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DOMINATION HYPERGRAPHS OF CERTAIN DIGRAPHS

Abstract. If $D = (V, A)$ is a digraph, its *domination hypergraph* $\mathcal{DH}(D) = (V, \mathcal{E})$ has the vertex set V and $e \subseteq V$ is an edge of $\mathcal{DH}(D)$ if and only if e is a minimal dominating set of D. We investigate domination hypergraphs of special classes of digraphs, namely tournaments, paths and cycles. Finally, using a special decomposition/composition method we construct edge sets of domination hypergraphs of certain digraphs.

Keywords: hypergraph, dominating set, directed graph.

Mathematics Subject Classification: 05C65, 05C20.

1. INTRODUCTION AND DEFINITIONS

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, graphs $G = (V(G), E(G))$ and digraphs $D = (V(D), A(D))$ considered here may have isolated vertices but no multiple edges. In the case of digraphs, loops are forbidden, because they are irrelevant for the investigation of the corresponding domination graphs or hypergraphs. In standard terminology we follow Berge [1].

Let $D = (V, A)$ be a digraph. A nonempty vertex set $V' \subseteq V$ *dominates* the digraph D if and only if every vertex $v \in V \setminus V'$ has a predecessor in V'. $\mathcal{D}(D) = (V, E)$ is the *domination graph* of the digraph $D = (V, A)$ if and only if it has the the same vertex set as D and

$$E = \{\{u, v\} \mid u \neq v \ \wedge \ \{u, v\} \subseteq V \text{ dominates } D\}.$$

Many results on domination in graphs (and digraphs) can be found in Haynes, Hedetniemi and Slater [8, 9]. A lot of the investigations of domination graphs of digraphs deal with tournaments, i.e., oriented complete graphs (cf. Cho et al. [2], Fisher et al. [3–7], McKenna et al. [10]).

The most interesting structural result on domination graphs of tournaments is due to Fisher, Lundgren, Merz and Reid:

Theorem 1.1 ([3, 6]). *The domination graph of a tournament is either a spiked odd cycle with or without isolated vertices, or a forest of caterpillars.*

Note that a *caterpillar* and a *spiked cycle* is a connected graph such that the removal of all end vertices results in a (possibly trivial) path and a cycle, respectively. In domination graphs, edges represent only dominating sets of cardinality two, but in many cases dominating sets of other cardinalities are of interest (cf. [8, 9]). Therefore the following definition is natural: If $D = (V, A)$ is a digraph its *domination hypergraph* $\mathcal{DH}(D) = (V, \mathcal{E})$ has the vertex set V and $e \subseteq V$ is an edge of $\mathcal{DH}(D)$ if and only if e is a minimal dominating set of D.

Figure 1 shows a tournament T_5 with five vertices and its domination hypergraph.

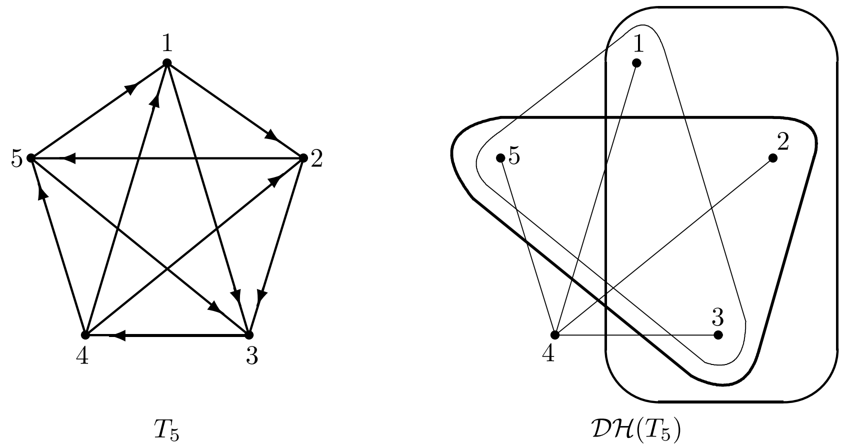

Fig. 1. A tournament T_5 and its domination hypergraph

In a digraph $D = (V, A)$ a vertex $v \in V$ is a *source* [*sink*] if and only if it has indegree $d^-(v) = 0$ [outdegree $d^+(v) = 0$]. By $N^+(v)$ and $N^-(v)$ we denote the set of all successors and the set of all predecessors of v in D, respectively.

Note that a loop in $\mathcal{DH}(D)$ represents a vertex of the digraph D dominating all other vertices, e.g. if D is a tournament containing such a vertex v then v is the (unique) source of D. Moreover, for any digraph D the domination graph $\mathcal{D}(D)$ is a subhypergraph of the domination hypergraph $\mathcal{DH}(D)$, if $\mathcal{DH}(D)$ has no loop.

Therefore, for tournaments Theorem 1.1 implies that the deletion of all hyperedges of cardinalities different from two in $\mathcal{DH}(T_n)$ leads to a forest of caterpillars or a spiked odd cycle (with or without isolated vertices). In Figure 1 the star with the center vertex 4 corresponds to this forest of caterpillars.

In Section 2 and 3 we investigate domination hypergraphs of special classes of digraphs, namely tournaments, paths and cycles. Finally, using a special decomposition/composition method we deal in Section 4 with the construction of the edge set of the domination hypergraph of a given digraph.

