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PSEUDOSPECTRAL METHOD FOR SEMILINEAR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. We present a convergence result for two spectral methods applied to an initial boundary value problem with functional dependence of Volterra type. Explicit condition of Courant-Friedrichs-Levy type is assumed on time step τ and the number N of collocation points. Stability statements and error estimates are written using continuous norms in weighted Jacobi spaces.

Keywords: pseudospectral collocation, CFS condition, convergence, error estimates.

Mathematics Subject Classification: 65M70, 35R10.

1. INTRODUCTION

We are concerned here with a numerical method for a semi-linear functional differential equation with initial boundary conditions. In recent years, many papers have been published, covering various approximation methods for partial functional differential equations, see [7] for a bibliography. The main stream of these works is connected with finite difference approximations. It is known that the explicit finite difference methods for hyperbolic or parabolic problems should obey a Courant-Friedrichs-Levy (CFL) condition to remain stable. Similar condition appears when one considers pseudospectral approximations for hyperbolic problems, consisting of the truncation and collocation of N-term spatial expansions, which are expressed in terms of Jacobi polynomials. This is covered in the papers [3, 4], of which the second one we follow closely. Stability estimates are given there with the use of discrete L^2-weighted norms, which we translate into continuous norms of this type, and combine with relevant approximation results from [6], to obtain convergence.

The work [4] may be also treated as introductory for one interested in Jacobi spectral methods for hyperbolic equations. For more information on spectral and pseudospectral methods for various partial differential problems, see the book [2].

The paper is organized as follows. In the next section we formulate the differential problem after defining function spaces for its solutions. The pseudospectral scheme is described in Section 3. The Section also explains how to deal with the functional dependence inside a given function. In Section 4 we prove stability results for homogeneous and inhomogeneous cases of our problem. A convergence proof is given in Section 5. The last section contains numerical examples.

2. DIFFERENTIAL PROBLEM

Let $\alpha,\ \beta > -1$. Denote $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ and $\Lambda = [-1,1]$. The Jacobi polynomials $\{P_k^{(\alpha,\beta)}\}_{k\geq 0}$ are the eigenfunctions of the Sturm-Liouville problem

$$\partial_x((1-x^2)\chi^{(\alpha,\beta)}(x)\partial_x v(x)) + \lambda\,\chi^{(\alpha,\beta)}(x)v(x) = 0, \qquad x\in\Lambda,$$

with the corresponding eigenvalues (see [8])

$$\lambda_k^{(\alpha,\beta)} = k(k+\alpha+\beta+1). \tag{2.1}$$

Write $\|v\|_{\alpha,\beta} = (\int_\Lambda |v(x)|^2\chi^{(\alpha,\beta)}(x)\,dx)^{1/2}$. We will work in the Jacobi-weighted Sobolev space, introduced in [6],

$$H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) = \left\{v : v \text{ is measurable and } \|v\|_{r,\chi^{(\alpha,\beta)},A} < \infty\right\}, \quad r\in\mathbb{N},$$

equipped with the norm and semi-norm, respectively,

$$\|v\|_{r,\chi^{(\alpha,\beta)},A} = \left(\sum_{k=0}^{r} |v|^2_{k,\alpha,\beta}\right)^{1/2}, \qquad |v|_{k,\alpha,\beta} = \|\partial_x^k v\|_{\alpha+k,\beta+k}, \quad 0\leq k\leq r.$$

For $t_0 \geq 0$, our domains are

$$E = (0,\infty)\times[-1,1], \quad E_0 = [-t_0,0]\times[-1,1], \quad E^* = E_0\cup E.$$

Let

$$X^r_{\alpha,\beta} = \{w : E\to\mathbb{R} \colon w(t,\cdot)\in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) \quad\text{and}\quad w(\cdot,x)\in C(0,\infty), \quad (t,x)\in E\},$$

where $C(0,\infty)$ is the space of all continuous functions defined on $(0,\infty)$. For $w\in X^r_{\alpha,\beta}$ we define

$$|w|_{\alpha,\beta;[t]} = \sup_{-t_0\leq s\leq t} \|w(s,\cdot)\|_{\alpha,\beta}.$$

Let $f : E\to\mathbb{R}$, $G : E\times X^r_{\alpha,\beta}\to\mathbb{R}$ and $\varphi : E_0\cup\partial_0 E\to\mathbb{R}$. be given functions. We require that $f > 0$ on E. We consider a problem consisting of a functional differential equation

$$\partial_t z(t,x) + f(t,x)\partial_x z(t,x) = G(t,x,z) \tag{2.2}$$

and an initial-boundary condition

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0, \qquad\qquad z(t,-1) = \varphi(t,-1), \quad t > 0. \tag{2.3}$$

Put

$$E_t^* = [-t_0, t] \times [-1, 1], \quad t \geq 0.$$

We will assume the Volterra condition on G, that is, we require that for each $(t,x) \in E$ there is a set $E[t,x]$ such that:

(i) $E[t,x] \subset E_t^*$,

(ii) if z, $\bar{z} \in X^r_{\alpha,\beta}$ and $z(s,y) = \bar{z}(s,y)$ for $(s,y) \in E[t,x]$ then $G(t,x,z) = G(t,x,\bar{z})$.

