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**A NOTE  
ON THE DISCRETE SCHRÖDINGER OPERATOR  
WITH A PERTURBED PERIODIC POTENTIAL**

**Abstract.** The aim of this paper is to study the spectrum of the one-dimensional discrete Schrödinger operator with a perturbed periodic potential. We obtain natural conditions under which this perturbation preserves the essential spectrum of the considered operator. Conditions on the number of isolated eigenvalues are given.

**Keywords:** one-dimensional Schrödinger operator, Jacobi operator, perturbation of periodic potential, essential spectrum, discrete part of the spectrum.

**Mathematics Subject Classification:** 47B39, 47B37.

1. INTRODUCTION

In the paper [19] Zelenko described the spectrum of the one-dimensional Schrödinger operator

$$H = -\frac{d^2}{dx^2} + \alpha^2(x)\tilde{V}(\alpha(x)x)$$

acting in the Hilbert space  $L_2(\mathbb{R})$  of all square summable functions on  $\mathbb{R}$ . In [19]  $\tilde{V}$  is a continuous real-valued periodic function and  $\alpha: \mathbb{R} \rightarrow (0, 1)$  is a continuous function such that

$$\lim_{|x| \rightarrow \infty} \alpha(x) = 1.$$

Zelenko obtains conditions under which the perturbation preserves the essential spectrum and an infinite number of isolated eigenvalues appear in a gap of the essential spectrum.

In the present paper we consider the discrete version of those issues. We describe the one-dimensional discrete Schrödinger operator  $H$  on  $l_2(\mathbb{Z}_+)$  with the perturbation

of a periodic potential that is a natural discrete version of the perturbation considered above. More precisely, we study the operator  $H$  defined by

$$(Hu)(n) = u(n-1) + u(n+1) + q(\alpha(n)n)u(n), \quad \text{for } n \in \mathbb{Z}_+, \quad (1.1)$$

for all  $u \in l_2(\mathbb{Z}_+)$  (we set  $u(-1) = 0$ , in what follows, this will be always assumed), where  $q$  is a continuous real-valued function with a period  $T$  (it is supposed to be a positive integer) and  $\{\alpha(n)\}_{n \in \mathbb{Z}_+}$  is a sequence of numbers belonging to  $(0, 1)$ . A physical interpretation of such perturbation is the following: the function  $q$  is the electric potential of an infinite atomic lattice and its perturbation corresponds to a local dilatation, for instance, some local heating. The function  $q$  could be also interpreted as potential of a one-dimensional crystal (cf. [8, Chapter 6]). We denote by  $H^0$  the one-dimensional discrete Schrödinger operator with a potential  $q$ , that is the operator acting as follows

$$(H^0u)(n) = u(n-1) + u(n+1) + q(n)u(n), \quad \text{for } n \in \mathbb{Z}_+, \quad (1.2)$$

for all  $u \in l_2(\mathbb{Z}_+)$ . This operator corresponds to an infinite lattice without any perturbation.

In [19–21] were obtained some properties of the spectrum of perturbed operators expressed in terms of the family of operators  $\{H_m\}_{m \in \mathbb{Z}_+}$  which comes from setting the value of a perturbing sequence. In the present paper we also consider a family of potentials defined in similar way, that is for  $m \in \mathbb{Z}_+$  we denote by  $H_m$  the one-dimensional discrete Schrödinger operator on  $l_2(\mathbb{Z}_+)$  with the potential

$$q_m = q(\alpha(m)n), \quad n \in \mathbb{Z}_+.$$

We cannot apply Zelenko's results directly, because we immediately encounter some technical problems that arise from differences between discrete and continuous cases. For example, contrary to the continuous case, there is no unitary equivalence between operators  $H_m$  and a scalar multiple of the operator  $H^0$  in the discrete version of the considered problem.