2. DOMINATION HYPERGRAPHS OF TOURNAMENTS

We start with two simple properties of domination hypergraphs of tournaments.

Proposition 2.1. (1) *If $T_n = (V, A)$ is a tournament with n vertices and $\mathcal{DH}(T_n) = (V, \mathcal{E})$, then every edge $e \in \mathcal{E}$ has a cardinality of at most $\lceil \frac{n}{2} \rceil$.*
(2) *For every $k \in \{1, \ldots, \lceil \frac{n}{2} \rceil\}$ there exists a tournament $T_n^k = (V, A)$ with n vertices such that its domination hypergraph $\mathcal{DH}(T_n^k) = (V, \mathcal{E})$ possesses an edge $e \in \mathcal{E}$ of cardinality k.*

Proof. (1) Assume, $e \in \mathcal{E}$ with $|e| > \lceil \frac{n}{2} \rceil$. Since e is a minimal dominating set, a vertex $v \in e$ must have the property that there exists a vertex $v' \in V \setminus e$ such that v is the only predecessor of v' in e or v is not dominated by $e \setminus \{v\}$. Then e contains at most one vertex of the second kind and $|V \setminus e|$ vertices of the first kind, a contradiction.
(2) In Figure 2 we give, for $k \leq \frac{n}{2}$ and $k = \frac{n+1}{2}$ (where n is odd), respectively, a tournament T_n^k with a minimal dominating set $e = \{1, 2, \ldots, k\}$. To simplify Figure 2 we draw only some of the arcs; the remaining arcs between the subtournaments A and B are going from B to A; inside the subtournaments the arcs can be arbitrarily directed. ☐

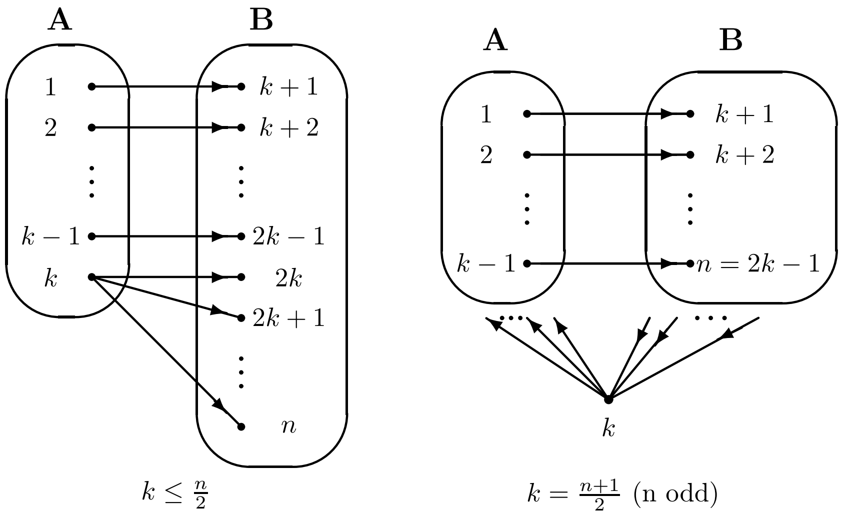

Fig. 2. Two tournaments with minimal dominating set $\{1, 2, \ldots, k\}$

Note that there are many examples of nonisomorphic tournaments T_n and T'_n with isomorphic domination hypergraphs (cf. Fig. 3).

An open problem is to find a characterization of domination hypergraphs of tournaments, but this seems to be difficult and we are far from a solution.

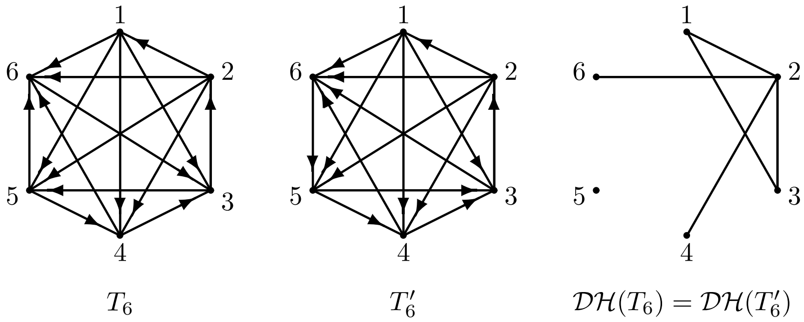

Fig. 3. Two nonisomorphic tournaments with the same domination hypergraph

In a hypergraph $\mathcal{H}$ a *trivial component* of $\mathcal{H}$ is referred to as an isolated vertex as well as a vertex of degree one being contained in a loop. It is known that domination graphs of tournaments can have several nontrivial components (cf. Theorem 1.1). By computer we tested hundreds of tournaments T_n having up to $n = 23$ vertices, but we did not find any domination hypergraph $\mathcal{DH}(T_n)$ which has more than one nontrivial component. Moreover, a computer-aided construction of the domination hypergraph of all tournaments up to $n = 9$ vertices showed, that more than one nontrivial component is impossible for $n \leq 9$ (cf. Wartner [12]). It seems that the "bigger" edges of $\mathcal{DH}(T_n)$ can guarantee the connectedness of the domination hypergraph (up to isolated vertices).

Conjecture. The domination hypergraph $\mathcal{DH}(T_n)$ of a tournament T_n consists of at most one nontrivial connected component.

We give a result concerning this conjecture.