Note that the Volterra condition means that the value of G at every point (t,x,z) depends on (t,x) and on the restriction of z to the set $E[t,x]$ only.

3. FORWARD EULER PSEUDOSPECTRAL SCHEME

Let $x^{(\alpha,\beta)}_{G,N,j}$, $x^{(\alpha,\beta)}_{R,N,j}$ and $x^{(\alpha,\beta)}_{L,N,j}$, $0 \leq j \leq N$, be the zeros of polynomials $P^{(\alpha,\beta)}_{N+1}(x)$, $(1+x)P^{(\alpha,\beta+1)}_N(x)$, $(1-x^2)\partial_x P^{(\alpha,\beta)}_N(x)$, respectively. They are arranged in decreasing order. Denote by $\omega^{(\alpha,\beta)}_{Z,N,j}$, $0 \leq j \leq N$, the corresponding Christoffel numbers such that

$$\int\limits_{\Lambda} p(x)\chi^{(\alpha,\beta)}(x)\,dx = \sum_{j=0}^{N} p(x^{(\alpha,\beta)}_{Z,N,j})\omega^{(\alpha,\beta)}_{Z,N,j} \quad \text{for each} \quad p \in \pi_{2N+k_Z}, \tag{3.1}$$

where π_N denotes the class of all polynomials of degree at most N, and integers k_Z are: 1, 0 and -1 for $Z = G$, R and L, respectively. The following relations are useful (see [5])

$$\begin{aligned} x^{(\alpha,\beta)}_{R,N,j} &= x^{(\alpha,\beta+1)}_{G,N-1,j}, & (1 + x^{(\alpha,\beta)}_{R,N,j})\,\omega^{(\alpha,\beta)}_{R,N,j} &= \omega^{(\alpha,\beta+1)}_{G,N-1,j}, & 0 \leq j \leq N-1, \\ x^{(\alpha,\beta)}_{L,N,j} &= x^{(\alpha+1,\beta+1)}_{G,N-2,j-1}, & (1 - (x^{(\alpha,\beta)}_{L,N,j})^2)\,\omega^{(\alpha,\beta)}_{L,N,j} &= \omega^{(\alpha+1,\beta+1)}_{G,N-2,j-1}, & 1 \leq j \leq N-1. \end{aligned} \tag{3.2}$$

For fixed $N \in \mathbb{N}$, we consider the spatial mesh $\{x_j\}_{j=1}^N$ on Λ,

$$x_j = x^{(\alpha,\beta)}_{L,N+1,j}, \quad 1 \leq j \leq N. \tag{3.3}$$

This mesh generates a family of Jacobi pseudospectral methods. We discretize the time variable in the usual way, putting $t^m = m\tau$, $m \in \mathbb{Z}$, where $\tau > 0$ is the time step. For fixed τ, it is convenient to choose $-m_0 \in \mathbb{Z}$ such that $m_0 = \lfloor t_0\tau^{-1} \rfloor$, that is, $t^{-m_0-1} < -t_0 \leq t^{-m_0}$. Following closely [4], we consider the pseudospectral collocation scheme for (2.2),

$$v_N^{m+1}(x_j) = v_N^m(x_j) - \tau f(t^m, x_j)\partial_x v_N(x_j) + \tau G(t^m, x_j, Tv_N), \quad 1 \leq j \leq N, \tag{3.4}$$

augmented with the homogeneous boundary condition

$$v_N^m(-1) = 0, \qquad -m_0 \leq m, \tag{3.5}$$

and with the initial condition

$$v_N^m(x_j) = \varphi(t^m, x_j), \qquad -m_0 \le m \le 0, \quad 1 \le j \le N. \tag{3.6}$$

Formulae (3.4)–(3.6) uniquely determine the sequence $\{v_N^m\}_{m \ge -m_0}$ of $v_N \in \pi_N$. Due to the functional dependence of G, a certain interpolation operator T is needed, which would map a sequence $\{v_N^m\}_{m \ge -m_0}$ of polynomials onto a member of $C(D, \mathbb{R})$. The simplest such operator is given by

$$(Tv_N)(t,x) = \begin{cases} v_N^m(x) + \theta[v_N^{m+1}(x) - v_N^m(x)], & \text{for} \quad t = t^m + \theta\tau, \quad \theta \in (0,1], \\ v_N^{-m_0}(x) & \text{for} \quad t \le t^{-m_0}, \end{cases} \tag{3.7}$$

where $(t,x) \in E^*$.

The scheme (3.4)–(3.6) is obtained from (2.2),(2.3) by using forward Euler time differencing, and pseudospectral spatial differencing. We first consider only the problems with $\varphi(\cdot, -1) \equiv 0$, for the sake of the stability proof, which is then carried over onto inhomogeneous problems.

