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DIFFERENTIAL DIFFERENCE INEQUALITIES RELATED TO PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. Initial boundary value problems for nonlinear parabolic functional differential equations are transformed by discretization in space variables into systems of ordinary functional differential equations. A comparison theorem for differential difference inequalities is proved. Sufficient conditions for the convergence of the method of lines is given. Nonlinear estimates of the Perron type for given operators with respect to functional variables are used. Results obtained in the paper can be applied to differential integral problems and to equations with deviated variables.

Keywords: parabolic functional differential equations, method of lines, stability and convergence.

Mathematics Subject Classification: 35R10, 35K20, 65N40.

1. INTRODUCTION

We are interested in establishing a method of approximation of solutions to nonlinear parabolic functional differential equations with solutions of associated systems of ordinary functional differential equations and in the estimation of the difference between the exact and approximate solutions. We ask under what conditions, solutions of ordinary functional differential equations tend to a solution of the original problem when the step-size tends to zero. The system of ordinary functional differential equations mentioned above are obtained by using a discretization in spatial variables of partial functional differential equations and are therefore called differential difference systems. This method of approximation of solutions to parabolic problems is called a numerical method of lines. The main problem in our investigations is to find a differential difference system which satisfies consistency conditions on all sufficiently regular solutions of the original equations and is stable. An error estimate implying the convergence of the numerical method of lines is obtained in the paper by using a comparison result for differential difference inequalities.

From the extensive literature concerning the numerical method of lines for classical differential equations we mention the monographs [3, 13, 15]. The papers [9, 11] began a theory of the method of lines for functional differential equations. Approximate solutions of nonlinear parabolic functional differential equations with initial boundary conditions of the Dirichlet type were investigated in [5, 9, 17]. An error estimate implying the convergence of line methods is obtained in these papers by using differential inequalities. In [6] the author studies the error due to the discretization in spatial variables of the Cauchy problem for parabolic equations. It is assumed that approximated solutions satisfy some growth-restricting conditions. In [10] the authors have established approximation solution theorems for nonlinear parabolic functional differential equations with initial boundary conditions of the Neumann type. A method of differential inequalities is used. Therefore, the authors have assumed in [10] that the right-hand sides of equations are nondecreasing with respect to the functional variable. The papers [2, 7] deal with the numerical method of lines for first order partial functional differential equations or systems. The method of lines is also treated as a tool for proving existence theorems for partial differential equations [1, 14, 16].

The aim of the paper is to construct a method of lines for nonlinear parabolic functional differential equations with general initial boundary conditions.

We now formulate our functional differential problems. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Write

$$Q_0 = [-b_0, 0] \times [-b, b], \quad Q = (0, a) \times [-b, b],$$

where $a > 0$, $b_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = (0, +\infty)$ and $b = (b_1, \ldots, b_n)$, $b_i > 0$ for $i = 1, \ldots, n$. Suppose that $\chi : [0, a) \to \mathbb{R}$ and $\psi_* = (\psi_1, \ldots, \psi_n) : [0, a) \times [-b, b] \to \mathbb{R}^n$ are given functions. Write $\psi(t, x) = (\chi(t), \psi_*(t, x))$ for $(t, x) \in Q$. We assume that $0 \le \chi(t) \le t$ for $t \in [0, a)$ and $-b \le \psi_*(t, x) \le b$ for $(t, x) \in Q$. For $(t, x) \in [0, a) \times [-b, b]$ we define

$$D[t, x] = \{(\tau, y) \in \mathbb{R}^{1+n} : \tau \le 0, \quad (t + \tau, x + y) \in Q_0 \cup Q\}.$$

It is clear that

$$D[t, x] = [-b_0 - t, 0] \times [-b - x, b - x].$$

The maximum norm in the space $C(D[t, x], \mathbb{R})$ will be denoted by $\| \cdot \|_{D[t,x]}$. For a function $z : Q_0 \cup Q \to \mathbb{R}$ and for a point $(t, x) \in [0, a) \times [-b, b]$ we define a function $z_{(t,x)} : D[t, x] \to \mathbb{R}$ as follows

$$z_{(t,x)}(\tau, y) = z(t + \tau, x + y), \quad (\tau, y) \in D[t, x].$$

Then $z_{(t,x)}$ is the restriction of z to the set $(Q_0 \cup Q) \cap ([-b_0, t] \times \mathbb{R}^n)$ and this restriction is shifted to the set $D[t, x]$. Write $I = [-b_0 - a, 0]$ and $B = I \times [-2b, 2b]$. Then $D[t, x] \subset B$ for $(t, x) \in [0, a) \times [-b, b]$. Let $M_{n \times n}$ be the class of all $n \times n$ symmetric matrices with real elements. For $x \in \mathbb{R}^n$, $U \in M_{n \times n}$ where $U = [u_{ij}]_{i,j=1,\ldots,n}$ we write

$$\|x\| = \sum_{i=1}^{n} |x_i|, \quad \|U\|_\infty = \max \Big\{ \sum_{j=1}^{n} |u_{ij}| \Big\}.$$

Put $\Xi = Q \times \mathbb{R} \times C(B, \mathbb{R}) \times \mathbb{R}^n \times M_{n\times n}$ and suppose that $F : \Xi \to \mathbb{R}$ is a given function. We consider the functional differential equation

$$\partial_t z(t,x) = F(t,x,z(t,x), z_{\psi(t,x)}, \partial_x z(t,x), \partial_{xx} z(t,x)) \tag{1.1}$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$, $\partial_{xx} z = [\partial_{x_i x_j} z]_{i,j=1,\ldots,n}$.

Now we formulate initial boundary conditions for (1.1). Write

$$S_i = \{x \in [-b,b] : x_i = b_i\}, \quad S_{n+i} = \{x \in [-b,b] : x_i = -b_i\},\ i = 1,\ldots,n$$

and

$$Q_1^+ = S_1, \quad Q_i^+ = S_i \setminus \bigcup_{j=1}^{i-1} S_j, \quad Q_i^- = S_{n+i} \setminus \bigcup_{j=1}^{n+i-1} S_j, \quad i = 1,\ldots,n.$$

Set

$$\partial_0 E_i^+ = [0,a) \times Q_i^+, \quad \partial_0 E_i^- = [0,a) \times Q_i^-,\ i - 1,\ldots,n$$

and

$$\partial_0 E = \bigcup_{i=1}^{n} (\partial_0 E_i^+ \cup \partial_0 E_i^-).$$

Suppose that $\beta,\ \gamma,\ \phi : \partial_0 E \to \mathbb{R}$, $\varphi : E_0 \to \mathbb{R}$ are given functions. The following initial boundary conditions are associated with (1.1)

$$z(t,x) = \varphi(t,x) \quad \text{on } Q_0, \tag{1.2}$$

$$\beta(t,x) z(t,x) + \gamma(t,x) \partial_{x_i} z(t,x) = \phi(t,x) \quad \text{on } \partial_0 E_i^+,\ i = 1,\ldots,n, \tag{1.3}$$

$$\beta(t,x) z(t,x) - \gamma(t,x) \partial_{x_i} z(t,x) = \phi(t,x) \quad \text{on } \partial_0 E_i^-,\ i = 1,\ldots,n. \tag{1.4}$$

A function $z : Q_0 \cup Q \to \mathbb{R}$ will be called a function of class C_* if z is continuous on $Q_0 \cup Q$, the partial derivatives $\partial_t z$, $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$, $\partial_{xx} z = [\partial_{x_i x_j} z]_{i,j=1,\ldots,n}$ exist on Q and the functions $\partial_t z, \partial_x z, \partial_{xx} z$ are continuous and bounded on Q. We consider solutions of (1.1)–(1.4) of class C_*. We will say that the function $F : \Xi \to \mathbb{R}$ satisfies the condition (V) if for each $(t,x,p,w,r,q) \in \Xi$, $\bar{w} \in C(B,\mathbb{R})$ such that $w(\tau,y) = \bar{w}(\tau,y)$ for $(\tau,y) \in D[\psi(t,x)]$ we have $F(t,x,p,w,r,q) = F(t,x,p,\bar{w},r,q)$. The condition (V) for F means that the value of F at the point $(t,x,p,w,r,q) \in \Xi$ depends on (t,x,p,r,q) and on the restriction of w to the set $D[\psi(t,x)]$ only.

Our focus is the numerical method of lines for problem (1.1)–(1.4). By making use of a discretization of the spatial variable, we associate with problem (1.1)–(1.4) a class of Cauchy problems for ordinary functional differential systems. Solutions of such systems are approximate solutions to (1.1)–(1.4). Then we estimate the difference between the exact and approximate solutions of the original problem and we prove that approximate solutions converge to the solutions of (1.1)–(1.4).

