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ON A PROPERTY OF ϕ-VARIATIONAL MODULAR SPACES

Abstract. Maligranda pointed out whether condition (B.1) is satisfied in the variational modular space X^*_ρ is an open problem. We will answer this open problem in $X^{*\prime}_\rho$, a subspace of X^*_ρ. As a consequence this modular space $X^{*\prime}_\rho$ can be F-normed.

Keywords: condition (B.1), modular, ϕ-function, ϕ-variation.

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1. INTRODUCTION

Modular spaces were originally defined by Nakano. We can refer to [7] for the theory of modular spaces in the sense of Nakano. Mazur and Orlicz [3] and Musielak and Orlicz [5] modified the definition of the modular space proposed by Nakano. In the definition of the modular space they wanted to avoid the lattice structure in the space X on which the modular is defined as well as the monotonicity axiom for the modular. Finally in [5] the following definition of the modular was given.

Let X be a real vector space. A functional $\rho : X \longrightarrow [0,\infty)$ is called a *modular* if it satisfies the conditions:

(M1) $\rho(x) = 0$ if and only if $x = 0$;
(M2) $\rho(-x) = \rho(x)$ for any $x \in X$;
(M3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $x, y \in X, \alpha, \beta \geq 0$ such that $\alpha + \beta = 1$.

Condition (M3) does not use any order structure of X but it guarantees that the function $f(\lambda) = \rho(\lambda x)$ is nondecreasing on $\mathbb{R}^+ = [0,\infty)$ for a fixed $x \in X$. For a new theory of modular spaces we refer to [6].

In this paper we will deal with the ϕ-variational modular ρ_ϕ, where ϕ is a ϕ-function.

A function $\phi : [0,\infty) \longrightarrow [0,\infty)$ is called a *ϕ-function*, if ϕ satisfies the following conditions:

(i) $\phi(0) = 0$ and $\phi(u) > 0$ for $u > 0$;

(ii) ϕ is increasing and continuous;

(iii) $\lim\limits_{u\to\infty} \phi(u) = \infty$.

If $\|\cdot\|$ is an F-norm in real vector space X, then $(X, \|\cdot\|)$ is called an F-space.

Let X be the space of all real-valued functions x on the compact interval $[a, b]$ such that $x(a) = 0$ and ϕ be a ϕ-function on $\mathbb{R}^+$. We define on X the ϕ-variation ρ_ϕ by

$$\rho_\phi(x) = \sup \sum_{k=1}^{n} \phi(|x(t_k) - x(t_{k-1})|),$$

where the supremum is taken over all partitions $\pi : a = t_0 < t_1 < \cdots < t_n = b$. For $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X$, $\alpha|x(t_k) - x(t_{k-1})| + \beta|y(t_k) - y(t_{k-1})| \leq \max\{|x(t_k) - x(t_{k-1})|, |y(t_k) - y(t_{k-1})|\}$. So $\rho_\phi(\alpha x + \beta y) \leq \rho_\phi(x) + \rho_\phi(y)$. Thus ρ_ϕ is a modular on X. This kind of variation was introduced by Young in 1938 and as a modular space X_{ρ_ϕ} (for the definition of this space see below) was considered first by Musielak and Orlicz in [4] and [5], (see also [1]).

Two vector subspaces of X can be defined:

$$X_{\rho_\phi} = \{x \in X : \lim_{\lambda\to 0^+} \rho_\phi(\lambda x) = 0\}$$

and

$$X^*_{\rho_\phi} = \{x \in X : \rho_\phi(\lambda x) < \infty \text{ for some } \lambda = \lambda(x) > 0\}.$$

It is obvious that $X_{\rho_\phi} \subset X^*_{\rho_\phi}$. We say that the modular ρ_ϕ (or the space $X^*_{\rho_\phi}$) satisfies condition (B.1) if $\lim\limits_{\lambda\to 0^+} \rho_\phi(\lambda x) = 0$ for any $x \in X^*_{\rho_\phi}$ (see [2,3,5] and [1]).

Condition (B.1) is necessary and sufficient in order that the following functional

$$\|x\| = \inf\left\{c : \rho_\phi\left(\frac{x}{c}\right) \leq c\right\} \quad (x \in X^*_{\rho_\phi})$$

is an F-norm on $X^*_{\rho_\phi}$. This functional was first defined by Mazur and Orlicz in [3]. It is easy to see that it satisfies the conditions:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|-x\| = \|x\|$ for all $x \in X^*_{\rho_\phi}$;

(c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X^*_{\rho_\phi}$.