4. STABILITY

Let $\{\omega_j\}_{j=1}^N$ be discrete positive weights, and let us write

$$\langle f, g \rangle_\omega = \sum_{j=1}^N \omega_j f(x_j) g(x_j), \qquad \|f\|_\omega = \langle f, f \rangle_\omega^{1/2}.$$

We will prove a priori estimates of $\|v_N^m\|_\omega$ in terms of estimates of the initial data $\|\varphi(t^m, \cdot)\|_\omega$, $m \le 0$, and inhomogeneous data $\|G(t^m, \cdot, v_N)\|_\omega$. Out of the four variants of discrete weights, presented in [4], we consider only the last one, namely:

$$\omega_j = (1 - x_j)\omega_{L,N+1,j}^{(\alpha,\beta)}, \qquad 1 \le j \le N, \tag{4.1}$$

where x_j are given by (3.3), and $\alpha, \beta \in (-1, 0)$.

Assumption H$[f]$**.** The functions $f : E \to (0, \infty)$, $G : E \times X_{\alpha+1,\beta}^r \to \mathbb{R}$, are such that:

1) there are positive constants b, B such that $b^{-1} < f(t,x) < B$ on E,
2) there is $L \ge 0$ such that for (t,x), $(\bar{t}, x) \in E$,

$$|f(t,x) - f(\bar{t}, x)| \le L\,|t - \bar{t}|.$$

Theorem 4.1. *Consider the pseudospectral scheme* (3.4)–(3.6)*, collocated at points* x_j *given by* (3.3)*. If the CFL condition:*

$$\eta_0 - \tau B\Big(\lambda_{N-1}(\alpha+1, \beta) + 2 \max_{1 \le j \le N} \frac{1}{1 + x_j}\Big) \ge 0 \tag{4.2}$$

holds, where

$$\eta_0 \equiv \eta_0(\alpha,\beta) = \begin{cases} -\beta, & \alpha+\beta+1 \ge 0, \\ 1+\alpha, & \alpha+\beta+1 \le 0, \end{cases} \qquad \alpha,\beta \in (-1,0),$$

and if the Assumption H$[f]$ *is satisfied, then the following estimates hold:*

$$\|v_N^m\|_\omega \le C\exp(at^m)\|v_N^0\|_\omega + \tau C\exp(at^m)\sum_{n=1}^{m}\exp(-at^n)\|G(t^{n-1},\cdot,Tv_N)\|_\omega, \quad m>0, \tag{4.3}$$

where the discrete weights ω_j *are given by* (4.1) *and* $a = \frac{1}{2}bL$.

Proof. We first modify the above discrete weights, so that they depend also on m:

$$\omega_j^m = \frac{1-x_j}{f(t^m,x_j)}\omega_{L,N+1,j}^{(\alpha,\beta)}.$$

The norm induced by these weights, $\|\cdot\|_{\omega^m}$, is equivalent to the norm $\|\cdot\|_\omega$:

$$\|v\|_{\omega^m} \le b^{1/2}\|v\|_\omega \quad \text{and} \quad \|v\|_\omega \le B^{1/2}\|v\|_{\omega^m},$$

in view of Assumption H$[f]$. Then we have

$$\|v_N^m - \tau f(t^m,\cdot)\partial_x v_N^m\|_{\omega^m}^2 \le \left[1 - 2\tau\left(\eta_0 - \tau B\big(\lambda_{N-1} + 2\max_{1\le j\le N}\frac{1}{1+x_j}\big)\right)\right]\|v_N^m\|_{\omega^m}^2.$$

We omit the proof of this estimate, which is almost identical to the corresponding part of the proof of Theorem 6.2 in [4]. Due to the CFL condition (4.2), it follows

$$\|v_N^m - \tau f(t^m,\cdot)\partial_x v_N^m\|_{\omega^m} \le \|v_N^m\|_{\omega^m}.$$

Now, let us express the norm $\|\cdot\|_{\omega^{m+1}}$ in terms of the norm $\|\cdot\|_{\omega^m}$:

$$\begin{aligned} \|w\|_{\omega^{m+1}}^2 &= \sum_{j=1}^{N} w^2(x_j)\frac{1-x_j}{f(t^{m+1},x_j)}\omega_{L,N+1,j}^{(\alpha,\beta)} \le \\ &\le \max_{1\le j\le N}\frac{f(t^m,x_j)}{f(t^{m+1},x_j)}\sum_{j=1}^{N} w^2(x_j)\frac{1-x_j}{f(t^m,x_j)}\omega_{L,N+1,j}^{(\alpha,\beta)} \le \\ &\le \Big(1 + b\max_{1\le j\le N}|f(t^m,x_j) - f(t^{m+1},x_j)|\Big)\|w\|_{\omega^m}^2 \le (1+bL\tau)\|w\|_{\omega^m}^2. \end{aligned}$$

Thus

$$\|w\|_{\omega^{m+1}} \le (1+a\tau)\|w\|_{\omega^m}.$$

Since taking the norm $\|\cdot\|_{\omega^m}$ on both sides of (3.4) gives

$$\begin{aligned} \|v_N^{m+1}\|_{\omega^m} &= \|v_N^m - \tau f(t^m,\cdot)\partial_x v_N^m + \tau G(t^m,\cdot,Tv_N)\|_{\omega^m} \le \\ &\le \|v_N^m - \tau f(t^m,\cdot)\partial_x v_N^m\|_{\omega^m} + \tau\|G(t^m,\cdot,Tv_N)\|_{\omega^m}, \end{aligned}$$

we get from the above estimates

$$\|v_N^{m+1}\|_{\omega^{m+1}} \le (1+a\tau)\|v_N^{m+1}\|_{\omega^m} \le (1+a\tau)\|v_N^m\|_{\omega^m} + \tau(1+a\tau)\|G(t^m,\cdot,Tv_N)\|_{\omega^m}.$$