The paper is organized as follows. In Section 2 we formulate a numerical method of lines for (1.1)–(1.4). In the next section we present a comparison result for differential difference inequalities. It will be a generalization of a corresponding result from [17]. A convergence result and error estimates are presented in Section 4. It is the main part of the paper. Numerical examples are given in the last part of the paper.

We will use general ideas for functional differential equations and inequalities which were introduced in [4].

2. DIFFERENTIAL DIFFERENCE EQUATIONS

Now we formulate the differential difference problem corresponding to (1.1)–(1.4). For any spaces X and Y we denote by $\mathcal{F}(X,Y)$ the class of all functions defined on X and taking values in Y. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. We define a mesh on $[-b,b]$ in the following way. Let $h=(h_1,\ldots,h_n)$, $h_i>0$ for $1\le i\le n$, stand for the steps of the mesh. For $m\in\mathbb{Z}^n,\ m=(m_1,\ldots,m_n)$, we define nodal points as follows: $x^{(m)}=(m_1h_1,\ldots,m_nh_n)=(x_1^{(m_1)},\ldots,x_n^{(m_n)})$. Let us denote by H the set of all h for which there exist $(M_1,\ldots,M_n)=M\in\mathbb{Z}^n$ such that $M_ih_i=b_i$ for $i=1,\ldots,n$. Write

$$\mathbb{R}^{1+n}_{t.h}=\{(t,x^{(m)}):t\in\mathbb{R},m\in\mathbb{Z}^n\}$$

and

$$Q_{0.h}=Q_0\cap\mathbb{R}^{1+n}_{t.h},\quad Q_h=Q\cap\mathbb{R}^{1+n}_{t.h},\quad B_h=B\cap\mathbb{R}^{1+n}_{t.h},$$
$$D_h[t,x^{(m)}]=D[t,x^{(m)}]\cap\mathbb{R}^{1+n}_{t.h}.$$

For a function $z:Q_{0.h}\cup Q_h\to\mathbb{R}$ and for a point $(t,x^{(m)})\in Q_{0.h}\cup Q_h$ we write $z^{(m)}(t)=z(t,x^{(m)})$. Let $\mathcal{F}_C(Q_{0.h}\cup Q_h,\mathbb{R})$ be a class of all functions $z:Q_{0.h}\cup Q_h\to\mathbb{R}$ such that $z(\cdot,x^{(m)})\in C([-b_0,a),\mathbb{R})$ for $-M\le m\le M$. In a similar way we define the spaces $\mathcal{F}_C(B_h,\mathbb{R})$ and $\mathcal{F}_C(Q_{0.h},\mathbb{R})$. For functions $z\in C(Q_0\cup Q,\mathbb{R}),\ z_h\in\mathcal{F}_C(Q_{0.h}\cup Q_h,\mathbb{R})$ we put

$$\|z\|_t=\max\{|z(\tau,y)|:\ (\tau,y)\in Q_0\cup Q,\ \tau\le t\},$$
$$\|z\|_{h.t}=\max\{|z(\tau,y)|:\ (\tau,y)\in Q_{0.h}\cup Q_h,\ \tau\le t\}$$

where $0\le t<a$. Difference operators for spatial variables are defined in the following way. Let $e_i=(0,\ldots,0,1,0,\ldots,0)\in\mathbb{R}^n$ with 1 at the $i-$th position. Write $J=\{(i,j):\ i,j=1,\ldots,n,\ i\ne j\}$. Suppose that we have defined the sets $J_+,\ J_-\subset J$ such that $J_+\cup J_-=J,\ J_+\cap J_-=\emptyset$. We assume that $(i,j)\in J_+$ if $(j,i)\in J_+$. In particular, it may happen that $J_+=\emptyset$ or $J_-=\emptyset$. A relation between the sets $J_+,\ J_-$ and equation (1.1) are given in Section 4.

Given $z\in\mathcal{F}_C(Q_{0.h}\cup Q_h,\mathbb{R})$ and $m,\ -(M-1)\le m\le M-1$, where $M-1=(M_1-1,\ldots,M_n-1)$. Write

$$\delta_i^+z^{(m)}(t)=\frac{1}{h_i}[z^{(m+e_i)}(t)-z^{(m)}(t)],\quad \delta_i^-z^{(m)}(t)=\frac{1}{h_i}[z^{(m)}(t)-z^{(m-e_i)}(t)],$$

$$i=1,\ldots,n,\ \text{and}\ \ \delta z^{(m)}(t)=(\delta_1z^{(m)}(t),\ldots,\delta_nz^{(m)}(t)),\ \text{where}$$

$$\delta_i z^{(m)}(t) = \frac{1}{2}[\delta_i^+ z^{(m)}(t) + \delta_i^- z^{(m)}(t)],\ i = 1, \dots, n.$$

The difference operators $\delta^{(2)} = [\delta_{ij}]_{i,j=1,\dots,n}$, are defined in the following way:

$$\delta_{ii}^{(2)} z^{(m)}(t) = \delta_i^+ \delta_i^- z^{(m)}(t) \quad \text{for} \quad i = 1, \dots, n$$

and

$$\delta_{ij}^{(2)} z^{(m)}(t) = \frac{1}{2}\big[\delta_i^+ \delta_j^- z^{(m)}(t) + \delta_i^- \delta_j^+ z^{(m)}(t)\big] \quad \text{for} \quad (i,j) \in J^-,$$
$$\delta_{ij}^{(2)} z^{(m)}(t) = \frac{1}{2}\big[\delta_i^+ \delta_j^+ z^{(m)}(t) + \delta_i^- \delta_j^- z^{(m)}(t)\big] \quad \text{for} \quad (i,j) \in J^+.$$

Solutions of differential difference equations are elements of the space $\mathcal{F}_C(Q_{0.h} \cup \cup Q_h, \mathbb{R})$. Since equation (1.1) contains the functional variable $z_{\psi(t,x)}$ which is an element of the space $C(B, \mathbb{R})$ then we need an interpolating operator $T_h : \mathcal{F}_C(Q_{0.h} \cup Q_h, \mathbb{R}) \to C(Q_0 \cup Q, \mathbb{R})$. In the next part of the paper we adopt additional assumptions on T_h. For $z \in \mathcal{F}_C(Q_{0.h} \cup Q_h, \mathbb{R})$ and $(t, x^{(m)}) \in Q_h$ we write $T_h z_{\psi^{(m)}(t)}$ instead of $(T_h z)_{\psi(t,x^{(m)})}$. Set

$$F_h[z]^{(m)}(t) = F(t, x^{(m)}, z^{(m)}(t), T_h z_{\psi^{(m)}(t)}, \delta z^{(m)}(t), \delta^{(2)} z^{(m)}(t))$$

and

$$\Lambda_{h.i}^+[z]^{(m)}(t) = \beta^{(m)}(t) z^{(m)}(t) + \gamma^{(m)}(t) \delta_i^- z^{(m)}(t),$$
$$\Lambda_{h.i}^-[z]^{(m)}(t) = \beta^{(m)}(t) z^{(m)}(t) - \gamma^{(m)}(t) \delta_i^+ z^{(m)}(t)$$

where $i = 1, \dots, n$. Given $\phi_h : \partial_0 E \to \mathbb{R}$, $\varphi_h : Q_{0.h} \to \mathbb{R}$. We consider the differential difference equation

$$\partial_t z^{(m)}(t) = F_h[z]^{(m)}(t) \tag{2.1}$$

with the initial boundary conditions

$$z^{(m)}(t) = \varphi_h^{(m)}(t) \quad \text{on } Q_{0.h}, \tag{2.2}$$

and

$$\Lambda_{h.i}^+[z]^{(m)}(t) = \phi_h^{(m)}(t) \text{ on } \partial_0 E_i^+, \quad \Lambda_{h.i}^-[z]^{(m)}(t) = \phi_h^{(m)}(t) \text{ on } \partial_0 E_i^-,\ 1 \le i \le n. \tag{2.3}$$

We prove that under natural assumptions on given functions there exists a solution of (2.1)–(2.3) and solutions of (2.1)–(2.3) approximate solutions of (1.1)–(1.4).

3. DIFFERENTIAL DIFFERENCE INEQUALITIES

We prove a comparison theorem for differential difference inequalities. The theorem states that a function $z : Q_{0.h} \cup Q_h \to \mathbb{R}$ satisfying the differential difference inequalities can be estimated by a suitable solution of an ordinary functional differential equation.