Proposition 2.2. *Let $T_n = (V, A)$ be a tournament with n vertices. Then every nontrivial connected component of the domination hypergraph $\mathcal{DH}(T_n) = (V, \mathcal{E})$ contains at least three edges.*

Proof. Let $e_1 \in \mathcal{E}$ and $x, x^- \in e_1$, where x^- is a predecessor of x in T_n. Because e_1 is a minimal dominating set in T_n and x is dominated by $x^- \in e_1$ there must exist at least one vertex $x^+ \in V \setminus e_1$ which is dominated by the vertex x in T_n but by no other vertex of e_1. Let $\{v_1, v_2, \ldots, v_t\} \subseteq N^+(x) \cap (V \setminus e_1)$ be the set of all such vertices; this implies $\forall z \in e_1 \setminus \{x\} \, \forall i \in \{1, \ldots, t\} : v_i \notin N^+(z)$. Obviously, $\widetilde{e_1} := (e_1 \setminus \{x\}) \cup \{v_1, \ldots, v_t\}$ is a dominating set in T_n. Hence there exists a minimal dominating set $e_2 \subseteq \widetilde{e_1}$, such that e_2 contains at least one of the vertices $v_1, \ldots, v_t$, say $v_1, \ldots, v_{t'}$, and in $e_2 \setminus \{v_1, \ldots, v_{t'}\}$ we have a predecessor of x. Without loss of generality we choose such a predecessor and refer to it again as x^-. Since $x^- \in e_1 \cap e_2$ the edge $e_2 \neq e_1$ is in the same component of $\mathcal{DH}(T_n)$ as e_1.

Note that $\emptyset \neq \{v_1, \ldots, v_{t'}\} \subseteq N^-(x^-)$, because $x \neq x^-$ is the only dominator (i.e. predecessor) of $v_1, \ldots, v_{t'}$ in e_1.

Now we apply the analogous procedure to e_2 and x^-: x^- is dominated by $v_1 \in e_2$; consequently there must exist at least one vertex in $V \setminus e_2$ which is dominated by the vertex x^- in T_n but by no other vertex of e_2. Let $\{w_1, w_2, \ldots, w_s\} \subseteq N^+(x^-) \cap (V \setminus e_2)$ be the set of all vertices of $V \setminus e_2$ being dominated by x^- but by no other vertex of e_2. This implies $\forall z \in e_2 \setminus \{x^-\} \, \forall i \in \{1, \ldots, s\} : w_i \notin N^+(z)$. (Of course, $x \in \{w_1, w_2, \ldots, w_s\}$ is possible.)

It follows that $\widetilde{e_2} := (e_2 \setminus \{x^-\}) \cup \{w_1, \ldots, w_s\}$ is a dominating set in T_n and has to contain a minimal dominating set e_3 of T_n. Owing to $x^- \notin e_3$ we obtain $e_1 \neq e_3 \neq e_2$. Since none of $w_1, \ldots, w_s$ dominates x^- at least one of the predecessors $v_1, \ldots, v_{t'} \in e_2$ of x^- in T_n has to be an element of e_3. Therefore, e_3, e_2 and e_1 are in the same component of $\mathcal{DH}(T_n)$. $\square$

It is easy to construct digraphs $D = (V, A)$ with $n \geq 3$ vertices such that the domination hypergraph $\mathcal{DH}(D) = (V, \mathcal{E})$ has a connected component with exactly three edges:

Let $\{x, y, z\} \subseteq V$ generate an oriented 3-cycle in D and each of the vertices x, y, z dominates $V \setminus \{x, y, z\}$. If none of x, y, z has a predecessor in $V \setminus \{x, y, z\}$, then $\mathcal{DH}(D) = (V, \{\{x, y\}, \{x, z\}, \{y, z\}\})$.

3. DOMINATION HYPERGRAPHS OF ORIENTED PATHS AND CYCLES

For special types of digraphs the domination hypergraphs can be easily found. A first example is any digraph with n vertices and a source v of outdegree $n-1$ (e.g. directed stars and directed wheels with center v or transitive tournaments); in this case $\mathcal{DH}(D) = (V, \{\{v\}\})$. Secondly, if $D = (V, A)$ has only three kinds of vertices: sources, sinks and possibly some isolated vertices, then $\mathcal{DH}(D) = (V, \{V'\})$, where V' contains all sources and all isolated vertices of D. Note that connected digraphs, where all vertices are sources or sinks are often referred to as *alternating digraphs*. In such digraphs every path is *alternating*, i.e. any two consecutive arcs in a path have opposite orientation.

Let $P_n = (V, A)$ and $C_n = (V, A \cup \{(n, 1)\})$ be the *oriented path* and the *oriented cycle* with n vertices, respectively, i.e. $V = \{1, 2, \ldots, n\}$ and $A = \{(1, 2), (2, 3), \ldots, (n-1, n)\}$.

Theorem 3.1. *Let $\mathcal{DH}(P_n) = (V, \mathcal{E})$ be the domination hypergraph of the oriented path $P_n = (V, A)$. Then:*

(1) $e \in \mathcal{E}$ *if and only if* $1 \in e \;\; \wedge$

$$\forall i \in \{1, 2, \ldots, n-1\} : |\{i, i+1\} \cap e| \geq 1 \wedge (i \leq n-2 \Rightarrow |\{i, i+1, i+2\} \cap e| \leq 2).$$

(2) $\forall e \in \mathcal{E} : \lceil \frac{n}{2} \rceil \leq |e| \leq \begin{cases} \frac{2}{3}n, & n \equiv 0 \textit{ mod } 3, \\ 2\lfloor \frac{n}{3} \rfloor + 1, & \textit{otherwise.} \end{cases}$

(3) *With* $a_n := |\,\mathcal{E}(\mathcal{DH}(P_n))|$, $n \in \mathbb{N}^+$, *we obtain* $a_1 = a_2 = 1$, $a_3 = 2$ *and* $a_n = a_{n-2} + a_{n-3}$, *for* $n \geq 4$.