The spectrum of the one-dimensional discrete Schrödinger operator on  $l_2(\mathbb{Z})$  (or, in another terminology, a Jacobi matrix) with a periodic potential and its perturbation was studied by Naïman ([12–15], see also [8, Chapter 6]). The discrete spectrum of perturbed Jacobi matrices on  $l_2(\mathbb{Z}_+)$  was also investigated in papers [2, 4, 5] by Cojuhari (for related results see [3, 6, 7]). In a more recent paper [1] estimate formulae for the number of the eigenvalues created by perturbation in the gaps of the unperturbed operator are obtained.

In this paper we focus on a particular form of the periodic potential perturbation as in (1.1). We prove, under some natural conditions, the invariance of the essential spectrum of  $H^0$  under the perturbation. We also provide some results concerning the discrete spectrum. We concentrate only on one of the spectrum gaps but our methods may be applied to obtain analogous results for any of them. Furthermore, we show that under some conditions the spectrum of the operator  $H^0$  can be expressed by a spectra of operators  $H_m$ . It should be noted that, by periodicity of the function  $q$ , one

may use the general theory of perturbation as an alternative way for describing the spectrum of  $H^0$  (cf. [10, 12]). Actually, spectral properties of the operator  $H^0$  could be described by the classical Floquet theory, but we prove that  $\sigma(H^0)$  coincides with the set  $\{\lambda \in \mathbb{C} : \limsup_{m \rightarrow \infty} \|R_\lambda(H_m)\| = \infty\}$  directly. Moreover, our approach is in a sense simpler.

The remaining part of the paper is organized as follows. In Section 2 we introduce some basic notations and definitions. Section 3 contains the main results of the paper. Proposition 3.1, provides us with conditions for preservation of the essential under the perturbation. We obtained this result using the Weyl theorem. Theorems 3.3 and 3.4 concern localization of the spectrum of the unperturbed operator  $H^0$  in terms of operators  $H_m$ . Next, in Theorems 3.7 and 3.9, the discrete spectrum of the perturbed is studied by means of the Kneser theorem (cf., for instance, [8, p. 126]).

## 2. PRELIMINARIES

To begin with, we introduce notations that we will use throughout the paper. In what follows,  $\mathbb{Z}_+$  stands for the set  $\{0, 1, 2, \dots\}$ . We write  $l_2(\mathbb{Z}_+)$  for the set of all square summable sequences on  $\mathbb{Z}_+$ . For an operator  $H$ , symbols  $\sigma(H)$ ,  $\sigma_e(H)$ ,  $\rho(H)$  stand for the spectrum, the essential spectrum and the resolvent set of  $H$  respectively.

In the paper we denote by  $H$  the one-dimensional discrete Schrödinger operator on  $l_2(\mathbb{Z}_+)$  with a perturbed periodic potential, that is the operator defined by

$$(Hu)(n) = u(n - 1) + u(n + 1) + q(\alpha(n)n)u(n), \text{ for } n \in \mathbb{Z}_+, \tag{2.1}$$

for all  $u \in l_2(\mathbb{Z}_+)$  (recall that we set  $u(-1) = 0$ ), where  $q: \mathbb{R} \rightarrow \mathbb{R}$  is a continuous periodic function (called the potential) with a period  $T \in \mathbb{Z}_+$  and  $\{\alpha(n)\}_{n \in \mathbb{Z}_+}$  is a sequence of numbers belonging to  $(0, 1)$ . Note that the sequence  $\{q(n)\}_{n \in \mathbb{Z}_+}$  is also periodic with the same period  $T$ . We write  $q_\alpha$  for the potential of the operator  $H$ , that is

$$q_\alpha(n) = q(\alpha(n)n), \text{ for } n \in \mathbb{Z}_+. \tag{2.2}$$

For convenience, we introduce the one-dimensional discrete Schrödinger operator on  $l_2(\mathbb{Z}_+)$  with periodic potential, namely the operator acting as follows

$$(H^0u)(n) = u(n - 1) + u(n + 1) + q(n)u(n), \text{ for } n \in \mathbb{Z}_+, \tag{2.3}$$

for all  $u \in l_2(\mathbb{Z}_+)$ .