For a function $z : Q_{0.h} \cup Q_h \to \mathbb{R}$ and for a point $(t, x^{(m)}) \in [0, a) \times [-b, b]$ we define a function $z_{[t,x^{(m)}]} : D[t, x^{(m)}] \to \mathbb{R}$ as follows

$$z_{[t,x^{(m)}]}(\tau, y) = z(t + \tau, x^{(m)} + y), \quad (\tau, y) \in D[t, x^{(m)}].$$

Then $z_{[t,x^{(m)}]}$ is the restriction of z to the set $(Q_{0.h} \cup Q_h) \cap ([-b_0, t] \times \mathbb{R}^n)$ and this restriction is shifted to the set $D[t, x^{(m)}]$. Write $X_h = Q_h \times \mathbb{R} \times \mathcal{F}_C(B_h, \mathbb{R})$ and suppose that

$$f : X_h \to M_{n\times n},\ f = [f_{ij}]_{i,j=1,\dots,n}, \quad g : X_h \to \mathbb{R}^n,\ g = (g_1, \dots, g_n),$$

are given functions. We will say that f and g satisfy the condition (V_h) if for each $(t, x^{(m)}, p, w) \in X_h$, $\bar{w} \in \mathcal{F}_C(B_h, \mathbb{R})$ such that $w(\tau, y) = \bar{w}(\tau, y)$ for $(\tau, y) \in D_h[\psi^{(m)}(t)]$ we have

$$f(t, x^{(m)}, p, w) = f(t, x^{(m)}, p, \bar{w}) \ \text{ and } \ g(t, x^{(m)}, p, w) = g(t, x^{(m)}, p, \bar{w}).$$

Write $I[t] = [-b_0 - t, 0]$ where $t \in [0, a)$. Then $I[t] \subset I$ for $t \in [0, a)$. Suppose that $\sigma : [0, a) \times \mathbb{R}_+ \times C(I, \mathbb{R}_+) \to \mathbb{R}_+$ is a given function. We will say that σ satisfies the condition (V_0) if for each $(t, p, \eta) \in [0, a) \times \mathbb{R}_+ \times C(I, \mathbb{R}_+)$, $\tilde{\eta} \in C(I, \mathbb{R}_+)$ such that $\eta(\tau) = \tilde{\eta}(\tau)$ for $\tau \in I[\chi(t)]$ we have $\sigma(t, p, \eta) = \sigma(t, p, \tilde{\eta})$.

Suppose that $(t, x^{(m)}) \in Q_h$ and $w \in \mathcal{F}_C(D_h[t, x^{(m)}], \mathbb{R})$. We denote by $U_h[w] : I[t] \to \mathbb{R}_+$ a function given by

$$U_h[w](\tau) = \max\{|w(\tau, y)| : (\tau, y) \in D_h[t, x^{(m)}]\},\ \tau \in I[t].$$

For $z \in \mathcal{F}_C(Q_{0.h} \cup Q_h, \mathbb{R})$ we write $U_h z_{\psi^{(m)}(t)}$ instead of $U_h[z_{\psi^{(m)}(t)}]$. Set

$$\mathbb{G}[z]^{(m)}(t) = \sum_{i=1}^{n} g_i(t, x^{(m)}, z^{(m)}(t), z_{\psi^{(m)}(t)})\delta_i z^{(m)}(t) + \\ + \sum_{i,j=1}^{n} f_{ij}(t, x^{(m)}, z^{(m)}(t), z_{\psi^{(m)}(t)})\delta_{ij} z^{(m)}(t).$$

In this section we consider the differential difference inequalities

$$\left|\partial_t z^{(m)}(t) - \mathbb{G}[z]^{(m)}(t)\right| \le \sigma(t, \left|z^{(m)}(t)\right|, U_h z_{\psi^{(m)}(t)}). \tag{3.1}$$

We prove that a function satisfying (3.1) can be estimated by a solution of a suitable ordinary functional differential equation.

Assumption H$[\sigma]$**.** The function $\sigma : [0, a) \times \mathbb{R}_+ \times C(I, \mathbb{R}_+) \to \mathbb{R}_+$ satisfies the condition (V_0) and:

1) σ is continuous and $\sigma(t, p, \cdot)$ is nondecreasing for every $(t, p) \in [0, a) \times \mathbb{R}_+$,
2) for each $\eta \in C([-b_0, 0], \mathbb{R}_+)$ there exists on $I \cup (0, a)$ the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t), \omega_{\chi(t)}), \quad \omega(t) = \eta(t) \quad \text{for } t \in [-b_0, 0]. \tag{3.2}$$

Assumption H$[\beta, \gamma]$**.** The functions $\beta : \partial_0 E \to (0, +\infty)$, $\gamma : \partial_0 E \to \mathbb{R}_+$ are continuous and bounded and they satisfy the conditions: $\beta(t, x) \geq 1$ and $\gamma(t, x) \geq 0$ for $(t, x) \in \partial_0 E$.

Let us state now a lemma for the functional differential inequalities.

Lemma 3.1. *Suppose that the function* $\sigma : [0, a) \times \mathbb{R}_+ \times C(I, \mathbb{R}_+) \to \mathbb{R}_+$ *satisfies condition* (V_0) *and:*

1) $\sigma(t, p, \cdot) : C(I, \mathbb{R}_+) \to \mathbb{R}_+$ *is nondecreasing,*
2) $\chi : [0, a) \to \mathbb{R}_+$ *and* $0 \leq \chi(t) \leq t$ *for* $t \in [0, a)$,
3) $u,\ v \in C(I \cup [0, a), \mathbb{R})$ *and* $u(t) < v(t)$ *for* $t \in [-b_0, 0]$,
4) *denote*

$$T_+ = \{t \in (0, a) : u(\tau) < v(\tau) \text{ for } \tau \in [-b_0, t) \text{ and } u(t) = v(t)\}$$

we assume that

$$D_- u(t) - \sigma(t, u(t), u_{\chi(t)}) < D_- v(t) - \sigma(t, v(t), v_{\chi(t)}) \text{ for } t \in T_+,$$

where D_- *is the left-hand lower Dini derivative.*

Under these assumptions we have $u(t) < v(t)$ *for* $t \in [0, a)$.

We omit the proof of the lemma.

Theorem 3.2. *Suppose that Assumptions* $\mathbf{H}[\sigma]$ *and* $\mathbf{H}[\beta, \gamma]$ *are satisfied and:*

1) *the functions* $f : X_h \to M_{n\times n}$, $g : X_h \to \mathbb{R}^n$, *satisfy the condition* (V_h) *and*

$$-\frac{1}{2}\,|g_i(P)| + \frac{1}{h_i} f_{ii}(P) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_j}\,|f_{ij}(P)| \geq 0, \quad i = 1, \ldots, n, \tag{3.3}$$

and

$$f_{ij}(P) \geq 0 \text{ for } (i, j) \in J^+, \quad f_{ij}(P) \leq 0 \text{ for } (i, j) \in J^-, \tag{3.4}$$

where $P = (t, x^{(m)}, p, w) \in X_h$,

2) $z \in \mathcal{F}_C(Q_{0.h} \cup Q_h, \mathbb{R})$ *and the derivative* $\partial_t z$ *exists on* Q_h,
3) *the initial estimate*

$$\left|z^{(m)}(t)\right| \leq \eta \quad \text{on } Q_{0.h}$$

and boundary inequalities

$$\left|\Lambda^+_{h.i}[z]^{(m)}(t)\right| \leq \omega(t, \eta) \text{ on } \partial_0 E_i^+, \quad \left|\Lambda^-_{h.i}[z]^{(m)}(t)\right| \leq \omega(t, \eta) \text{ on } \partial_0 E_i^-,$$

$i = 1, \ldots, n$ *are satisfied where* $\eta \in \mathbb{R}_+$ *and* $\omega(\cdot, \eta)$ *is the maximal solution of* (3.2),

4) *denoted*

$$\Sigma = \{(t, x^{(m)}) \in Q_h \setminus \partial_0 E : \left|z^{(m)}(t)\right| > \omega(t, \eta)\}$$

we assume that the differential difference inequality (3.1) *is satisfied for* $(t, x^{(m)}) \in \Sigma$.