Proof. (1). The three conditions are obvious.
(2). The lower bound is reached by $e = \{1, 3, \dots, 2\lceil\frac{n}{2}\rceil - 1\}$. The upper bound we obtain with $e = \{1, 2, 4, 5, 7, 8, \dots, n-2, n-1\}$, if $n \equiv 0 \bmod 3$, and $e = \{1, 2, 4, 5, 7, 8, \dots, 3\lfloor\frac{n}{3}\rfloor - 2, 3\lfloor\frac{n}{3}\rfloor - 1, 3\lfloor\frac{n}{3}\rfloor + 1\}$, otherwise.
(3). The proof will be done by induction. The case $n \leq 3$ is clear, let us consider $n \geq 4$. We will construct a bijection from the disjoint union of the system $\mathcal{E}_{n-2} = \mathcal{E}(\mathcal{DH}(P_{n-2}))$ of the minimal dominating sets of P_{n-2} and the system $\mathcal{E}_{n-3} = \mathcal{E}(\mathcal{DH}(P_{n-3}))$ onto $\mathcal{E}_n = \mathcal{E}(\mathcal{DH}(P_n))$. First, starting with P_{n-2}, we construct a bijection from $\mathcal{E}_{n-2}$ onto a subset $\mathcal{E}' \subseteq \mathcal{E}_n$; then the edges of $\mathcal{E}_n \setminus \mathcal{E}'$ will be constructed from $\mathcal{E}_{n-3}$ analogously.

Algorithm A:
Let $\mathcal{DH}(P_{n-2}) = (V \setminus \{n-1, n\}, \mathcal{E}_{n-2})$ and $\mathcal{DH}(P_{n-3}) = (V \setminus \{n-2, n-1, n\}, \mathcal{E}_{n-3})$. Then the edge set of $\mathcal{DH}(P_n) = (V, \mathcal{E}_n)$ can be constructed as follows:

1. **Let** $\mathcal{E} := \emptyset$.
2. **Let** $e \in \mathcal{E}_{n-2}$.
2.1. **If** $n-2 \in e$, **then** $\mathcal{E} := \mathcal{E} \cup \{e \cup \{n\}\}$ **else** $\mathcal{E} := \mathcal{E} \cup \{e \cup \{n-1\}\}$.
2.2. $\mathcal{E}_{n-2} := \mathcal{E}_{n-2} \setminus \{e\}$.
2.3. **If** $\mathcal{E}_{n-2} \neq \emptyset$, **then go to** 2.
3. **Let** $e \in \mathcal{E}_{n-3}$.
3.1. **If** $n-3 \in e$, **then** $\mathcal{E} := \mathcal{E} \cup \{e \cup \{n-2, n\}\}$ **else** $\mathcal{E} := \mathcal{E} \cup \{e \cup \{n-2, n-1\}\}$.
3.2. $\mathcal{E}_{n-3} := \mathcal{E}_{n-3} \setminus \{e\}$.
3.3. **If** $\mathcal{E}_{n-3} \neq \emptyset$, **then go to** 3, **else** $\mathcal{E} = \mathcal{E}_n$.
4. **Stop.**

In step 2.1 and step 3.1 every edge e of $\mathcal{E}_{n-2}$ and of $\mathcal{E}_{n-3}$, respectively, is taken exactly once to construct an edge of $\mathcal{E} = \mathcal{E}_n$.
Assume, two of the sets $e_1 \cup \{n\}$, $e_2 \cup \{n-1\}$ (cf. 2.1) and $e_3 \cup \{n-2, n\}$, $e_4 \cup \{n-2, n-1\}$ (cf. 3.1) coincide. Obviously, this is only possible for a pair $e_1 \cup \{n\}$, $e_3 \cup \{n-2, n\}$ and $e_2 \cup \{n-1\}$, $e_4 \cup \{n-2, n-1\}$, respectively. Consider the case $e_1 \cup \{n\} = e_3 \cup \{n-2, n\}$. Because of $n-2 \in e_1$ (see 2.1) we obtain $n-3 \notin e_1$, since e_1 is minimal dominating, i.e. $n-3 \notin e_1 \cup \{n\}$. Step 3.1 includes $n-3 \in e_3 \subset e_3 \cup \{n-2, n\}$, therefore $e_1 \cup \{n\} = e_3 \cup \{n-2, n\}$ is impossible.

It remains to investigate $e_2 \cup \{n-1\} = e_4 \cup \{n-2, n-1\}$. Step 2.1 implies $n-2 \notin e_2 \cup \{n-1\}$, but $n-2 \in e_4 \cup \{n-2, n-1\}$ (see 3.1). Consequently, all of the sets $e_1 \cup \{n\}$, $e_2 \cup \{n-1\}$, $e_3 \cup \{n-2, n\}$ and $e_4 \cup \{n-2, n-1\}$ are pairwise distinct, i.e. our algorithm describes a bijection of the disjoint union of $\mathcal{E}_{n-2}$ and $\mathcal{E}_{n-3}$ onto $\mathcal{E}$.