Recurrent application of this inequality produces

$$\|v_N^m\|_{\omega^m} \le (1+a\tau)^m\|v_N^0\|_{\omega^0} + \tau\sum_{n=0}^{m-1}(1+a\tau)^{m-n-1}\|G(t^n,\cdot,Tv_N)\|_{\omega^n},$$

which, in view of the equivalence of $\|\cdot\|_{\omega^m}$ and $\|\cdot\|_\omega$, gives (4.3) with $C = B^{1/2}b^{1/2}$. □

Remark 4.2. The expression in parentheses in the CFL condition (4.2) may be written in terms of N. Assume that $\alpha,\ \beta \in (-1,0)$. Then the formula (2.1) gives $\lambda_{N-1}(\alpha+1,\beta) < N^2-1$. Moreover, using a result on the distribution of Jacobi-Gauss nodes ([6], Lemma 4.1, (ii)), we may write $(1+x_j)^{-1} \le k(N+\frac{3}{2})^2$, where $k \le \frac{96p}{12\pi^2pq-\pi^4q^2} < 1$, $p = (\alpha+\beta+3)^2$, $q = (\beta+2)^2$.

We introduce now an inhomogeneous boundary condition

$$v_N^m(-1) = \varphi(t^m,-1), \qquad m \ge -m_0. \tag{4.4}$$

Theorem 4.3. *Consider the scheme* (3.4), (3.6), (4.4). *If all the conditions of the preceding theorem are satisfied, then there is* $C_1 > 0$ *such that*

$$\begin{aligned}\|v_N^m\|_\omega \le C\exp(at^m)\|v_N^0\|_\omega + \tau C\exp(at^m)\sum_{n=1}^{m}\exp(-at^n)\|G(t^{n-1},\cdot,Tv_N)\|_\omega +\\ + \tau C_1\exp(at^m)N\sum_{n=1}^{m}\exp(-at^n)|\varphi(t^{n-1},-1)|,\end{aligned} \tag{4.5}$$

with weights ω_j *given by* (4.1).

Proof. We use the substitution formula introduced in [3],

$$V_N^m(x) = v_N^m(x) - \frac{\partial_x P_{N+1}^{(\alpha,\beta)}(x)}{\partial_x P_{N+1}^{(\alpha,\beta)}(-1)}\varphi(t^m,-1).$$

Since $v_N^m(x_j) = V_N^m(x_j)$, $1 \le j \le N$, $m \ge -m_0$, the π_N-polynomial V_N satisfies (3.4)–(3.6) with $G(t,x,w)$ replaced in (3.4) by

$$G(t,x,w) + f(t,x)\frac{\partial_x^2 P_{N+1}^{(\alpha,\beta)}(x)}{\partial_x P_{N+1}^{(\alpha,\beta)}(-1)}\varphi(t,-1).$$

The norm $\|\cdot\|_\omega$ of the above fraction may be estimated by $c(\alpha,\beta)\cdot N$, where $c(\alpha,\beta)$ depends only on α,β, see [4], Lemma 4.2. Hence (4.5) follows with $C_1 = c(\alpha,\beta)B$. □

5. APPROXIMATION AND CONVERGENCE

Thanks to the exactness of the Gauss-Lobatto quadrature, and to the positiveness of quadrature weights, we have for $p \in \pi_N$

$$\|p\|_\omega^2 = \sum_{j=1}^{N} (1 - x_{L,N+1,j}^{(\alpha,\beta)}) \omega_{L,N+1,j}^{(\alpha,\beta)} \, p^2(x_{L,N+1,j}^{(\alpha,\beta)}) \le$$

$$\le \sum_{j=0}^{N+1} (1 - x_{L,N+1,j}^{(\alpha,\beta)}) \omega_{L,N+1,j}^{(\alpha,\beta)} \, p^2(x_{L,N+1,j}^{(\alpha,\beta)}) = \|p\|_{\alpha+1,\beta}^2.$$

On the other hand, let us note that relations (3.2) give $\omega_j = \omega_{R,N,j-1}^{(\alpha+1,\beta)}$ and $x_j = x_{R,N,j-1}^{(\alpha+1,\beta)}$, $1 \le j \le N$, so that the exactness of the Gauss-Radau quadrature implies $\|p\|_\omega = \|p\|_{\alpha+1,\beta}$ for $p \in \pi_N$ satisfying $p(-1) = 0$. For $p \in \pi_N$ let us put $\tilde{p}(x) = p(x) - p(-1)$. Then $\tilde{p}(-1) = 0$ and hence $\|\tilde{p}\|_\omega = \|\tilde{p}\|_{\alpha+1,\beta}$. Consequently,

$$\|p\|_{\alpha+1,\beta} \le \|\tilde{p}\|_{\alpha+1,\beta} + k|p(-1)| = \|\tilde{p}\|_\omega + k|p(-1)| \le$$

$$\le \|p\|_\omega + \|p(-1)\|_\omega + k|p(-1)| \le \|p\|_\omega + 2k|p(-1)|,$$

where $k = \|1\|_{\alpha+1,\beta}$.