Under these assumptions we have

$$\left|z^{(m)}(t)\right| \leq \omega(t,\eta) \quad \text{for } (t,x^{(m)}) \in Q_h. \tag{3.5}$$

Proof. Let

$$\bar{w}(t) = \max\{\left|z^{(m)}(\tau)\right| : -b_0 \leq \tau \leq t,\ -M \leq m \leq M\}, \quad t \in [0,a).$$

Then $\bar{w} \in C([0,a),\mathbb{R}_+)$ and estimation (3.5) is equivalent to $\bar{w}(t) \leq \omega(t,\eta)$, $t \in [0,a)$. Let $0 < \tilde{a} < a$ be fixed. For $\varepsilon > 0$ we denote by $\omega(\cdot,\eta,\varepsilon)$ the right-hand maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t,\omega(t),\omega_{\chi(t)}) + \varepsilon, \quad \omega(t) = \eta + \varepsilon \quad \text{for } t \in [-b_0, 0].$$

There is $\tilde{\varepsilon} > 0$ such that for $0 < \varepsilon < \tilde{\varepsilon}$ the function $\omega(\cdot,\eta,\varepsilon)$ is defined on $[0,\tilde{a})$ and $\lim_{\varepsilon\to 0} \omega(t,\eta,\varepsilon) = \omega(t,\eta)$ uniformly on $(0,\tilde{a})$. We prove that

$$\bar{w}(t) < \omega(t,\eta,\varepsilon) \quad \text{for } t \in [0,\tilde{a}). \tag{3.6}$$

It follows that $\bar{w}(t) < \omega(t,\eta,\varepsilon)$ for $t \in [-b_0, 0]$. Write

$$\Sigma_\varepsilon = \{t \in (0,\tilde{a}) : \bar{w}(\tau) < \omega(\tau,\eta,\varepsilon) \text{ for } \tau \in [0,t) \text{ and } \bar{w}(t) = \omega(t,\eta,\varepsilon)\}.$$

We prove that

$$D_-\bar{w}(t) < \sigma(t,\bar{w}(t),\bar{w}_{\chi(t)}) + \varepsilon \quad \text{for } t \in \Sigma_\varepsilon.$$

Suppose that $t \in \Sigma_\varepsilon$. There is $x^{(m)} \in [-b,b]$ such that $\bar{w}(t) = \left|z^{(m)}(t)\right|$. Then $(t,x^{(m)}) \in \Sigma$. Thus two possibilities can happen, either (i) $\bar{w}(t) = z(t,x^{(m)})$ or (ii) $\bar{w}(t) = -z(t,x^{(m)})$. Lets consider the first case. We prove that $(t,x^{(m)}) \notin \partial_0 E$. Suppose that there exists $i \in \{1,\ldots,n\}$ such that $x_i = b_i$. It follows from assumption 3) that

$$\Lambda^+_{h.i}[z]^{(m)}(t) < \omega(t,\eta,\varepsilon)$$

and consequently

$$\gamma^{(m)}(t)\delta_i^- z^{(m)}(t) < 0. \tag{3.7}$$

But $\gamma^{(m)}(t) \geq 0$ and $\delta_i^- z^{(m)}(t) \geq 0$ which contradicts (3.7). Hence we have $x_i \neq b_i$ for all $i \in \{1,\ldots,n\}$. Analogously we prove that $x_i \neq -b_i$. It follows from (3.1) that

$$D_-\bar{w}(t) \leq \partial_t z^{(m)}(t) \leq \sigma(t,\bar{w}(t),\bar{w}_{\chi(t)}) + \sum_{i=1}^n g_i\big(t,x^{(m)},z^{(m)}(t),T_h z_{\psi(t,x^{(m)})}\big)\delta_i z^{(m)}(t) +$$
$$+ \sum_{i,j=1}^n f_{ij}\big(t,x^{(m)},z^{(m)}(t),T_h z_{\psi(t,x^{(m)})}\big)\delta^{(2)}_{ij} z^{(m)}(t).$$

Put $\bar{P} = (t, x^{(m)}, z^{(m)}(t), T_h z_{\psi(t,x^{(m)})})$ and

$$S_0(\bar{P}) = \sum_{i,j\in J} \frac{1}{h_i h_j} \left|f_{ij}(\bar{P})\right| - 2\sum_{i=1}^{n} \frac{1}{h_i^2} f_{ii}(\bar{P}),$$

$$S_i^+(\bar{P}) = \frac{1}{2h_i} g_i(\bar{P}) + \frac{1}{h_i^2} f_{ii}(\bar{P}) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_i h_j} \left|f_{ij}(\bar{P})\right|,$$

$$S_i^-(\bar{P}) = -\frac{1}{2h_i} g_i(\bar{P}) + \frac{1}{h_i^2} f_{ii}(\bar{P}) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_i h_j} \left|f_{ij}(\bar{P})\right|, \quad S_{ij} = \frac{1}{2h_i h_j} \left|f_{ij}(\bar{P})\right|,$$

where $i,\ j = 1,\dots,n$. It follows from (3.4) and from the definitions of δ and $\delta^{(2)}$ that

$$\begin{aligned} D_-\bar{w}(t) \le \sigma(t, \bar{w}(t), \bar{w}_{\chi(t)}) &+ S_0(\bar{P}) z^{(m)}(t) + \\ &+ \sum_{i=1}^{n} z^{(m+e_i)}(t) S_i^+(\bar{P}) + \sum_{i=1}^{n} z^{(m-e_i)}(t) S_i^-(\bar{P}) + \\ &+ \sum_{(i,j)\in J^+} S_{ij}(\bar{P}) \big[z^{(m+e_i+e_j)}(t) + z^{(m-e_i-e_j)}(t)\big] - \\ &- \sum_{(i,j)\in J^-} S_{ij}(\bar{P}) \big[z^{(m+e_i-e_j)}(t) + z^{(m-e_i+e_j)}(t)\big]. \end{aligned} \tag{3.8}$$

It follows easily that

$$S_0(\bar{P}) + \sum_{i=1}^{n} [S_i^+(\bar{P}) + S_i^-(\bar{P})] + \sum_{(i,j)\in J} S_{ij}(\bar{P}) = 0. \tag{3.9}$$

Since $S_i^+(\bar{P}) \ge 0$, $S_i^-(\bar{P}) \ge 0$, $S_{ij}(\bar{P}) \ge 0$, $i,\ j = 1,\dots,n$, relations (3.8) and (3.9) show that

$$D_-\bar{w}(t) \le \sigma(t, \bar{w}(t), \bar{w}_{\chi(t)}) < \sigma(t, \bar{w}(t), \bar{w}_{\chi(t)}) + \varepsilon.$$

Applying Lemma 3.1 we obtain (3.6) on $[0, \tilde{a})$. The other case can be proved similarly. Letting $\varepsilon \to 0$ we obtain (3.5) on $Q_h \cap ((0, \tilde{a}) \times \mathbb{R}^n)$. Since $0 < \tilde{a} < a$ is arbitrary then we obtain (3.5) on Q_h. □

4. METHOD OF LINES FOR INITIAL BOUNDARY VALUE PROBLEMS

Let $\Gamma : \mathcal{F}(B, \mathbb{R}) \to C(I, \mathbb{R}_+)$ be defined by

$$\Gamma[w](t) = \max\{|w(t,x)| : x \in [-2b, 2b]\},\ t \in I.$$

For functions $z,\ u,\ v \in \mathcal{F}_C(Q_{0.h} \cup Q_h, \mathbb{R})$ we put

$$P[z,u,v]^{(m)}(t,\tau) = (t, x^{(m)}, z^{(m)}(t), (T_h z)_{\psi^{(m)}(t)}, \delta v^{(m)}(t) + \\ + \tau\delta(u-v)^{(m)}(t), \delta^{(2)} v^{(m)}(t) + \tau\delta^{(2)}(u-v)^{(m)}(t)),$$

where $0 \le \tau \le 1$ and

$$W[z,u,v]^{(m)}(t) = F(t, x^{(m)}, z^{(m)}(t), T_h z_{\psi^{(m)}(t)}, \delta u^{(m)}(t), \delta^{(2)} u^{(m)}(t)) - \\ - F(t, x^{(m)}, v^{(m)}(t), T_h v_{\psi^{(m)}(t)}, \delta u^{(m)}(t), \delta^{(2)} u^{(m)}(t)),$$

$$\tilde{W}[z,u,v]^{(m)}(t) = F(t, x^{(m)}, z^{(m)}(t), T_h z_{\psi^{(m)}(t)}, \delta u^{(m)}(t), \delta^{(2)} u^{(m)}(t)) - \\ - F(t, x^{(m)}, z^{(m)}(t), T_h z_{\psi^{(m)}(t)}, \delta v^{(m)}(t), \delta^{(2)} v^{(m)}(t)).$$

Assumption $\mathbf{H}_*[\sigma]$. The function $\sigma : [0,a) \times \mathbb{R}_+ \times C(I, \mathbb{R}_+) \to \mathbb{R}_+$ satisfies Assumption $\mathbf{H}[\sigma]$ and the maximal solution of (3.2) with $\eta(t) = 0$ for $t \in [-b_0, 0]$ is $\hat{\omega}(t) = 0$ for $t \in [-b_0, a)$.