It is easy to see that our construction in 2.1 and 3.1 leads to minimal dominating sets of P_n (we add no superfluous vertices to the minimal dominating sets e of P_{n-2} and P_{n-3}, respectively). Hence, the set $\mathcal{E}$ constructed in the algorithm is a subset of $\mathcal{E}_n$.

Vice versa, let $e \in \mathcal{E}_n$ be minimal dominating in P_n. In the case $n \in e$ we obtain $n-1 \notin e$, $n-2 \in e$ and if $n-3 \notin e$, then $e \setminus \{n\} \in \mathcal{E}_{n-2}$. Otherwise $e \setminus \{n-2, n\} \in \mathcal{E}_{n-3}$.

Now consider $n \notin e$, i.e. $n-1 \in e$. If $n-2 \notin e$, then $n-3 \in e$ as well as $e \setminus \{n-1\} \in \mathcal{E}_{n-2}$, otherwise $e \setminus \{n-2, n-1\} \in \mathcal{E}_{n-3}$ with $n-3 \notin e$. Therefore, $\mathcal{E}_n$ is a subset of the set $\mathcal{E}$ from the algorithm, i.e. $\mathcal{E}_n = \mathcal{E}$. □

Observe that an arbitrary path can be decomposed into oriented subpaths. Thus, using Theorem 3.1 and the decomposition principle described in Section 4, the domination hypergraph of an arbitrary path can be determined.

Theorem 3.2. *Let $\mathcal{DH}(C_n) = (V, \mathcal{E})$ be the domination hypergraph of the oriented cycle $C_n = (V, A)$. Then it holds:*

(1) $e \in \mathcal{E}$ *if and only if*

$$\forall\, i \in \{1, 2, \dots, n\} : |\{i, i+1\} \cap e| \geq 1 \ \wedge\ |\{i, i+1, i+2\} \cap e| \leq 2) \quad (\text{modulo } n).$$

(2) $$\forall\, e \in \mathcal{E} : \lceil \tfrac{n}{2} \rceil \leq |e| \leq \begin{cases} \frac{2}{3}n, & n \equiv 0 \ mod\ 3, \\ \frac{2}{3}(n-1), & n \equiv 1 \ mod\ 3, \\ \frac{2}{3}(n-2)+1, & n \equiv 2 \ mod\ 3. \end{cases}$$

(3) *With $a_n := |\,\mathcal{E}(\mathcal{DH}(C_n))|$, $n \geq 2$, we obtain*
$a_2 = 2$, $a_3 = 3$, $a_4 = 2$ and $a_n = a_{n-2} + a_{n-3}$, for $n \geq 5$.

Proof. (1). This is evident.
(2). For the lower bound we can take the same dominating set as in (2) of the proof of Theorem 3.1. The upper bound can be obtained with

$$e = \{1, 2, 4, 5, \dots \begin{cases} \dots, n-2, n-1\}, & n \equiv 0 \text{ mod } 3, \\ \dots, n-6, n-5, n-3, n-1\}, & n \equiv 1 \text{ mod } 3, \\ \dots, n-4, n-3, n-1\}, & n \equiv 2 \text{ mod } 3. \end{cases}$$

(3). Again, we prove this part by induction. The values a_n for $n \leq 4$ can be easily verified, so we assume $n \geq 5$. In the following algorithm a bijection from the disjoint union of the system $\mathcal{E}_{n-2} = \mathcal{E}(\mathcal{DH}(C_{n-2}))$ of the minimal dominating sets of C_{n-2} and the system $\mathcal{E}_{n-3} = \mathcal{E}(\mathcal{DH}(C_{n-3}))$ onto $\mathcal{E}_n = \mathcal{E}(\mathcal{DH}(C_n))$ will be constructed.

Algorithm B:

Let $\mathcal{DH}(C_{n-2}) = (V \setminus \{n-1, n\}, \mathcal{E}_{n-2})$ and $\mathcal{DH}(C_{n-3}) = (V \setminus \{n-2, n-1, n\}, \mathcal{E}_{n-3})$. Then the edge set of $\mathcal{DH}(C_n) = (V, \mathcal{E}_n)$ can be constructed as follows:

1. **Let** $\mathcal{E} := \emptyset$.
2. **Let** $e \in \mathcal{E}_{n-2}$.
2.1. **If** $n-2 \in e$, **then** $\mathcal{E} := \mathcal{E} \cup \{e \cup \{n\}\}$.
2.2. **If** $n-2 \notin e$, **then** $\mathcal{E} := \mathcal{E} \cup \{e \cup \{n-1\}\}$.
2.3. $\mathcal{E}_{n-2} := \mathcal{E}_{n-2} \setminus \{e\}$.
2.4. **If** $\mathcal{E}_{n-2} \neq \emptyset$, **then go to** 2.
3. **Let** $e \in \mathcal{E}_{n-3}$.
3.1. **If** $1 \in e \ \wedge\ n-3 \notin e$, **then** $\mathcal{E} := \mathcal{E} \cup \{e \cup \{n-2, n-1\}\}$.
3.2. **If** $1 \in e \ \wedge\ n-3 \in e$, **then** $\mathcal{E} := \mathcal{E} \cup \{e \cup \{n-2, n\}\}$.
3.3. **If** $1 \notin e$, **then** $\mathcal{E} := \mathcal{E} \cup \{e \cup \{n-1, n\}\}$.
3.4. $\mathcal{E}_{n-3} := \mathcal{E}_{n-3} \setminus \{e\}$.
3.5. **If** $\mathcal{E}_{n-3} \neq \emptyset$, **then go to** 3 **else** $\mathcal{E} = \mathcal{E}_n$.
4. **Stop.**