We cite below two approximation results (Theorems 4.6 and 4.7) of [6], in somewhat simplified versions. As the main simplification, we will understand by $c(\alpha,\beta)$ merely a constant depending only on α, β, while in [6], the $c(\alpha,\beta)$ is given explicitly every time.

Theorem 5.1. *For any $w \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, $r \ge 1$, we have*

$$\|I_{R,N,\alpha,\beta} w - w\|_{\alpha,\beta} \le c(\alpha,\beta)(N(N+\alpha+\beta))^{-r/2} |w|_{r,\alpha,\beta}. \tag{5.1}$$

Theorem 5.2. *If one of the following conditions hold:*

$$\alpha = \beta \ge -\frac{1}{2} \qquad \text{or} \quad \alpha \ge \beta + 1, \quad \beta \ge 0 \qquad \text{or} \quad \frac{1}{2} \le \alpha \le \beta + 1, \tag{5.2}$$

then for any w such that $\partial_x w \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, $r \ge 0$, we have

$$|I_{R,N,\alpha,\beta} w - w|_{1,\alpha,\beta} \le c(\alpha,\beta)(N(N+\alpha+\beta))^{(1-r)/2} |\partial_x w|_{r,\alpha,\beta}, \tag{5.3}$$

$$\|I_{R,N,\alpha,\beta} w - w\|_{\alpha,\beta} \le c(\alpha,\beta)(N(N+\alpha+\beta))^{(1-r)/2} |\partial_x w|_{r,\alpha,\beta}. \tag{5.4}$$

Suppose that conditions of Theorem 5.2 are satisfied and $r \ge 1$. Then, from the triangle inequality joining (5.3), and (5.1) with $\partial_x w$ instead of w, we get the following

Corollary 5.3. *If (5.2) holds, then for any w such that $\partial_x w \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, $r \ge 1$, we have*

$$\|\partial_x I_{R,N,\alpha,\beta} w - I_{R,N,\alpha,\beta} \partial_x w\|_{\alpha,\beta} \le c(\alpha,\beta)(N(N+\alpha+\beta))^{(1-r)/2} |\partial_x w|_{r,\alpha,\beta}. \tag{5.5}$$

Assumption H[G]**.** Suppose that G satisfies one of the following conditions with $L \geq 0$.

(i) $|G(t,x,w) - G(t,x,\bar{w})| \leq L\|w - \bar{w}\|_{\alpha+1,\beta;[t]}$, $\qquad (t,x) \in E,\ w,\bar{w} \in X^0_{\alpha+1,\beta}$,

or, for all $N \in \mathbb{N}$,

(ii) $\|G(t,\cdot,w) - G(t,\cdot,\bar{w})\|_\omega \leq L \sup_{-t_0 \leq s \leq t} \|(w-\bar{w})(s,\cdot)\|_\omega$, $\quad (t,x) \in E,\ w,\bar{w} \in X^0_{\alpha+1,\beta}$.

The following types of functional dependence are admissible by the Assumption H[G].

Example 5.4. Let $\gamma : (0,\infty) \to (0,\infty)$ be such that $\gamma(t) \leq t$ and let $g : E \times \mathbb{R} \to \mathbb{R}$ satisfy the Lipschitz condition $|g(t,x,p) - g(t,x,\bar{p})| \leq L|p - \bar{p}|$ on $E \times \mathbb{R}$ for some $L \geq 0$. Then G given by

$$G(t,x,w) = g(t,x,w(\gamma(t),x))$$

satisfies the condition (ii) of Assumption H[G].

Example 5.5. Let γ, g be as in Example 5.4. Then G given by

$$G(t,x,w) = g(t,x,\|w(\gamma(t),\cdot)\|_{\alpha+1,\beta})$$

satisfies the condition (i) of Assumption H[G].

Theorem 5.6. *Let $-\frac{1}{2} \leq \alpha \leq \beta < 0$. Suppose that the Assumption* H[G] *and all conditions of the Theorem 4.3 are satisfied and let $u \in X^0_{\alpha+1,\beta}$ be the unique solution of* (2.2), (2.3)*, such that $\partial_x u \in X^r_{\alpha+1,\beta}$, where $r \geq 2$, and, for some $A \geq 0$, $|\partial_t u(t,x)| \leq 2A$ on E. Moreover, let $\{v^m_N\}_{m \geq -m_0}$ be the solution of the scheme* (3.4), (3.6), (4.4) *with $\bar{\varphi}$ instead of φ, and assume that*

$$\partial_x \varphi(t,\cdot)\ ,\ \partial_x \bar{\varphi}(t,\cdot) \in H^r_{\chi^{(\alpha+1,\beta)},A}(\Lambda), \qquad t \in [-t_0,0] \tag{5.6}$$

and

$$\sup_{-t_0 \leq t} |(\bar{\varphi} - \varphi)(t,-1)| \leq \psi(\tau,N).$$

Then there is $C \geq 0$ (independent of τ, N) such that, for $m \geq 0$,

$$\begin{aligned}&\|(v^m_N - u)(t^m,\cdot)\|_{\alpha+1,\beta} \leq C\exp(\bar{a}t^m)\times \\ &\quad\times \Big(\big|(\varphi - \bar{\varphi})(t,\cdot)\big|_{\alpha+1,\beta;\,[0]} + (N(N+\alpha+\beta+1))^{(1-r)/2} + \tau + N\psi(\tau,N)\Big),\end{aligned} \tag{5.7}$$

where $\bar{a} = a + (1 + a\tau)CL\max\{1,k\}$, $a = \frac{1}{2}bL$.