Assumption $\mathbf{H}_*[F]$. The function $F : \Xi \to \mathbb{R}$ of the variables (t, x, p, w, q, r), $q = (q_1, \ldots, q_n)$, $r = [r_{ij}]_{i,j=1,\ldots,n}$ satisfies the conditions:

1) there exist the derivatives

$$\partial_q F = (\partial_{q_1} F, \ldots, \partial_{q_n} F), \quad \partial_r F = [\partial_{r_{ij}} F]_{i,j=1,\ldots,n}$$

and the functions $\partial_q F(t,x,p,w,\cdot),\ \partial_r F(t,x,p,w,\cdot)$ are continuous for each $(t,x,p,w) \in Q \times \mathbb{R} \times C(B, \mathbb{R})$,

2) the matrix $\partial_r F$ is symmetric and

$$-\frac{1}{2}\left|\partial_{q_i} F(P)\right| + \frac{1}{h_i}\partial_{r_{ii}} F(P) - \sum_{\substack{j=1 \\ j\ne i}}^{n} \frac{1}{h_j}\left|\partial_{r_{ij}} F(P)\right| \ge 0, \quad i = 1, \ldots, n, \tag{4.1}$$

$$\partial_{r_{ij}} F(P) \ge 0 \text{ for } (i,j) \in J^+, \quad \partial_{r_{ij}} F(P) \le 0 \text{ for } (i,j) \in J^-, \tag{4.2}$$

where $P \in \Xi$,

3) Assumption $\mathbf{H}_*[\sigma]$ is satisfied and the estimate

$$|F(t,x,u,w,q,r) - F(t,x,\tilde{u},\tilde{w},q,r)| \le \sigma(t, |u-\tilde{u}|, \Gamma[w-\tilde{w}])$$

holds on Ξ.

Assumption $\mathbf{H}[T_h]$. The operator $T_h : \mathcal{F}_C(Q_{0.h} \cup Q_h, \mathbb{R}) \to C(Q_0 \cup Q, \mathbb{R})$ satisfies the conditions:

1) for any functions $z,\ \tilde{z} \in \mathcal{F}_C(Q_{0.h} \cup Q_h, \mathbb{R})$ we have

$$\|T_h[z] - T_h[\tilde{z}]\|_t \le \|z - \tilde{z}\|_{h.t},\ 0 \le t < a,$$

2) if $z : Q_0 \cup Q \to \mathbb{R}$ is of class C^2 then there is $\gamma_* : H \to \mathbb{R}_+$ such that

$$\|T_h[z_h] - z\|_t \le \gamma_*(h), \quad 0 \le t < a, \quad \text{and} \quad \lim_{h\to 0} \gamma_*(h) = 0,$$

where z_h is the restriction of z to the set $Q_{0.h} \cup Q_h$.

Remark 4.1. The condition 1) of Assumption $\mathbf{H}[T_h]$ states that T_h satisfies the Lipschitz condition with the constant $L = 1$. It follows from condition 2) that $T_h[z_h]$ is an approximation of z and the error of the approximation is estimated by $\gamma_*(h)$.

Remark 4.2. We have assumed that the functions

$$G_{ij}(t,x,p,w,q,r) = \text{sign}\, \partial_{r_{ij}} F(t,x,p,w,q,r), \quad (i,j) \in J,$$

are constants. Relations (4.2) can be considered as the definitions of J^+ and J^-.

Assumption $\mathbf{H}[z_0]$. The function $z_0 \in \mathcal{F}(Q_{0.h} \cup Q_h, \mathbb{R})$ satisfies the conditions:

1) $z_0^{(m)}(t) = \varphi_h^{(m)}(t)$ on $Q_{0.h}$, and for $1 = 1, \ldots, n$ we have

$$\Lambda^+_{h.i}[z_0]^{(m)}(t) = \phi_h^{(m)}(t) \text{ on } \partial_0 E_i^+, \quad \Lambda^-_{h.i}[z_0]^{(m)}(t) = \phi_h^{(m)}(t) \text{ on } \partial_0 E_i^-,$$

2) there exists the derivative $\partial_t z_0$ on Q_h and

$$\left|\partial_t z_0^{(m)}(t) - F_h[z_0]^{(m)}(t)\right| \le \gamma_0(t) \quad \text{on } Q_h,$$

where $\gamma_0 \in C([0,a), \mathbb{R}_+)$,

3) the maximal solution ω_0 of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t), \omega_{\chi(t)}) + \gamma_0(t), \quad \omega(t) = 0 \text{ for } t \in [-b_0, 0]$$

is defined on $[-b_0, a)$.

Theorem 4.3. *Suppose that Assumptions $\mathbf{H}_*[F]$, $\mathbf{H}[z_0]$ and condition 1) of Assumption $\mathbf{H}[T_h]$ are satisfied. Under these assumptions there exists exactly one solution of problem (2.1)–(2.3). The solution is defined on $Q_{0.h} \cup Q_h$.*

Proof. The proof will we divided into three steps.
Step 1. Let us define the sequence $\{\omega_k\}_{k=0}^{\infty}$, $\omega_k : [-b_0, a) \to \mathbb{R}_+$, $k \ge 0$, in the following way:

(i) the function ω_0 is given by Assumption $\mathbf{H}[z_0]$,
(ii) if ω_k is given then $\omega_{k+1}(t) = 0$ for $t \in [-b_0, 0]$ and

$$\omega_{k+1}(t) = \int_0^t \sigma(\tau, \omega_k(\tau), (\omega_k)_{\chi(\tau)}) d\tau \text{ for } t \in [0, a).$$

We have that

$$\omega_{k+1}(t) \le \omega_k(t) \quad \text{for } t \in [-b_0, a),\ k \ge 0. \tag{4.3}$$

and from the Dini theorem we have $\lim_{k\to\infty} \omega_k(t) = 0$ uniformly on $[0, \tilde{a}]$, for each $\tilde{a} \in (0, a)$. We omit the simple proof of the above properties of $\{\omega_k\}$.

Step 2. Now we define the sequence $\{z_k\}_{k=0}^{\infty}$, $z_k : Q_{0.h} \cup Q_h \to \mathbb{R}$, $k \ge 0$, in the following way:

(i) z_0 is given in Assumption $\mathbf{H}[z_0]$,

(ii) if $z_k : Q_{0.h} \cup Q_h \to \mathbb{R}$ is a known function then z_{k+1} is the solution of the problem

$$\partial_t z^{(m)}(t) = F(t, x^{(m)}, z_k^{(m)}(t), T_h(z_k)_{\psi^{(m)}(t)}, \delta z^{(m)}(t), \delta^{(2)} z^{(m)}(t)),$$

$-M+1 \le m \le M-1$, with initial boundary conditions

$$z^{(m)}(t) = \varphi_h^{(m)}(t) \text{ on } Q_{0.h},$$

$$\Lambda_{h.i}^{+}[z]^{(m)}(t) = \phi_h^{(m)}(t) \text{ on } \partial_0 E_i^{+}, \quad \Lambda_{h.i}^{-}[z]^{(m)}(t) = \phi_h^{(m)}(t) \text{ on } \partial_0 E_i^{-},$$

$i = 1, \ldots, n.$

Step 3. We prove that

$$\left| z_{k+l}^{(m)}(t) - z_k^{(m)}(t) \right| \le \omega_k(t) \quad \text{on } Q_{0.h} \cup Q_h \text{ for } k,\ l \in \mathbb{N}. \tag{4.4}$$

First we prove (4.4) for $k = 0$ and $l \in \mathbb{N}$. It is easy to show that (4.4) is satisfied for $k,\ l = 0$. Now we assume (4.4) for $k = 0$ and some $l \in \mathbb{N}$. We prove that

$$\left| z_{l+1}^{(m)}(t) - z_0^{(m)}(t) \right| \le \omega_0(t) \quad \text{on } Q_{0.h} \cup Q_h.$$

It is easy to show that

$$\begin{aligned} \left| z_{l+1}^{(m)}(t) - z_0^{(m)}(t) \right| &\le \omega_0(t) \quad \text{on} \quad Q_{0.h}, \\ \left| \Lambda_{h.i}^{+}[z_{l+1} - z_0]^{(m)}(t) \right| &\le \omega_0(t) \quad \text{on} \quad \partial_0 E_i^{+}, \\ \left| \Lambda_{h.i}^{-}[z_{l+1} - z_0]^{(m)}(t) \right| &\le \omega_0(t) \quad \text{on} \quad \partial_0 E_i^{-}. \end{aligned}$$