In step 2 and step 3 every edge e of $\mathcal{E}_{n-2}$ and of $\mathcal{E}_{n-3}$, respectively, is used exactly once to construct an edge of $\mathcal{E} = \mathcal{E}_n$. To demonstrate that all edges generated in the algorithm are pairwise distinct and that $\mathcal{E} = \mathcal{E}_n$ (cf. 3.5.), we study the structure of these edges. For this purpose we use the notation $\tilde{e}_1 = e_1 \cup \{n\}$, $\tilde{e}_2 = e_2 \cup \{n-1\}$, $\tilde{e}_3 = e_3 \cup \{n-2, n-1\}$, $\tilde{e}_4 = e_4 \cup \{n-2, n\}$ and $\tilde{e}_5 = e_5 \cup \{n-1, n\}$ for the edge "$e \cup \{\ldots\}$" in 2.1, 2.2, 3.1, 3.2 and 3.3, respectively.

Hence it can be verified that

$$\tilde{e}_1 = \begin{cases} \{1, 3, \ldots, n-4, n-2, n\}, & \text{if } 1 \in e_1, \\ \{2, \ldots, n-2, n\}, & \text{if } 1 \notin e_1. \end{cases}$$

$\tilde{e}_2 = \{1, \ldots, n-3, n-1\}$.

$\tilde{e}_3 = \{1, \ldots, n-4, n-2, n-1\}$.

$\tilde{e}_4 = \{1, 3, \ldots, n-5, n-3, n-2, n\}$.

$\tilde{e}_5 = \{2, \ldots, n-3, n-1, n\}$.

Now it is easy to see that $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_5$ are pairwise distinct. Note that minimal dominating sets having the same structure $\tilde{e}_i$, $i \in \{1, \ldots, 5\}$, are distinct, since their origins are distinct minimal dominating sets of C_j, $j \in \{n-2, n-3\}$.

The deletion of the vertices $n-1, n$ or $n-2, n-1, n$ in a dominating set of C_n always results in a dominating set of C_{n-2} or of C_{n-3}. Therefore, in order to show that Algorithm B yields all dominating sets of C_n, it suffices to ensure that all possible configurations of the first vertices $1, 2, 3$ and the last vertices $n-5, n-4, \ldots, n$ occur in $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_5$. This can be done by checking that in $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_5$ there are no three immediately consecutive vertices of C_n and that it is impossible to add other vertices or to replace vertices at the "beginning" or the "end" of the dominating sets $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_5$. □

4. A DECOMPOSITION PRINCIPLE FOR THE CONSTRUCTION OF MINIMAL DOMINATING SETS

In this section we describe how to combine minimal dominating sets of certain subdigraphs of a given digraph $D = (V, A)$ to a minimal dominating set of D. The construction makes use of (possibly existing) articulation vertices to decompose a digraph D into smaller subdigraphs. Iterating this procedure we can try to obtain simple subdigraphs (e.g. paths, cycles etc.) for which minimal dominating sets can be easily found. Step by step these dominating sets can be combined to build a minimal dominating set of D.

Let $z \in V$ be an *articulation vertex* of the connected digraph $D = (V, A)$, i.e. $D \setminus \{z\}$ consists of more than one pairwise distinct connected components $D_1, D_2, \ldots, D_k$. Choose $l \in \{1, 2, \ldots, k-1\}$ and set $S_1 := D[\bigcup_{i=1}^{l} V(D_i) \cup \{z\}]$ and $S_2 := D[\bigcup_{i=l+1}^{k} V(D_i) \cup \{z\}]$, where $D[V']$ denotes the subdigraph induced by the vertex set $V' \subseteq V$ in D. We say that S_1 and S_2 can be obtained from D by *splitting* the articulation vertex z. Obviously, the reversal operation of splitting the articulation vertex z is the union of S_1 and S_2, i.e. $D = S_1 \cup S_2 := (V(S_1) \cup V(S_2), A(S_1) \cup A(S_2))$. Note that the splitting of an articulation vertex z for $k > 2$ is not unique.

Now we split the articulation vertex z of the digraph $D = (V, A)$ and obtain the subdigraphs S_1 and S_2. In the following, for such a situation we write $D = S_1 \overset{z}{\cup} S_2$.

For given minimal dominating sets e_1 of S_1 and e_2 of S_2 we want to construct a minimal dominating set $e = e(e_1, e_2)$ of the digraph $D = S_1 \overset{z}{\cup} S_2$. Before tackling this problem, we need some basic properties of minimal dominating sets of D.

If e is a minimal dominating set of the digraph $D = (V, A)$, there are two reasons for a vertex v to be in e: $N^-(v) \cap e = \emptyset$ or there exists a vertex $v^+ \in N^+(v) \setminus e$ such that $N^-(v^+) \cap e = \{v\}$.
In the first case, v is in e since it is not dominated by another vertex of e, and we say that v is in e *for itself*. In the second case, v is in e because it is needed to dominate another vertex $v^+ \notin e$, i.e. v is in e *as a dominator* (of v^+). Of course, both cases do not exclude each other.

