reduces to LP with bounded variables. In this case, the relation (4.5) is satisfied and the relations (4.2), (4.8), (4.9) are respectively as follows

$$0 \leq x^*_{B_h} - \delta \frac{\beta_{hi}\beta_{k.}}{1 + \delta\beta_{ki}}(b - \sum_{j \in N} A_{.j}t_j) \leq \mathbf{u}_{B_h}, \qquad h = 1, \ldots, m,$$

$$\max_{j \in N} \left\{ \frac{-\Delta'_j}{W} : W > 0 \right\} \leq \delta \leq \min_{j \in N} \left\{ \frac{-\Delta'_j}{W} : \ W < 0 \right\}, \quad \text{if} \quad x_j = 0,$$

$$\max_{j \in N} \left\{ \frac{-\bar{\Delta}'_j}{W'} : W' < 0 \right\} \leq \delta \leq \min_{j \in N} \left\{ \frac{-\bar{\Delta}'_j}{W'} : \ W' > 0 \right\}, \quad \text{if } x_j = u_j,$$

where

$$t_j = 0 \ \text{ or } u_j, \quad W = \left(\mathbf{c}_B\beta_{.i}\beta_{k.}\mathbf{A}_{.j} + \beta_{ki}\Delta'_j\right), \quad \text{and} \quad W' = \left(\mathbf{c}_B\beta_{.i}\beta_{k.}\mathbf{A}_{.j} + \beta_{ki}\bar{\Delta}'_j\right).$$

3. If both $g_j(x_j)$ and $f_j(x_j), j = 1, 2, \ldots, n$, are linear functions then the $PLFP$ reduces to LFP and this means that $c^j_{\nu_j} = c_j$, $d^j_{\nu_j} = d_j$, $\Delta'_j = c_j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}$, $\Delta''_j = d_j - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}$ and $\eta_j(\mathbf{x}^*) = \Delta'_j - Z(\mathbf{x}^*)\Delta''_j$. Therefore, $\bar{\mathbf{x}}$ is an optimal solution (by $u_j = \infty, \ \forall j$) if

$$\max_{1 \leq h \leq m} \left\{ \frac{-x^*_{B_h}}{H} : H > 0 \right\} \leq \delta \leq \min_{1 \leq h \leq m} \left\{ \frac{-x^*_{B_h}}{H} : \ H < 0 \right\},$$

$$\max_{j \in N} \left\{ \frac{-D(\mathbf{x}^*)\eta_j(\mathbf{x}^*)}{W} : W > 0 \right\} \leq \delta \leq \min_{j \in N} \left\{ \frac{-D(\mathbf{x}^*)\eta_j(\mathbf{x}^*)}{W} : \ W < 0 \right\},$$

where $H = x^*_{B_h}\beta_{ki} - \beta_{hi}\beta_{k.}\mathbf{b}$,

$$W = \left(D(\mathbf{x}^*)\mathbf{c}_B - P(\mathbf{x}^*)\mathbf{d}_B\right)\beta_{.i}\beta_{k.}\mathbf{A}_{.j} + \left(\mathbf{c}_B\Delta''_j - \mathbf{d}_B\Delta'_j\right)\beta_{.i}\beta_{k.}(\mathbf{b} + \beta_{ki}D(\mathbf{x}^*)\eta_j(\mathbf{x}^*).$$

Example 4.3. Consider Example 3.3. Let the basis matrix $\mathbf{B}$ be replaced by $\overline{\mathbf{B}}$, where $\overline{\mathbf{B}} = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 3 & 1+\delta \\ 1 & 3 & 2 \end{pmatrix}$.

Using Theorem 4.1 we obtain the following interval for δ

$$\frac{-10}{11} \leq \delta \leq \frac{10}{21}.$$

5. SUMMARY

The sensitivity analysis of optimal solutions has been presented in this paper. Two cases were considered: (i) change in the coefficients of a non-basic variable, (ii) change in the coefficients of a basic variable. In each case the underlying theory for sensitivity analysis has been presented to in order to obtain bounds for each perturbation and also to special cases as LP, LFP and PLP.

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