Proof. The exact solution u fulfils the equation

$$u(t^{m+1},x_j) = u(t^m,x_j) + \tau\partial_x u(t^m,x_j)f(t^m,x_j) + \tau G(t^m,x_j,u) + \tau\Gamma^m_j(\tau), \tag{5.8}$$

where $|\Gamma_j^m(\tau)| \leq A\tau$. Consider the interpolation operator I_N, mapping functions defined on Λ, onto π_N, given by:

$$I_N w(x_j) = w(x_j), \quad 1 \leq j \leq N, \quad \text{and} \quad I_N w(-1) = w(-1).$$

The operator I_N is in fact a Gauss-Radau interpolation operator, since, as we have already stated, $x_j = x_{R,N,j-1}^{(\alpha+1,\beta)}$, $1 \leq j \leq N$ and $-1 = x_{R,N,N}^{(\alpha+1,\beta)}$. Put $\varepsilon_N = v_N - I_N u$ and $y^m = \max_{-m_0 \leq n \leq m} \|\varepsilon_N^n\|_{\omega^n}$. Then we have

$$y^0 = \max_{-m_0 \leq n \leq 0} \|I_N(\bar{\varphi} - \varphi)(t^n, \cdot)\|_{\omega^n}.$$

Moreover, using (5.4), we can derive

$$\begin{aligned} b^{-1/2}\|I_N w\|_{\omega^n} \leq \|I_N w\|_{\omega} \leq \|I_N w\|_{\alpha+1,\beta} \leq \|I_N w - w\|_{\alpha+1,\beta} + \|w\|_{\alpha+1,\beta} \leq \\ \leq C_0 (N(N+\alpha+\beta+1))^{(1-r)/2} + \|w\|_{\alpha+1,\beta} \end{aligned}$$

with some $C_0 \geq 0$, provided that $\partial_x w \in H^r_{\chi^{(\alpha+1,\beta)},A}(\Lambda)$. Substituting $w = \bar{\varphi} - \varphi$, and taking into account condition (5.6), we get

$$y^0 \leq b^{1/2} C_0 \left(N(N+\alpha+\beta+1)\right)^{(1-r)/2} + b^{1/2} \big|\bar{\varphi} - \varphi\big|_{\alpha+1,\beta;\,[0]}. \tag{5.9}$$

The point of the proof is to construct an appropriate difference inequality for y^m, $m \geq 0$. Gaining a relevant estimate of ε_N is sufficient here, since, by the triangle inequality,

$$\|(v_N - u)(t, \cdot)\|_{\alpha+1,\beta} \leq \|\varepsilon_N(t, \cdot)\|_{\alpha+1,\beta} + \|(I_N u - u)(t, \cdot)\|_{\alpha+1,\beta},$$

and the latter term is appropriately bounded due to (5.4).

Subtracting by sides (3.4) and (5.8), we get

$$\varepsilon_N^{m+1}(x_j) = \varepsilon_N^m(x_j) + \tau f(t^m, x_j)\partial_x \varepsilon_N^m(x_j) + \tau \widetilde{G}(t^m, x_j, u, v_N), \qquad 1 \leq j \leq N, \tag{5.10}$$

and

$$\varepsilon_N^m(-1) = (\bar{\varphi} - \varphi)(t^m, -1),$$

and

$$\varepsilon_N^m(x_j) = (\bar{\varphi} - \varphi)(t^m, x_j), \quad -t_0 \leq t^m \leq 0, \quad 1 \leq j \leq N,$$

where

$$\widetilde{G}(t^m, x_j, u, v_N) = \partial_x(u - I_N u)(x_j) f(t^m, x_j) + G(t^m, x_j, T v_N) - G(t^m, x_j, u) - \Gamma_j^m(\tau).$$

Proceeding as in the proof of Theorem 4.3, we obtain, for $m \geq 0$,

$$\|\varepsilon_N^{m+1}\|_{\omega^{m+1}} \leq (1+a\tau)\|\varepsilon_N^m\|_{\omega^m} + \tau(1+a\tau)\Big(\|\widetilde{G}(t^m, \cdot, u, v_N)\|_{\omega^m} + C_1 N |(\bar{\varphi}-\varphi)(t^m, -1)|\Big) \tag{5.11}$$

with some $C_1 \geq 0$. Let us now estimate the norm of $\widetilde{G}$. Due to Assumption H[f], we have

$$\|\widetilde{G}(t^m, \cdot\,, u, v_N)\|_\omega \leq B\|\partial_x(u - I_N u)\|_\omega + \|G(t^m, \cdot, Tv_N) - G(t^m, \cdot, u)\|_\omega + kA\tau.$$