In order to obtain some information on the relations between a minimal dominating set e of $D = S_1 \overset{z}{\cup} S_2$ and minimal dominating sets of S_1 and S_2 we investigate $e^1 := e \cap S_1$ and $e^2 := e \cap S_2$. The following cases can occur:

1. $z \notin e$.
   a) e^1 and e^2 are minimal dominating for S_1 and S_2, respectively.
   Note that z is dominated by at least one vertex of e^1 and at least one vertex of e^2.
   b) Only one of the sets e^1 and e^2 is minimal dominating for S_1 and S_2, respectively.
   Let $\{i, j\} = \{1, 2\}$ and e^i be minimal dominating for S_i. Then e^j is minimal dominating for $S_j \setminus \{z\}$ and z is not dominated by e^j.
   c) Neither e^1 nor e^2 is minimal dominating for S_1 and S_2, respectively.
   This case is impossible.
2. $z \in e$.
   a) e^1 and e^2 are minimal dominating for S_1 and S_2, respectively. Obviously, z is needed in e^1 as well as in e^2 to be a dominating set.
   b) Only one of e^1 and e^2 is minimal dominating for S_1 and S_2, respectively.
   Let $\{i, j\} = \{1, 2\}$ and e^i be minimal dominating for S_i. Again, z is needed in e^i to be a dominating set. Note that e^j dominates S_j, but e^j is not minimal dominating. Hence, $\exists\, e_0^j \subset e^j \,:\, e_0^j$ is minimal dominating for S_2. The assumption $z \in e_0^j$ leads to a dominating set $e^i \cup e_0^j \subset e_1 \cup e_2$ for D, in contradiction to the fact that $e_1 \cup e_2$ is minimal dominating for D. Therefore we obtain $e_0^j = e^j \setminus \{z\}$, because $e_0^j \subset e^j \setminus \{z\}$ would lead to the same contradiction as the assumption $z \in e_0^j$.
   Consequently, case (2b) is equivalent to e^i and $e^j \setminus \{z\}$ is minimal dominating for S_i and S_j, respectively.
   c) Neither e^1 nor e^2 is minimal dominating for S_1 and S_2, respectively.
   Since e^i is dominating for S_i, the deletion of some vertices of e^i would lead to minimal dominating sets $e_0^i \subset e^i$ of S_i $(i = 1, 2)$. It follows that $e_0^1 \cup e_0^2 \subset e_1 \cup e_2$ is dominating for D, a contradiction. That is, case (2c) cannot occur.

Note that if we rename e_0^j to e^j in case (2b) we have:

(1) : $z \notin e^1 \cup e^2$,
(2a): $z \in e^1 \cap e^2$,
(2b): $z \in e^1 \triangle e^2 := (e^1 \cup e^2) \setminus (e^1 \cap e^2)$.

Summarizing the considerations above, we obtain the following.

Theorem 4.1. *Every minimal dominating set e of a digraph $D = S_1 \overset{z}{\cup} S_2$ is the union of certain minimal dominating sets e_1 and e_2 of S_1 (or $S_1 \setminus \{z\}$) and S_2 (or $S_2 \setminus \{z\}$), respectively. In detail, $e = e_1 \cup e_2$, where the following situations can appear:*

(i) *e_1 and e_2 is minimal dominating for S_1 and S_2, respectively, or*
(ii) *with $\{i,j\} = \{1,2\}$, e_i and e_j is minimal dominating for S_i and $S_j \setminus \{z\}$, respectively, where $z \notin e_i \cup e_j$ and $N^-(z) \cap e_j = \emptyset$.*

Proof. Let e be a minimal dominating set for D, $e^1 := e \cap S_1$ and $e^2 := e \cap S_2$. Setting $e_1 := e^1$ and $e_2 := e^2$ (with the exception $e_j := e^j \setminus \{z\}$ for (2b)), in the cases (1a), (2a) and (2b) we obtain (i). Case (1b) corresponds to (ii). □

CONSTRUCTION OF A MINIMAL DOMINATING SET e OF $D = S_1 \overset{z}{\cup} S_2$

Let e_1 and e_2 be minimal dominating for S_1 and S_2, respectively.

1. $z \notin e_1 \cup e_2$.
   If there is an $i \in \{1,2\}$ and a vertex $z^- \in N^-(z) \cap e_i$ such that in e_i the vertex z^- is needed only as the dominator of z (i.e. z^- is the only dominator of z in e_i and z^- itself has a dominator in e_i), then this z^- is unique in e_i. Moreover, $e_i \setminus \{z^-\}$ is minimal dominating for $S_i \setminus \{z\}$. If such vertices z^- exist in e_1 or e_2, we delete exactly one of them in $e_1 \cup e_2$:

$$e := \begin{cases} (e_1 \cup e_2) \setminus \{z^-\}, & \text{if } z^- \text{ exists,} \\ e_1 \cup e_2, & \text{otherwise.} \end{cases}$$

   Then in e there is at most one other vertex z_0^- of this kind: if $z^- \in e_i$, then $z_0^- \in e_j$, where $\{i,j\} = \{1,2\}$. Obviously, in e the vertex z_0^- is needed to dominate z, we cannot delete it in e.
2. $z \in e_1 \cup e_2$.
   a) $z \in e_1 \cap e_2$.
      In this case, z is needed in both minimal dominating sets, e_1 and e_2. Therefore z is needed in the dominating set $e_1 \cup e_2$, too. Because e_1 and e_2 are minimal dominating, no other vertex of $e_1 \cup e_2$ can be deleted, i.e. a minimal dominating set of D is
$$e := e_1 \cup e_2\,.$$
   b) $z \in e_1 \triangle e_2$.
      Let $z \in e_i$ and $\{i,j\} = \{1,2\}$.