From conditions (c) it follows that the addition of elements is a continuous operation in $X^*_{\rho_\phi}$. However, as it is easy to see, the operation of the multiplication by scalars of the elements from $X^*_{\rho_\phi}$ is continuous if and only if condition (B.1) is satisfied. This follows by the fact that for any sequence $(x_n)_{n=1}^\infty$ in $X^*_{\rho_\phi}$ we have that $\|x_n\| \to 0$ as $n \to \infty$ if and only if $\rho_\phi(\lambda x_n) \to 0$ as $n \to \infty$ for any $\lambda > 0$. Therefore, the variational modular space $X^*_{\rho_\phi}$ is an F-space if and only if condition (B.1) is satisfied.

In contrast to the Orlicz space, it remains an open problem whether condition (B.1) is satisfied in the space $X^*_{\rho_\phi}$ (see [1]).

A criterion for local boundedness of a variational modular space $X^*_{\rho_\phi}$ is given in [8]. Local boundedness of $X^*_{\rho_\phi}$ guarantees that the modular ρ_ϕ satisfies condition (B.1). As a consequence such a modular space can be F-normed. Thus the result in [8] partly answers the above open problem.

In this paper, we will try to remove the criterion in [8] which guarantees that $X^*_{\rho_\phi}$ can be F-normed. We will add a weak condition on X^*_{ρ}, and we can answer the open problem in [1] partly.

2. MAIN RESULT

The following Lemma 2, which was proved by J. Musielak and W. Orlicz (see [1,4] or [5]), will play an important role in the proof of our main result in this paper. Let us begin with Lemma 2.1.

Lemma 2.1. *Let $f \in X^*_{\rho_\phi}$, Then the set of points of discontinuity of f is countable.*

Proof. Let $f \in X^*_{\rho_\phi}$. Then there exists $k > 0$ such that

$$\rho_\phi(kf) = \sup_\pi \sum_{i=1}^n \phi(k|f(t_i) - f(t_{i-1})|) < \infty, \tag{2.1}$$

where supremum is taken over all partitions $\pi : a = t_0 < t_1 < \cdots < t_n = b$. For each $x \in [a, b]$, we define the *jump*

$$J_f(x) = \lim_{r \to 0} \sup\{\phi(k|f(y) - f(x)|)\},$$

where the sup is taken for $y \in [x - r, x + r]$.

By (2.1), we have

(i) For any given $\varepsilon > 0$, there is only a finite number of points $x \in [a, b]$, such that

$$J_f(x) > \varepsilon.$$

By the continuity at 0 of ϕ, we have

(ii) f is continuous at x if and only if $J_f(x) = 0$.

Let

$$A_i = \{x \in [a, b] : J_r(x) > \frac{1}{i}\}, \quad A = \{x \in [a, b] : x \text{ is discontinuity point of } f\}.$$

Then by (i) and (ii), we have

$$A = \bigcup_{i=1}^\infty A_i.$$

This implies that the set A which is the set of points, where f is not continuous is a countable set. □

Lemma 2.2 (Helly's extraction theorem)**.** *If* $\sup_n \rho_\phi(kx_n) < \infty$ *for some* $k > 0$, *then there exists a function* $x \in X_{\rho_\phi}$ *such that* $x_n(t) \to x(t)$ *for every* $t \in [a, b]$ (see J. Musielak and W. Orlicz, 1959 or page 13 in [1]).

We give a sketch of the proof of Lemma 2.2 here: We write $v_n(t) = \rho_\phi(kx_n; a, t) = \sup \sum_{i=1}^{m} \phi(k|x_n(t_i) - x_n(t_{i-1})|)$, where the supremum is taken over all partitions π: $a = t_0 < t_1 < \cdots < t_m = t \le b$. The functions $v_n(t)$ are non-decreasing and bounded by K in $[a, b]$. Thus, we conclude from the well-known Helly extracting theorem for sequences of monotonic functions that the sequence $v_n(t)$ includes subsequences of $v_{n_i}(t)$ convergent to a non-decreasing function $v(t)$ at every point t of the interval $[a, b]$. Using the diagonal method we extract from the sequence of indices n_i a subsequence n_{i_j} such that $x_{n_{i_j}}$ is convergent at every rational point of the interval $[a, b]$ and at the points a, b. Writing $x_{n_{i_j}}(t) = x_j^*(t)$ and $v_{n_{i_j}}(t) = v_j^*(t)$, we obtain $v_j^*(t) \to v(t)$ for every $t \in [a, b]$ and $x_j^* \to x(t)$ for every rational $t \in [a, b]$ and for $t = a, t = b$. Now, let us assume that t_0 is a non-rational point of continuity of the function $v(t)$ in (a, b). We can prove that the numerical sequence $x_j^*(t_0)$ is convergent. The set of points of discontinuity of the function $v(t)$ being at most enumerable, the diagonal method enables us to extract from the sequence $x_j^*(t)$ a subsequence convergent to a function $x(t)$ at every point of the interval $[a, b]$. Evidently, $\rho_\phi(kx) \le K$.