We prove that the function $z_{l+1} - z_0$ satisfies the differential difference inequalities

$$\left| \partial_t (z_{l+1} - z_0)^{(m)}(t) - \tilde{G}[z_{l+1} - z_0]^{(m)}(t) \right| \le \sigma(t, \omega_0(t), (\omega_0)_{\chi(t)}) + \gamma_0(t) \tag{4.5}$$

for $(t, x^{(m)}(t)) \in Q_{0.h} \cup Q_h$, where

$$\begin{aligned} \tilde{G}[z]^{(m)}(t) = \sum_{i=1}^{n} \int_0^1 \partial_{q_i} F(P[z_0, z_{l+1}, z_0]^{(m)}(t, \tau)) \big[\delta_i z^{(m)}(t)\big] d\tau + \\ + \sum_{i,j=1}^{n} \int_0^1 \partial_{r_{ij}} F(P[z_0, z_{l+1}, z_0]^{(m)}(t, \tau)) \big[\delta_{ij}^{(2)} z^{(m)}(t)\big] d\tau. \end{aligned} \tag{4.6}$$

It follows that

$$\partial_t[z_{l+1}-z_0]^{(m)}(t)=W[z_l,z_{l+1},z_0]^{(m)}(t)+\tilde{W}[z_0,z_{l+1},z_0]^{(m)}(t)+F_h[z_0]^{(m)}(t)-\partial_t z_0^{(m)}(t).$$

Applying the Hadamard mean value theorem and Assumption $\mathbf{H}[\sigma]$ and Assumption $\mathbf{H}[T_h]$ we have

$$\left|W[z_l,z_{l+1},z_0]^{(m)}(t)\right| \le \sigma(t,\omega_0(t),(\omega_0)_{\chi(t)}),$$
$$\tilde{W}[z_0,z_{l+1},z_0]^{(m)}(t)=\tilde{G}[z_{l+1}-z_0]^{(m)}(t),$$

where $\tilde{G}[z]$ is given by (4.6). The above relations and Assumption $\mathbf{H}[z]$ imply (4.5). It follows from Theorem 3.2 that

$$\left|z_{l+1}^{(m)}(t)-z_0^{(m)}(t)\right| \le \omega_0(t) \quad \text{for } (t,x^{(m)})\in Q_{0.h}\cup Q_h.$$

Now let us assume (4.4) for certain $k\in\mathbb{N}$ and every $l\in\mathbb{N}$. We prove that

$$\left|z_{k+1+l}^{(m)}(t)-z_{k+1}^{(m)}(t)\right| \le \omega_{k+1}(t)$$

for $(t,x^{(m)})\in Q_{0.h}\cup Q_h$. It is easy to show that

$$\left|z_{k+l+1}^{(m)}(t)-z_{k+1}(t)\right| \le \omega_{k+1}(t) \quad \text{on} \quad Q_{0.h},$$
$$\left|\Lambda_{h.i}^{+}[z_{k+l+1}-z_{k+1}]^{(m)}(t)\right| \le \omega_{k+1}(t) \quad \text{on} \quad \partial_0 E_i^{+},$$
$$\left|\Lambda_{h.i}^{-}[z_{k+l+1}-z_{k+1}]^{(m)}(t)\right| \le \omega_{k+1}(t) \quad \text{on} \quad \partial_0 E_i^{-}.$$

We prove that the function $z_{k+1+l}-z_{k+1}$ satisfies the differential difference inequalities

$$\left|\partial_t(z_{k+1+l}-z_{k+1})^{(m)}(t)-G_*[z_{k+1+l}-z_{k+1}]^{(m)}(t)\right| \le \sigma(t,\omega_k(t),(\omega_k)_{\chi(t)}), \tag{4.7}$$

where

$$\begin{aligned} G_*[z]^{(m)}(t) = {} & \sum_{i=1}^{n}\int_0^1 \partial_{q_i}F(P[z_k,z_{k+1+l},z_{k+1}]^{(m)}(t,\tau))\big[\delta_i z^{(m)}(t)\big]d\tau+ \\ & +\sum_{i,j=1}^{n}\int_0^1 \partial_{r_{ij}}F(P[z_k,z_{k+1+l},z_{k+1}]^{(m)}(t,\tau))\big[\delta_{ij}^{(2)}z^{(m)}(t)\big]d\tau. \end{aligned} \tag{4.8}$$

It follows from (2.1) that

$$\partial_t(z_{k+1+l}-z_{k+1})^{(m)}(t)=W[z_{k+l},z_{k+1+l},z_k]^{(m)}(t)+\tilde{W}[z_k,z_{k+1+l},z_{k+1}]^{(m)}(t).$$

Applying Hadamard mean value theorem and Assumptions $\mathbf{H}[\sigma]$ and $\mathbf{H}[z_0]$ we have

$$\left|W[z_{k+l},z_{k+1+l},z_k]^{(m)}(t)\right| \le \sigma(t,\omega_k(t),(\omega_k)_{\chi(t)}),$$
$$\tilde{W}[z_k,z_{k+1+l},z_{k+1}]^{(m)}(t)=G_*[z_{k+1+l}-z_{k+1}]^{(m)}(t),$$

where $G_*[z]$ is given by (4.8). The above relations imply (4.7). It follows from Theorem 3.2 and by induction that (4.4) is satisfied. Hence we have that the sequence $\{z_k\}_{k=0}^{\infty}$ is a Cauchy sequence. By the definition of the sequence $\{z_k\}_{k=0}^{\infty}$ it follows that

$$z_{k+1}^{(m)}(t) = \varphi_h^{(m)}(0) + \int_0^t F\big(\tau, x^{(m)}, z_k^{(m)}(\tau), T_h(z_k)_{\psi^{(m)}(\tau)}, \delta z_{k+1}^{(m)}(\tau), \delta^{(2)} z_{k+1}^{(m)}(\tau)\big) d\tau$$

for $(t, x^{(m)}) \in Q_h$. From this and (4.4) we conclude that there exists a solution for (2.1)–(2.3) and it is defined on $Q_{0.h} \cup Q_h$.

If z_h and $\tilde{z}_h$ satisfy (2.1)–(2.3) then the function $z_h - \tilde{z}_h$ satisfies the initial boundary condition

$$(z_h - \tilde{z}_h)^{(m)}(t) = 0 \quad \text{on } Q_{0.h},$$

$$\Lambda_{h.i}^{+}[z_h - \tilde{z}_h]^{(m)}(t) = 0 \text{ on } \partial_0 E_i^{+}, \quad \Lambda_{h.i}^{-}[z_h - \tilde{z}_h]^{(m)}(t) = 0 \text{ on } \partial_0 E_i^{-}$$

for $= 1, \ldots, n$ and differential difference inequalities

$$\Big|\partial_t (z_h - \tilde{z}_h)^{(m)}(t) - \hat{G}[z_h - \tilde{z}_h]^{(m)}(t)\Big| \le \sigma\big(t, \Big|(z_h - \tilde{z}_h)^{(m)}(t)\Big|, U_h(z_h - \tilde{z}_h)_{\psi^{(m)}(t)}\big),$$

where

$$\hat{G}[z]^{(m)}(t) = \sum_{i=1}^{n} \int_0^1 \partial_{q_i} F(P[\tilde{z}_h, z_h, \tilde{z}_h]^{(m)}(t,\tau)) \big[\delta_i z^{(m)}(t)\big] d\tau +$$
$$+ \sum_{i,j=1}^{n} \int_0^1 \partial_{r_{ij}} F(P[\tilde{z}_h, z_h, \tilde{z}_h]^{(m)}(t,\tau)) \big[\delta_{ij}^{(2)} z^{(m)}(t)\big] d\tau.$$

It follows from Theorem 3.2 that $z_h = \tilde{z}_h$. This completes the proof. □

Theorem 4.4. *Suppose that Assumptions* $\mathbf{H}[\beta, \gamma]$, $\mathbf{H}_*[F]$ *and* $\mathbf{H}[T_h]$ *are satisfied and:*

1) $v : Q_0 \cup Q \to \mathbb{R}$ *is a solution of* (1.1)–(1.4) *and* v *is of class* C_* *and* v_h *is a restriction of* v *to the set* $Q_{0.h} \cup Q_h$,
2) $u_h : Q_{0.h} \cup Q_h \to \mathbb{R}$ *is a solution of problem* (2.1)–(2.3), *and there is* $\tilde{c} > 0$ *such that* $h_i \le \tilde{c} h_j$ *for* $i, j = 1, \ldots, n$,
3) *there is* $\gamma_* : H \to \mathbb{R}_+$ *such that*

$$\Big|\varphi^{(m)}(t) - \varphi_h^{(m)}(t)\Big| \le \gamma_*(h) \quad \text{on } Q_{0.h},$$
$$\Big|\phi^{(m)}(t) - \phi_h^{(m)}(t)\Big| \le \gamma_*(h) \quad \text{on } \partial_0 E,$$

and $\lim_{h \to 0} \gamma_*(h) = 0$.