(b1) z is in e_i only for itself.
Since z has a dominator in e_j, we can delete z:

$$e := (e_1 \cup e_2) \setminus \{z\}.$$

(b2) z is needed in e_i as a dominator.
We do not delete z; otherwise we had to add some vertices of $N^+(z) \cap e_i$ or predecessors of such vertices, but this could be difficult.
Let e_i remain unchanged and modify e_j to obtain a new dominating set e'_j of S_j :

(i) Delete $z^- \in N^-(z) \cap e_j$, if z^- is needed in e_j only as the dominator of z (obviously, at most one such z^- can exist).

(ii) As long as they exist in e_j we delete successively all vertices z^+ and z' of the following kind:
- $z^+ \in N^+(z) \cap e_j$, if $z^+ \in e_j$ only for itself,
- $z' \in N^-(z^+) \cap e_j$, if the only reason for $z' \in e_j$ is that z' is needed in e_j as the only dominator of certain vertices $z^+ \in N^+(z) \cap e_j$.

(iii) Add z to e_j.

Note that the vertices being deleted in (i) and (ii) became superfluous because of (iii).
If we denote the resulting set by e'_j we see that e'_j is dominating but not necessarily minimal dominating for S_j. For example, this situation occurs if z has a dominator in $e'_j \setminus \{z\}$ but z is not needed in e'_j to dominate a vertex of $V(S_j) \setminus e'_j$ (i.e. all vertices of $V(S_j) \setminus e'_j$ have a dominator in $e'_j \setminus \{z\}$). Now a minimal dominating set of D is $e := e_i \cup e'_j$.

END OF CONSTRUCTION.

Considering digraphs D with articulation vertices, this construction principle enables us to reduce the search for minimal dominating sets in D to the analogous problem in (smaller) subdigraphs. As an example we consider a special class of directed cacti.

A connected digraph $D = (V, A)$ is a (directed) *cactus* if and only if every arc $e \in A$ is contained in at most one cycle. A directed cactus $D = (V, A)$ is referred to as a *cycle-oriented cactus* if and only if each cycle in D is oriented.

By stepwise splitting of articulation vertices a cycle-oriented cactus $D = (V, A)$ can be decomposed into a system of oriented paths and cycles. Because we know all minimal dominating sets of oriented paths and cycles (cf. Theorems 3.1 and 3.2), starting with such minimal dominating sets we are able to compose minimal dominating sets of the cactus D.

As an example, we construct a minimal dominating set of the cycle-oriented cactus $D = (V, A)$ shown in Figure 4.

We decompose $D = (V, A)$ into the oriented paths $P_1 = (\{1, 2, 3\}, \{(1, 2), (2, 3)\})$, $P_2 = (\{5, 8, 9\}, \{(5, 8), (8, 9)\})$ and $P_3 = (\{9, 10\}, \{(10, 9)\})$, and the oriented cycle $C = (\{3, 4, 5, 6, 7\}, \{(3, 4), (4, 5), (5, 6), (6, 7), (7, 3)\})$. The sets $e_{P_1} = \{1, 3\}, e_{P_2} = \{5, 9\}, e_{P_3} = \{10\}$ and $e_C = \{3, 4, 6\}$ are minimal dominating sets for P_1, P_2, P_3 and C, respectively. With (2a) (see the construction described above) from e_{P_1} and e_C we obtain the set $e_{P_1C} = \{1, 3, 4, 6\}$ which is minimal dominating for $P_1 \overset{3}{\cup} C$. Considering e_{P_1C} and e_{P_2} by (2b)(ii) we get $e_{P_1CP_2} = \{1, 3, 5, 6, 9\}$ as a minimal dominating set of $P_1 \overset{3}{\cup} C \overset{5}{\cup} P_2$. Finally, (2b)(i) leads from $e_{P_1CP_2}$ and e_{P_3} to the minimal dominating set $e = \{1, 3, 5, 6, 10\}$ of D.

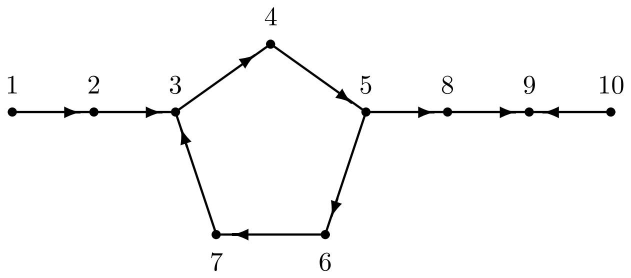

Fig. 4. A cycle-oriented cactus

Note that our decomposition/composition principle together with Theorems 3.1 and 3.2 allows to construct all minimal dominating sets of a cycle-oriented cactus. More generally, the domination hypergraph of a digraph $D = S_1 \overset{z_1}{\cup} S_2 \overset{z_2}{\cup} \ldots \overset{z_k}{\cup} S_{k+1}$ can be obtained from the domination hypergraphs of $S_1, \ldots, S_{k+1}$.

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