For the potential  $q$  and the sequence  $\{\alpha(n)\}_{n \in \mathbb{Z}_+}$ , we will consider the ensuing conditions:

- (1) The potential  $q$  satisfies the Hölder condition, that is

$$\exists \gamma > 0 \exists L > 0 \forall x, y \in \mathbb{R} : |q(x) - q(y)| \leq L|x - y|^\gamma.$$

- (2) The sequence  $\{\alpha(n)\}_{n \in \mathbb{Z}_+}$  satisfies one of the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha(n) = 1$ ,

- (b)  $\lim_{n \rightarrow \infty} (1 - \alpha(n))n = 0$ ,  
 or  
 (c)  $\lim_{n \rightarrow \infty} (1 - \alpha(n))^\gamma n^{2+\gamma} = 0$ , where  $\gamma$  is the constant from the Hölder condition (1).

We also consider the family of potentials  $\{q_m\}_{m \in \mathbb{Z}_+}$  defined by

$$q_m(n) = q(\alpha(m)n), \quad n \in \mathbb{Z}_+. \quad (2.4)$$

We write  $H_m$  for the Schrödinger operator on  $l_2(\mathbb{Z}_+)$  with the potential  $q_m$ . More precisely, for  $m \in \mathbb{Z}_+$  we denote by  $H_m$  the operator acting as follows

$$(H_m u)(n) = u(n-1) + u(n+1) + q_m(n)u(n), \quad \text{for } n \in \mathbb{Z}_+,$$

for all  $u \in l_2(\mathbb{Z}_+)$ .

### 3. RESULTS

The following proposition provides some natural conditions for the preservation of the essential spectrum of  $H^0$  after the perturbation.

**Proposition 3.1.** *Suppose that conditions (1) and (b) are satisfied. Then the essential spectrum of the operator  $H$  coincides with the spectrum of  $H^0$ , that is  $\sigma_e(H^0) = \sigma_e(H)$ .*

*Proof.* Observe that

$$H = H^0 + (q_\alpha - q).$$

Moreover, from the definition of the potential  $q_\alpha(n) = q(\alpha(n)n)$  we obtain

$$|q_\alpha(n) - q(n)| \leq L(1 - \alpha(n))^\gamma, \quad (3.1)$$

where  $L, \gamma$  are constants as in (1). From (3.1) and assumption (b) we infer that the perturbation  $(q_\alpha - q)$  is a compact operator on  $l_2(\mathbb{Z})$ . By the Weyl theorem (cf. [17]) we obtain  $\sigma(H^0) = \sigma_e(H)$ , which completes the proof.  $\square$

**Remark 3.2.** It should be noted that Proposition 3.1 does not cover all cases of preservation of the essential spectrum. Indeed, it suffices to take a perturbing sequence of the form

$$\alpha(n) = 1 - \frac{T}{n} \quad \text{for } n > 0,$$

where  $T$  is a period of  $q$ . Then  $q = q_\alpha$ , so in fact, there is no perturbation at all, but the sequence  $\{\alpha(n)\}_{n \in \mathbb{Z}_+}$  does not satisfy the assumptions of Proposition 3.1.

The following example illustrates the importance of assumption (b) in Proposition 3.1.

**Example 1.** Define a function  $q$  by

$$q(x) = \begin{cases} 2 - 3(x - 2k), & x \in [2k, \frac{1}{2} + 2k] \text{ for some } k \in \mathbb{Z}_+, \\ x - 2k, & x \in [\frac{1}{2} + 2k, 2(k + 1)] \text{ for some } k \in \mathbb{Z}_+. \end{cases}$$

Then

$$q(n) = \begin{cases} 1, & n = 2k + 1, \text{ for some } k \in \mathbb{Z}_+, \\ 2, & n = 2k \text{ for some } k \in \mathbb{Z}_+. \end{cases}$$