Our main result is

Theorem 2.3. *Let* X *be the space of real-valued functions in the interval* $[a, b]$ *such that* $x(a) = 0$ *and let*

$$\rho_\phi(x) = \sup_\pi \sum_{i=1}^{n} \phi(|x(t_k) - x(t_{k-1})|),$$

where the supremum is taken over all partitions $\pi : a = t_0 < t_1 < \cdots < t_n = b$. *Define*

$$X^*_{\rho_\phi}{}' = \left\{x : x \in X^*_\rho \text{ and all the discontinuity points of } x \text{ are isolated}\right\}.$$

Then condition (B.1) *is satisfied in the space* $X^*_{\rho_\phi}{}'$.

Proof. For $x \in X^*_{\rho_\phi}{}'$, there exists $k > 0$ such that

$$\rho_\phi(kx) = \sup \sum_{i=1}^{n} \phi(k|x(h_i) - x(h_{i-1})|) < +\infty,$$

where sup is taken over all partitions $\pi : a = h_0 < h_1 < \cdots < h_n = b$. By Lemma 2.1 and the definition of $X^*_{\rho_\phi}{}'$, we know that $x(t)$ have at most countable discontinuity points and all this discontinuity points are isolated. Without loss of generality, we can suppose the countable discontinuity points are $t_0 < t_1 < \cdots < t_n < \cdots$ and the endpoints of $[a, b]$ are continuity points. Now we construct x_n as follows.

For convenience, we denote a, t_0, t_1, b as $s_1^1, s_2^1, s_3^1, s_4^1$, which construct a partition of the closed interval $[a, b]$. Define

$$x_1(t) = \begin{cases} x(s_1^1), & a = s_1^1 \le t < s_2^1, \\ x(s_2^1), & s_2^1 \le t < s_3^1, \\ x(b), & s_3^1 \le t < s_4^1 = b. \end{cases}$$

Next, we divide every interval of $[a, t_0], [t_0, t_1], [t_1, t_2]$ and $[t_2, b]$ into two non-overlapping intervals of the same length. All the intermediate points and all endpoints of the closed intervals construct a partition of the interval $[a, b]$. we denote all the points as $a = s_1^2 < s_2^2 < \cdots < s_9^2 = b$. Define

$$x_2(t) = \begin{cases} x(s_i^2), & s_i^2 \le t < s_{i+1}^2, i = 1, 2, \cdots, 7, \\ x(b), & s_8^2 \le t \le s_9^2 = b. \end{cases}$$

Then we divide every interval of $[a, t_0], [t_0, t_1], [t_1, t_2], [t_2, t_3]$ and $[t_3, b]$ into three non-overlapping intervals of the same length. All the intermediate points and all the end points of the closed interval construct a partition of the interval $[a, b]$. we denote all the points as $a = s_1^3 < s_2^3 < \cdots < s_{16}^3 = b$. Define

$$x_3(t) = \begin{cases} x(s_i^2), & s_i^2 \le t < s_{i+1}^2, i = 1, 2, \cdots, 14, \\ x(b), & s_{15}^2 \le t \le s_{16}^2 = b. \end{cases}$$

Generally, we divided every interval of $[a, t_0], [t_0, t_1], \cdots, [t_{n-1}, t_n]$ and $[t_n, b]$ into n non-overlapping intervals of the same length. All the intermediate points and all the endpoints of the closed interval construct a partition of the interval $[a, b]$. we denote all the points as $a = s_1^n < s_2^n < \cdots < s_{(n+1)^2}^n = b$. Define

$$x_n(t) = \begin{cases} x(s_i^n), & s_i^n \le t < s_{i+1}^n, i = 1, 2, \cdots, (n+1)^2 - 2, \\ x(b), & s_{(n+1)^2-1}^n \le t \le s_{(n+1)^2}^n = b. \end{cases}$$