By Corollary 5.3, there is $C_2 \geq 0$ such that

$$\|\partial_x u - \partial_x I_N u\|_\omega = \|I_N \partial_x u - \partial_x I_N u\|_\omega \leq \|I_N \partial_x u - \partial_x I_N u\|_{\alpha+1,\beta} \leq \\ \leq C_2(N(N+\alpha+\beta+1))^{(1-r)/2}.$$

For abbreviation, let L^* stand for $\max\{L, kL\}$. Thanks to Assumption H[G], we are able to majorize the term $\|G(t^m, \cdot, Tv_N) - G(t^m, \cdot, u)\|_\omega$ either by

$$L^*\|Tv_N - u\|_{\alpha+1,\beta;[t^m]},$$

or by

$$L^* \sup_{-t_0 \leq s \leq t^m} \|(Tv_N - u)(s, \cdot)\|_\omega.$$

We write $Tv_N - u$ as $T\varepsilon_N + T(I_N u - u) + (Tu - u)$ and apply the triangle inequality.

Suppose that the condition (i) from this Assumption holds true. Then, from the definition of operator T follows

$$\|T\varepsilon_N\|_{\alpha+1,\beta;[t^m]} = \sup_{-t_0 \leq s \leq t^m} \|T\varepsilon_N(s, \cdot)\|_{\alpha+1,\beta} \leq \\ \leq \max_{-m_0 \leq n \leq m} \|\varepsilon_N^n\|_{\alpha+1,\beta} \leq B^{1/2} y^m + 2B^{1/2} k\psi(\tau, N).$$

By the same token, and by use of (5.4), there is $C_3 \geq 0$ such that

$$\|T(I_N u - u)\|_{\alpha+1,\beta;[t^m]} \leq C_3(N(N+\alpha+\beta+1))^{(1-r)/2}.$$

Finally,

$$\|Tu - u\|_{\alpha+1,\beta;[t^m]} \leq kA\tau.$$

Suppose now that (ii) holds. If this is the case, we need to estimate the quantities $\|\varepsilon_N^n\|_\omega$, but these are upper bounded by $\|\varepsilon_N^n\|_{\alpha+1,\beta}$, since $\varepsilon_N^n \in \pi_N$. The ω-norm of $(I_N u - u)(s, \cdot)$ is equal to zero, and $\|(Tu - u)(s, \cdot)\|_\omega$ has the same estimate as $\|(Tu - u)(s, \cdot)\|_{\alpha+1,\beta}$.

Taking both variants of Assumption H[G] into account, we find that

$$\|G(t^m, \cdot, Tv_N) - G(t^m, \cdot, u)\|_\omega \leq \\ \leq L^*\Big(B^{1/2} y^m + 2B^{1/2} k\psi(\tau, N) + C_3(N(N+\alpha+\beta+1))^{(1-r)/2} + kA\tau\Big).$$

With the notation $\xi_{\alpha,\beta}(\tau, N) = (N(N+\alpha+\beta+1))^{(1-r)/2} + \tau + N\psi(\tau, N)$, it is easy to see that there exists a constant $C_4 \geq 0$ such that

$$\|\widetilde{G}(t^m, \cdot, u, v_N)\|_{\omega^m} \leq b^{1/2}\|\widetilde{G}(t^m, \cdot, u, v_N)\|_\omega \leq CL^* y^m + C_4 \xi_{\alpha,\beta}(\tau, N),$$

where $C = B^{1/2} b^{1/2}$. Hence, and by (5.11),

$$\|\varepsilon_N^{m+1}\|_{\omega^{m+1}} \leq \Big(1 + \tau[a + (1 + a\tau)CL^*]\Big)\, y^m + \tau(1 + a\tau)C_4\xi_{\alpha,\beta}(\tau, N), \quad m \geq 0.$$

Consequently,

$$y^{m+1} \leq (1 + \bar{a}\tau)\, y^m + \tau(1 + \bar{a}\tau)C_4\xi_{\alpha,\beta}(\tau, N), \quad m \geq 0,$$

where $\bar{a} = a + (1 + a\tau)CL^*$, which is the recurrent inequality we have searched for. Solving this inequality and using (5.9), we obtain

$$\begin{aligned}
\|\varepsilon_N^m\|_{\alpha+1,\beta} &\leq \|\varepsilon_N^m\|_\omega + 2k\psi(\tau, N) \leq \\
&\leq B^{1/2}\|\varepsilon_N^m\|_{\omega^m} + 2k\psi(\tau, N) = \\
&= B^{1/2} y^m + 2k\psi(\tau, N) \leq \\
&\leq B^{1/2}\exp(\bar{a}t^m)\Big(y^0 + \frac{C_4}{\bar{a}}\xi_{\alpha,\beta}(\tau, N)\Big) + 2k\psi(\tau, N) \leq \\
&\leq B^{1/2}\exp(\bar{a}t^m)\Big(b^{1/2}\big|\bar{\varphi} - \varphi\big|_{\alpha+1,\beta;\,[0]} + (b^{1/2}C_0 + C_4\bar{a}^{\,-1})\xi_{\alpha,\beta}(\tau, N)\Big) + \\
&\quad + 2k\psi(\tau, N),
\end{aligned}$$

and the assertion (5.7) follows. □

6. NUMERICAL EXAMPLES

Take $\alpha = \beta = -\frac{1}{2}$. Then simply $x_j = \cos\frac{\pi j}{N+1}$ and $\|w\|_\omega = \frac{\pi}{N+1}\sum_{j=1}^{N}(1 - x_j)w^2(x_j)$. Consider the problem (2.2), (2.3) with $t_0 = 0$, $f(t, x) = 1 + t + x^2$, $G(t, x, z) = z(t, x)[x^2 - 2 + 2tx(1 + t + x^2)] + \rho(z(\frac{t}{2}, x))$, $\varphi \equiv 1$, where

$$\rho(y) = \begin{cases} y^2, & |y| \leq 2, \\ 4, & \text{otherwise.} \end{cases}$$