Setting  $\alpha(n) = 1 - \frac{1}{2n}$  for every non-zero  $n$ , we obtain

$$q_\alpha(n) = \begin{cases} 2, & n = 0, \\ \frac{1}{2}, & n = 2k + 1, \text{ for some } k \in \mathbb{Z}_+, \\ \frac{3}{2}, & n = 2(k + 1) \text{ for some } k \in \mathbb{Z}_+. \end{cases}$$

Moreover,  $H$  is a one dimensional perturbation of the operator  $H^0 - \frac{1}{2}I$ . As a result,

$$\sigma_e(H) = \{\lambda \in \mathbb{C} : \lambda + \frac{1}{2} \in \sigma_e(\tilde{H})\},$$

so the essential spectrum is not preserved.

Now let  $R_\lambda(H_m) = (H_m - \lambda I)^{-1}$  be the resolvent of the operator  $H_m$  for  $\lambda \in \rho(H_m)$ . In addition to this, we put  $\|R_\lambda(H_m)\| = \infty$  for  $\lambda \in \sigma(H_m)$ . Consider the set

$$\Gamma = \{\lambda \in \mathbb{C} : \limsup_{m \rightarrow \infty} \|R_\lambda(H_m)\| = \infty\}. \tag{3.2}$$

It turns out that under some conditions on the perturbation, the set  $\Gamma$  coincides with the spectrum of the operator  $H^0$ .

**Theorem 3.3.** *Suppose that conditions (1) and (a) are satisfied. Then*

$$\sigma(H^0) \subset \Gamma, \tag{3.3}$$

where  $\Gamma$  is the set defined by (3.2).

*Proof.* We show that  $\mathbb{C} \setminus \Gamma \subset \mathbb{C} \setminus \sigma(H^0)$ . For this purpose, take a  $\lambda \in \mathbb{C} \setminus \Gamma$ . Then there exists  $\delta > 0$  such that

$$\|R_\lambda(H_m)\| < \delta \tag{3.4}$$

for all  $m \in \mathbb{Z}_+$ . Let  $u \in l_2(\mathbb{Z}_+)$  be a finite sequence for which there exists  $N \in \mathbb{Z}_+$  with the property that  $u(n) = 0$  for all  $n > N$ . Let  $m \in \mathbb{Z}_+$  be chosen to satisfy

$$1 - \alpha(m) < \frac{1}{(2\delta L)^{\frac{1}{\gamma}} N}, \tag{3.5}$$

where  $L, \gamma$  are constants as in (1). Condition (a) guarantees that such an  $m$  exists. Then

$$\|(q - q_m)u\|^2 = \sum_{n=0}^N (q(n) - q(\alpha(m)n))^2 |u(n)|^2 \leq L^2((1 - \alpha(m))N)^{2\gamma} \|u\|^2 \quad (3.6)$$

and from (3.5) we obtain

$$\|(q - q_m)u\| \leq \frac{1}{2\delta} \|u\|. \quad (3.7)$$

Next, observe that

$$H^0 u - \lambda u = H_m u - \lambda u + (q - q_m)u,$$

which implies

$$u = R_\lambda(H_m) (H^0 u - \lambda u - (q - q_m)u).$$

The latter equality yields

$$\|u\| \leq \|R_\lambda(H_m)\| \|H^0 u - \lambda u\| + \|R_\lambda(H_m)\| \|(q - q_m)u\|. \quad (3.8)$$

Then (3.8) combined with (3.4) and (3.7) gives

$$\|u\| \leq \delta \|H^0 u - \lambda u\| + \delta \frac{1}{2\delta} \|u\|,$$

which in turn leads to

$$\|u\| \leq 2\delta \|H^0 u - \lambda u\|.$$

Since the constant  $2\delta$  does not depend on the choice of  $u$  and the set of all finite sequences is dense in  $l_2(\mathbb{Z}_+)$ , we conclude that  $\lambda \in \mathbb{C} \setminus \sigma(H^0)$ . This finishes the proof.  $\square$