Under these assumptions for each $\tilde{a}\in(0,a)$ *there exists* $\varepsilon>0$ *and* $\omega(\cdot,h):[-b_0,\tilde{a}]\to\mathbb{R}_+$ *such that for* $h\in H$, $\|h\|<\varepsilon$ *we have*

$$\left|(u_h - v_h)^{(m)}(t)\right| \le \omega(t,h) \quad \text{on } Q_h\cap([0,\tilde{a}]\times\mathbb{R}^n)$$

and $\lim_{h\to 0}\omega(t,h)=0$ *uniformly on* $t\in[0,\tilde{a}]$.

Proof. Let $\Gamma_h : Q_h\to\mathbb{R}$ be defined by the relation

$$\partial_t v_h^{(m)}(t) = \mathbb{F}_h[v_h]^{(m)}(t) + \Gamma_h^{(m)}(t) \quad \text{on } Q_h.$$

It follows that there is $\tilde{\gamma}: H\to\mathbb{R}_+$ such that

$$\left|\Gamma_h^{(m)}(t)\right| \le \tilde{\gamma}(h) \quad \text{on } Q_h \quad \text{and } \lim_{h\to 0}\tilde{\gamma}(h)=0.$$

An easy computation shows that $v_h - u_h$ satisfies the differential difference inequalities

$$\begin{aligned}&\left|\partial_t(v_h-u_h)^{(m)}(t) - \mathbb{G}[v_h-u_h]^{(m)}(t)\right| \le \\ &\le \sigma\Big(t, \left|(v_h-u_h)^{(m)}(t)\right|, U_h(v_h-u_h)_{\psi^{(m)}(t)}\Big) + \tilde{\gamma}(h) \quad \text{on } Q_h,\end{aligned}$$

where

$$\begin{aligned}\mathbb{G}[z]^{(m)}(t) = &\sum_{i=1}^n \int_0^1 \partial_{q_i}F(P[v_h,u_h,v_h]^{(m)}(t,\tau))\big[\delta_i z^{(m)}(t)\big]d\tau + \\ &+\sum_{i,j=1}^n \int_0^1 \partial_{r_{ij}}F(P[v_h,u_h,v_h]^{(m)}(t,\tau))\big[\delta_{ij}^{(2)} z^{(m)}(t)\big]d\tau.\end{aligned}$$

It is clear that there is $\gamma: H\to\mathbb{R}_+$ such that

$$\left|(v_h-u_h)^{(m)}(t)\right| \le \gamma(h) \quad \text{on} \quad Q_{0.h}$$

and

$$\left|\Lambda_{h.i}^{+}[v_h-u_h]^{(m)}(t)\right| \le \gamma(h) \text{ on } \partial_0 E_i^+, \quad \left|\Lambda_{h.i}^{-}[v_h-u_h]^{(m)}(t)\right| \le \gamma(h) \text{ on } \partial_0 E_i^-,$$

where $i=1,\dots,n$. Let us consider the Cauchy problem

$$\omega'(t) = \sigma(t,\omega(t),\omega_{\chi(t)}) + \tilde{\gamma}(h), \quad \omega(t)=\gamma(h) \text{ on } [-b_0,0]. \tag{4.9}$$

Suppose that $\tilde{a}\in(0,a)$ is fixed. There is $\varepsilon>0$ such that the maximal solution $\omega(\cdot,h)$ of (4.9) is defined on $[-b_0,\tilde{a}]$ for $\|h\|<\varepsilon$ and $\lim_{h\to 0}\omega(t,h)=0$ uniformly on $t\in[0,\tilde{a}]$. It follows from Theorem 3.2 that

$$\left|u_h^{(m)}(t) - v_h^{(m)}(t)\right| \le \omega(t,h) \quad \text{on } Q_h\cap([0,\tilde{a}]\times\mathbb{R}^n).$$

This is the derived conclusion. □

Remark 4.5. Suppose that all the assumptions of Theorem 4.4 are satisfied and $\sigma : [0,a) \times \mathbb{R}_+ \times C(I,\mathbb{R}_+) \to \mathbb{R}_+$ is given by

$$\sigma(t,p,\eta) = L(p + \|\eta\|_I),$$

where $\|\cdot\|_I$ is the maximum norm in $C(I,\mathbb{R}_+)$. Then

$$\left|u_h^{(m)}(t) - v_h^{(m)}(t)\right| \le \tilde{\alpha}(h) \quad \text{on } Q_h$$

where

$$\tilde{\alpha}(h) = \gamma(h)e^{2La} + \frac{\tilde{\gamma}(h)}{2L}(e^{2La} - 1) \qquad \text{if} \quad L > 0, \tag{4.10}$$

$$\tilde{\alpha}(h) = \gamma(h) + a\tilde{\gamma}(h) \qquad \text{if} \quad L = 0. \tag{4.11}$$

Now we give a result on the error estimate for the numerical method of lines. Let us consider the interpolating operator $T_h : \mathcal{F}_C(Q_{0.h} \cup Q_h, \mathbb{R}) \to C(Q_0 \cup Q, \mathbb{R})$ defined in the following way. Suppose that $w \in \mathcal{F}_C(Q_{0.h} \cup Q_h, \mathbb{R})$. For each $(t,x) \in Q_0 \cup Q$ there exists $m \in \mathbb{Z}^n$ such that $x^{(m)} \le x \le x^{(m+1)}$ where $m+1 = (m_1+1, \dots, m_n+1)$ and $(t, x^{(m)}), (t, x^{(m+1)}) \in Q_{0.h} \cup Q_h$. Write

$$T_h[w](t,x) = \sum_{s \in S_+} w^{(m+s)}(t) \Big(\frac{x - x^{(m)}}{h}\Big)^s \Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-s}, \tag{4.12}$$

where

$$S_+ = \{s = (s_1, \dots, s_n) : s_i \in \{0,1\},\ 1 \le i \le n\},$$

$$\Big(\frac{x - x^{(m)}}{h}\Big)^s = \prod_{i=1}^n \Big(\frac{x_i - x_i^{(m_i)}}{h}\Big)^{s_i},$$

$$\Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-s} = \prod_{i=1}^n \Big(1 - \frac{x_i - x_i^{(m_i)}}{h}\Big)^{1-s_i}$$

and we put $0^0 = 1$ in the above definitions. It is easy to see that $T_h[w] \in C(Q_0 \cup Q, \mathbb{R})$. We consider problem (2.1)–(2.3) with T_h defined by (4.12).

Lemma 4.6. *Suppose that Assumption* $\mathbf{H}[\beta,\gamma]$ *holds and:*

1) *the functions* $F : \Xi \to \mathbb{R}$ *satisfies Assumption* $\mathbf{H}_*[F]$ *with* $\sigma(t,p,\eta) = L(p + \|\eta\|_I)$, *where* $L \in \mathbb{R}_+$, *and there is* $\tilde{L} \in \mathbb{R}_+$ *such that*

$$\|\partial_q F(\tilde{P})\| \le \tilde{L}, \quad \|\partial_r F(\tilde{P})\|_\infty \le \tilde{L}, \quad \text{on } \Xi,$$

2) $v : Q_0 \cup Q \to \mathbb{R}$ *is a solution of* (1.1)–(1.4) *and* v *is of class* C_* *and for each* $t \in [0,a)$ *the function* $v(t,\cdot) : [-b,b] \to \mathbb{R}$ *is of class* C^3,
3) *there is* $\tilde{C} > 0$ *such that* $\|\tilde{v}_i(t,x)\|_\infty \le \tilde{C}$ *on* Q, $i = 1, \dots, n$, *where*

$$\tilde{v}_i(t,x) = \partial_{x_i} V(t,x), \quad V(t,x) = \partial_{xx} v(t,x),\ i = 1, \dots, n, \tag{4.13}$$

4) *conditions* 2), 3) *of Theorem 4.4 are satisfied.*

Then there are $C_1, C_2 \in \mathbb{R}_+$ *such that estimates* (4.10), (4.11) *are satisfied with*

$$\left|(u_h - v_h)^{(m)}(t)\right| \leq C_1\|h\| + C_2\|h\|^2 \quad \text{on } Q_h. \tag{4.14}$$