For every n, we denote $\{s_{i_1}^n, s_{i_2}^n, \cdots, s_{i_l}^n\}$ as a subset of $\{s_1^n, s_2^n, \cdots, s_{(n+1)^2}^n\}$, and $\{s_{i_1}^n, s_{i_2}^n, \cdots, s_{i_l}^n\}$ is a partition of $[a, b]$. If $s_{(n+1)^2-1}^n \in \{s_{i_1}^n, s_{i_2}^n, \cdots, s_{i_l}^n\}$, we replace $s_{(n+1)^2-1}^n$ with b in $\{s_{i_1}^n, s_{i_2}^n, \cdots, s_{i_l}^n\}$. If $\{s_{(n+1)^2-1}^n, b\} \subset \{s_{i_1}^n, s_{i_2}^n, \cdots, s_{i_l}^n\}$, we delete $s_{(n+1)^2-1}^n$ in $\{s_{i_1}^n, s_{i_2}^n, \cdots, s_{i_l}^n\}$. Thus we guarantee the following equality

$$\sum_{j=1}^{l} \phi(k|x_n(s_{i_j}^n) - x_n(s_{i_{j-1}}^n)|) = \sum_{j=1}^{l} \phi(k|x(s_{i_j}^n) - x(s_{i_{j-1}}^n)|),$$

and do not affect computation of $\sum\limits_{j=1}^{l} \phi(k|x_n(s_{i_j}^n) - x_n(s_{i_{j-1}}^n)|)$ with original $\{s_{i_1}^n, s_{i_2}^n, \cdots, s_{i_l}^n\}$. We use π representing all partitions of $[a, b]$ and π_1 representing all partitions of $[a,b]$ which are generated by $\{s^n_{i_1}, s^n_{i_2}, \cdots, s^n_{i_l}\} \subset \{s^n_1, s^n_2, \cdots, s^n_{(n+1)^2}\}$. Then

$$\begin{aligned}\rho_\phi(kx_n) = \sup_\pi \sum_{j=1}^{m} \phi(k|x_n(h_j) - x_n(h_{j-1})|) = \\ = \sup_{\pi_1} \sum_{j=1}^{l} \phi(k|x_n(s^n_{i_j}) - x_n(s^n_{i_{j-1}})|) = \\ = \sup_{\pi_1} \sum_{j=1}^{l} \phi(k|x(s^n_{i_j}) - x(s^n_{i_{j-1}})|) \leq \\ \leq \rho_\phi(kx) < +\infty,\end{aligned}$$

here we get the second equality by the definition of x_n. So

$$\sup_n \rho_\phi(kx_n) \leq \rho_\phi(kx) < +\infty.$$

By Lemma 2.2, there exists $y \in X_{\rho_\phi}$, such that $x_n(t) \to y(t)$ for every $t \in [a,b]$. On the other hand, by the construction of $x_n(t)$, we know $x_n(t) \to x(t)$ for every $t \in [a,b]$. So $x(t) = y(t)$ for every $t \in [a,b]$ and $x = y \in X_{\rho_\phi}$. This implies condition (B.1) is satisfied in ${X^*_{\rho_\phi}}'$. □

Corollary 2.4. *${X^*_{\rho_\phi}}'$ can always be F-normed.*

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REFERENCES

[1] L. Maligranda, *Orlicz Space and Interpolation*, Seminars in Mathematics, Univ. Estadual de Campinas, Campinas SP, Brasil, 1989.

[2] L. Maligranda, W. Orlicz, *On some properties of functions of generalized variation*, Monatsh. Math. **104** (1987), 53–65.

[3] S. Mazur, W. Orlicz, *On some classes of linear spaces*, Studia Math. **17** (1958), 97–119.

[4] J. Musielak, W. Orlicz, *On generalized variation*, Studia Math. **28** (1959), 11–41.

[5] J. Musielak, W. Orlicz, *On modular spaces,* Studia Math. **18** (1959), 49–65.

[6] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math., 1034, Springer-Varlag, 1983.

[7] H. Nakano, *Modulared Semi-ordered Spaces*, Tokyo, 1950.

[8] Wang Jin Cai, H. Hudzik, *A note on ϕ-variational modular spaces*, Commentationes Mathematicae **41** (2001), 93–98.

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