**Theorem 3.4.** *Suppose that conditions (1), (b) are satisfied. Then*

$$\Gamma \subset \sigma(H^0). \quad (3.9)$$

*Proof.* Take a  $\lambda \in \Gamma$ . Then there exists a sequence  $\{k_n\}_{n \in \mathbb{Z}_+} \subset \mathbb{Z}_+$  such that  $k_n \rightarrow \infty$  and

$$\|R_\lambda(H_{k_n})\| \rightarrow \infty \quad (3.10)$$

when  $n$  tends to  $\infty$ . From (3.10) we deduce that

$$\exists \{u_n\} \subset l_2(\mathbb{Z}_+): \|u_n\| = 1, \quad \|H_{k_n} u_n - \lambda u_n\| \rightarrow 0.$$

Let  $\widetilde{u}_n \in l_2(\mathbb{Z}_+)$  be a sequence with a finite support satisfying the condition

$$\|u_n - \widetilde{u}_n\| \leq \frac{1}{n} \text{ for all } n > 0.$$

From the following inequalities

$$\begin{aligned} \|(\lambda I - H_{k_n})\widetilde{u}_n\| &\leq \|(\lambda I - H_{k_n})u_n\| + \|(\lambda I - H_{k_n})(u_n - \widetilde{u}_n)\| \leq \\ &\leq \|(\lambda I - H_{k_n})u_n\| + (\|H_{k_n}\| + |\lambda|)\|u_n - \widetilde{u}_n\| \end{aligned}$$

we infer that  $(\lambda I - H_{k_n})\widetilde{u}_n \rightarrow 0$ , when  $n$  tends to  $\infty$ . Notice that, without any loss of generality,  $\alpha(n)$  could be chosen to be rational. Denote by  $T_n$  the period of  $q_{k_n}$ . For all non-zero  $n \in \mathbb{Z}_+$ , we define  $v_n$  by

$$v_n(k + m_n T_n) = \widetilde{u}_n(k), \quad k \in \mathbb{Z}_+,$$

where  $m_n$  is chosen to satisfy  $M_n = \min\{n : n \in \text{supp } v_n\} \geq k_n$ . Observe that

$$\|v_n\| \geq 1 - \frac{1}{n}$$

and

$$\begin{aligned} \|\lambda v_n - H^0 v_n\| &\leq \|\lambda v_n - H_{k_n} v_n\| + \|(q - q_{k_n})v_n\| \leq \\ &\leq \|\lambda \widetilde{u}_n - H^0 \widetilde{u}_n\| + L \sup_{k \geq M_n} ((1 - \alpha(k))k)^\gamma \|v_n\|. \end{aligned}$$

As  $n$  tends to  $\infty$ , we obtain

$$\|\lambda v_n - H^0 v_n\| \rightarrow 0,$$

because  $\sup_{k \geq M_n} ((1 - \alpha(k))k) \rightarrow 0$ , when  $M_n \rightarrow \infty$ . This yields  $\lambda \in \sigma(H^0)$ .  $\square$

**Corollary 3.5.** *Suppose that conditions (1), (b) are satisfied. Then*

$$\Gamma = \sigma(H^0). \tag{3.11}$$

**Remark 3.6.** Under assumptions (1) and (a) the sequence  $\{H_m\}_{m \in \mathbb{Z}_+}$  strongly converges to the operator  $H^0$ , that is

$$H_m \xrightarrow{s} H^0 \text{ when } m \rightarrow \infty.$$

Indeed, it follows from (3.6).

Next, assuming that  $\sigma_e(H) = \sigma_e(H^0)$ , we investigate the number of eigenvalues of the operator  $H$  lying in the set  $(-\infty, \min \sigma_e(H))$ . Our method of proof relies on the well known Kneser theorem.

Since the function  $q$  is periodic, it follows that values of

$$\liminf_{k \rightarrow \infty} k^2 \omega_k, \quad \limsup_{k \rightarrow \infty} k^2 \omega_k,$$

where

$$\omega_k = -2 + q(k) - \lambda_0,$$

depend only on signs of  $\omega_k$  for  $k = 1, \dots, T$ . Our next theorems are based on this simple observation.