We have tested the pseudospectral method for this problem against the known solution $u(t, x) = \exp(t(x^2 - 1))$. The errors were measured at the cut $T = 0.25$. We denote $\|v_N - u\| = \|(v_N - u)(T, \cdot)\|$ in the Tables below. From Table 1 it can be seen that the term with magnitude N^{1-r} (see convergence theorem) is practically negligible for $N \geq 8$, since there is no gain in accuracy, when augmenting N to 16 and farther.

Hence the idea of fixing $N = 8$, during the second series of experiments, and using τ to control the accuracy. We aimed to compare the pseudospectral method and the Euler finite difference scheme. The classical method has accuracy $O(\tau + N^{-1})$, so instead of running it with the same number N of spatial nodes, we found it more interesting to measure the time needed to obtain results not worse than that of the pseudospectral method. Logs of this “accuracy vs. time” test are gathered in the Table 2.

Due to the classical CFL condition, we have been always choosing τ, N so that $\tau N = 4/5$, in the finite difference scheme. The $\|\cdot\|_\omega$ norm of the error in the classical case is defined by $\|z_{\tau,N} - u\|_\omega^2 = \sum_{j=1}^{N-1} \chi^{(1/2,-1/2)}(x_j)(z_{\tau,N} - u)^2(x_j)$, where $z_{\tau,N}$ is the solution of the Euler finite difference scheme, on the mesh with spatial nodes $x_j = -1+2jN^{-1}$, $j = 0,\dots,N-1$. The maximum norm $\|\cdot\|_0$ is defined as the maximal absolute value in either case. It can be seen from the Table 2 that the classical finite difference scheme is faster only when quite low accuracy ($> 2^{-14} \approx 10^{-4}$) is needed. Moreover, the data storage, required by the Euler scheme for this specific problem, grows locally faster (with respect to accuracy ε, like ε^{-2} vs. ε^{-1}) than this required by the collocation scheme. To make the error smaller than 2^{-21} using the first one, one should expect, looking at the right half of the Table 2, to have to store about 2^{34} of real numbers. Thus, restricting ourselves to sequential (not parallel) computing, we have skipped the last two experiments with the finite difference scheme.

Let us take now the integral-differential problem (2.2), (2.3) with f as in the previous example, $G(t,x,w) = (1-x^2)^{7/4} \cdot \left[(1-x^2) - \frac{11}{2}tx(1+x^2+I(t,w))\right]$ and $\varphi \equiv 0$, where $I^2(t,w) = \int_\Lambda w^2(t/2,x)\chi^{(1/2,-1/2)}(x)\,dx$. We have tested the pseudospectral method against the known solution $u = t(1-x^2)^{11/4}$ and obtained the results, gathered in the Table 3.

Table 1. Error of pseudospectral method for $\tau = 2^{-14}$

N	$\|v_N - u\|_\omega$
2	$2.929\,222 \cdot 10^{-3}$
4	$9.113\,179 \cdot 10^{-5}$
8	$5.102\,305 \cdot 10^{-6}$
16	$5.102\,451 \cdot 10^{-6}$
32	$5.102\,455 \cdot 10^{-6}$

Table 2. Comparison of time costs for prescribed accuracy (the $-\log_2$ of errors is taken)

Pseudospectral collocation ($N = 8$)				Euler FDM ($N = 5\cdot 2^k$, $\tau N = \frac{4}{5}$)			
$-\log_2 \tau$	$\|v_N - u\|_\omega$	$\|v_N - u\|_0$	time [s]	k	$\|z_{\tau,N} - u\|_\omega$	$\|z_{\tau,N} - u\|_0$	time [s]
6	9.563 177	9.360 599	0.25	4	9.741 110	8.456 797	0.002
10	13.579 380	13.370 646	0.25	8	13.660 191	13.322 350	0.08
14	17.580 419	17.372 826	0.26	12	17.655 178	17.314 159	17.11
18	21.577 699	21.397 366	0.50	–	–	–	–
22	25.003 930	25.183 749	4.19	–	–	–	–

Table 3. Error of the pseudospectral method for the differential-integral problem

$-\log_2 \tau$	N	$\|v_N - u\|_\omega$
5	8	$1.522\,379 \cdot 10^{-4}$
7	16	$3.183\,423 \cdot 10^{-6}$
9	32	$7.629\,330 \cdot 10^{-8}$
11	64	$1.765\,044 \cdot 10^{-9}$
13	128	$4.003\,189 \cdot 10^{-11}$
15	256	$8.967\,462 \cdot 10^{-13}$

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