**Theorem 3.7.** *Suppose that  $\sigma_e(H) = \sigma_e(H^0)$  and set  $\mu_0 = \min \sigma_e(H^0)$ . Suppose that  $\omega_k \geq 0$  for  $k = 1, \dots, T$  and there exists an index  $k_0$  satisfying  $q(k_0) = 2 + \mu_0$  and the following inequality*

$$\liminf_{k \rightarrow \infty} k^2 (q(\alpha(k)k) - q(k)) > -\frac{1}{4} \tag{3.12}$$

*holds. Then the operator  $H$  has at most a finite number of eigenvalues smaller than  $\mu_0$ .*

*Proof.* The potential  $q$  is periodic, so our assumptions imply

$$\liminf_{k \rightarrow \infty} k^2 \omega_k = 0. \quad (3.13)$$

Observe that

$$k^2(-2 + q(\alpha(k)k) - \mu_0) = k^2 \omega_k + k^2(q(\alpha(k)k) - q(k)). \quad (3.14)$$

Set  $\eta_k = -2 + q(\alpha(k)k) - \mu_0$ . From (3.14) and (3.13) we infer that

$$\liminf_{k \rightarrow \infty} k^2 \eta_k \geq \liminf_{k \rightarrow \infty} k^2(q(\alpha(k)k) - q(k)).$$

Assumptions of Theorem 3.7 yield

$$\liminf_{k \rightarrow \infty} k^2 \eta_k > -\frac{1}{4}.$$

It follows from the Kneser theorem that there is at most finite number eigenvalues smaller than  $\mu_0$ .  $\square$

**Corollary 3.8.** *Suppose that conditions (1), (c) are satisfied and set  $\mu_0 = \min \sigma_e(H^0)$ . Suppose that  $\omega_k \geq 0$  for  $k = 1, \dots, T$  and there exists an index  $k_0$  such that  $q(k_0) = 2 + \mu_0$ . Then the operator  $H$  has at most a finite number of eigenvalues smaller than  $\mu_0$ .*

*Proof.* Note that (3.12) automatically holds if conditions (1), (c) are satisfied. Indeed, it is a consequence of the inequality

$$|k^2(q(\alpha(k)k) - q(k))| \leq Lk^{2+\gamma}(1 - \alpha(k))^\gamma.$$

Then from Theorem 3.1 we obtain  $\sigma_e(H) = \sigma_e(H^0)$ . The assertion follows directly from Theorem 3.7.  $\square$

**Theorem 3.9.** *Suppose  $\sigma_e(H) = \sigma_e(H^0)$  and  $\mu_0 = \min \sigma_e(H^0)$ . Suppose that  $\omega_k \leq 0$  for  $k = 1, \dots, T$  and there exists an index  $k_0$  satisfying  $q(k_0) = 2 + \mu_0$  and the following inequality*

$$\limsup_{k \rightarrow \infty} k^2(q(\alpha(k)k) - q(k)) < -\frac{1}{4},$$

*holds. Then operator the  $H$  has an infinite number of eigenvalues smaller than  $\mu_0$ .*

*Proof.* The potential  $q$  is periodic, so by assumptions we obtain

$$\limsup_{k \rightarrow \infty} k^2 \omega_k = 0. \quad (3.15)$$

Analogously to the proof of Theorem 3.7, we observe that

$$k^2 \eta_k = k^2 \omega_k + k^2(q(\alpha(k)k) - q(k)),$$

where  $\eta_k = -2 + q(\alpha(k)k) - \mu_0$ . Equality (3.15) and assumptions of Theorem 3.9 imply

$$\limsup_{k \rightarrow \infty} k^2 \eta_k \leq \limsup_{k \rightarrow \infty} k^2(q(\alpha(k)k) - q(k)) < -\frac{1}{4}.$$

This, in view of the Kneser theorem, leads to the desired conclusion.  $\square$



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