Proof. We apply (4.10), (4.11) to prove (4.14). It follows that there is $C_0 \in \mathbb{R}_+$ such that

$$\left|\Lambda^+_{h.i}[v_h - u_h]^{(m)}(t)\right| \leq \gamma_*(h) + C_0\|h\| \text{ on } \partial_0 E_i^+,$$

$$\left|\Lambda^-_{h.i}[v_h - u_h]^{(m)}(t)\right| \leq \gamma_*(h) + C_0\|h\| \text{ on } \partial_0 E_i^-,$$

where $i = 1, \ldots, n$. Write

$$\Gamma_h^{(m)}(t) = F(t, x^{(m)}, v^{(m)}, v_{\psi^{(m)}(t)}, \partial_x v^{(m)}(t), \partial_{xx} v^{(m)}(t)) - \\ - F(t, x^{(m)}, v_h^{(m)}, (T_h v_h)_{\psi^{(m)}(t)}, \delta v_h^{(m)}(t), \delta^{(2)} v_h^{(m)}(t)).$$

There is $\tilde{C} \in \mathbb{R}_+$ such that

$$\|\partial_x v^{(m)}(t) - \delta v_h^{(m)}(t)\| \leq \tilde{C}\|h\|^2, \quad \|\partial_{xx} v^{(m)}(t) - \delta^{(2)} v_h^{(m)}(t)\|_\infty \leq \tilde{C}\|h\|$$

on Q_h. It follows from Theorem 5.27 in [4] that there is $\bar{C} \in \mathbb{R}_+$ such that

$$\|v_{\psi^{(m)}(t)} - (T_h v_h)_{\psi^{(m)}(t)}\|_{D[\psi^{(m)}(t)]} \leq \bar{C}\|h\|^2, \text{ on } Q_h.$$

The above relations imply

$$\left|\Gamma_h^{(m)}(t)\right| \leq \tilde{L}\tilde{C}\|h\| + (L\bar{C} + \tilde{L}\tilde{C})\|h\|^2 \text{ on } Q_h.$$

Then we obtain (4.14) from (4.10), (4.11). □

Remark 4.7. Let us consider problem (2.1)–(2.3) with $\gamma(t,x) = 0$ on $\partial_0 E$ and T_h given by (4.12). Then we have parabolic functional differential equations with initial boundary conditions of the Dirichlet type. Suppose that all the assumptions of Lemma 4.6 are satisfied and assume additionally that:

(i) $\varphi_h(t,x) = \varphi(t,x)$ on Q_h and $\Phi_h(t,x) = \Phi(t,x)$ on $\partial_0 E$,
(ii) for each $t \in [0,a)$ the function $v(t,\cdot) : [-b,b] \to \mathbb{R}$ is of class C^4 and the functions $\partial_{x_j}\tilde{v}_i,\ i,j = 1,\ldots,n$ where $\tilde{v}_i$ are given by (4.13), are bounded on Q.

Then there is $C^* \in \mathbb{R}_+$ such that

$$\left|u_h^{(m)} - v_h^{(m)}\right| \leq C^*\|h\|^2 \text{ on } Q_h. \tag{4.15}$$

Note that we have in this case the relations:

$$\left|\Lambda^+_{h.i}[v_h - u_h]^{(m)}(t)\right| = 0 \text{ on } \partial_0 E_i^+, \quad \left|\Lambda^-_{h.i}[v_h - u_h]^{(m)}(t)\right| = 0 \text{ on } \partial_0 E_i^-,$$

where $i = 1, \ldots, n$ and there is $\tilde{C} \in \mathbb{R}_+$ such that

$$\|\partial_{xx} v^{(m)}(t) - \delta^{(2)} v_h^{(m)}(t)\|_\infty \leq \tilde{C}\|h\|^2 \text{ on } Q_h.$$

Then we obtain (4.15) from (4.10), (4.11).

5. NUMERICAL EXAMPLES

Suppose that we apply a difference method for (2.1)–(2.3). The superposition of the numerical method of lines and difference method for ordinary differential functional equations leads to a difference scheme for the original problem. It is not our aim to show results on such difference schemes. We give examples and a short comment only.

Suppose that we apply the explicit Euler method to solve numerically (2.1)–(2.3). Then we get an explicit difference scheme for (1.1)–(1.4) which is not convergent. Corresponding examples are not published because the observation is very natural. Suppose additionally that $\gamma(t,x) = 0$ on $\partial_0 E$. Then we have a parabolic differential functional equation with an initial boundary condition of the Dirichlet type. In this special case there are explicit difference schemes of the Euler type which are convergent (see [8, 12]).

Note that we consider general initial boundary conditions and that the derivatives $\partial_{x_i} z$, $1 \le i \le n$, appear in (1.3)–(1.4).

We show by examples that there are difference schemes for (1.1)–(1.4) which are convergent. We apply the implicit Euler method for (2.1)–(2.3) and we get an implicit difference scheme for the original problem. As far as we are aware theorems on the convergence of such difference schemes are not known. We do not formulate hypothesis on error estimates for implicit difference schemes.

Example 5.1. Write $Q = [0,1) \times [-1,1] \times [-1,1]$. Consider the differential equation with deviated variables

$$\begin{aligned} \partial_t z(t,x,y) = \partial_{xx} z(t,x,y) + \partial_{xy} z(t,x,y) + \partial_{yy} z(t,x,y) + \\ + z\Big(t, \frac{x+y}{2}, \frac{x-y}{2}\Big) + f(t,x,y) z(t,x,y) \end{aligned} \tag{5.1}$$

and the initial boundary conditions

$$z(0,x,y) = 1, \quad (x,y) \in [-1,1] \times [-1,1], \tag{5.2}$$

$$\partial_x z(t,-1,y) = -2te^{t(1-y^2)}, \quad \partial_x z(t,1,y) = 2te^{t(1-y^2)}, \quad t \in [0,1],\ y \in [-1,1], \tag{5.3}$$

$$\partial_y z(t,x,-1) = 2te^{t(x^2-1)}, \quad \partial_y z(t,x,1) = -2te^{t(x^2-1)}, \quad t \in [0,1],\ x \in [-1,1], \tag{5.4}$$

where

$$f(t,x,y) = x^2 - y^2 - 4t^2(x^2 + y^2 - xy) - e^{xy - x^2 + y^2}.$$

The solution of (5.1)–(5.4) is known, it is

$$v(t,x,y) = e^{t(x^2 - y^2)}.$$

We have transformed the above problem into a system of ordinary differential functional equations. The system such obtained is solved numerically by using the implicit Euler method. We use the interpolating operator T_h defined in [4].

Let us denote by $\varepsilon_h^{(r)}$ the arithmetical mean of the errors with fixed $t = t^{(r)}$. In Table 1 we give experimental values for $\varepsilon_h^{(r)}$ and $h_0 = h_1 = h_2 = \frac{1}{300}$ where (h_0, h_1, h_2) are step-sizes with respect to (t, x, y) respectively.

Table 1. Errors (ε_h)

t	0.10	0.20	0.30	0.40	0.50	0.60	0.70
ε_h	0.001 974	0.002 038	0.002 303	0.002 767	0.003 417	0.00 423	0.005 174

Example 5.2. Write

$$Q = [0, 1) \times [0, 1] \times [0, 1].$$

Consider the differential equation with deviated variables

$$\begin{aligned}\partial_t z(t,x,y) = \partial_{xx} z(t,x,y) - \frac{1}{10}\partial_{xy} z(t,x,y) + \partial_{yy} z(t,x,y) + \\ + \partial_x z(t,x,y) + \partial_y z(t,x,y) + \pi^2 \int_0^x z(t,s,y)ds + \\ + \pi^2 \int_0^y z(t,x,s)ds + 2\pi^2 z(t,x,y) + f(t,x,y)\end{aligned} \tag{5.5}$$

and the initial boundary conditions

$$z(0,x,y) = 0, \quad (x,y) \in [0,1] \times [0,1], \tag{5.6}$$

$$\partial_x z(t,0,y) = 0, \quad \partial_x z(t,1,y) = 0, \quad t \in [0,1], \quad y \in [0,1], \tag{5.7}$$

$$\partial_y z(t,x,0) = 0, \quad \partial_y z(t,x,1) = 0, \quad t \in [0,1], \quad x \in [0,1], \tag{5.8}$$

where

$$f(t,x,y) = \cos t \cos \pi x \cos \pi y + \frac{1}{10}\pi^2 \sin t \sin \pi x \sin \pi y.$$

The solution of (5.5)–(5.8) is known, it is

$$v(t,x,y) = \sin t \cos \pi x \cos \pi y.$$

We apply the theory presented in Section 4 to the above problems. A system of ordinary differential equations is solved by using implicit Euler method. We use the interpolating operator T_h given in [4]. In Table 2 we give experimental values for the arithmetical means of the errors $\varepsilon_h^{(r)}$ with fixed $t = t^{(r)}$. We put $h_0 = h_1 = h_2 = \frac{1}{800}$ in our calculations.

Table 2. Errors (ε_h)

t	0.10	0.15	0.20	0.25	0.30	0.35
ε_h	0.001 069	0.001 674	0.002 394	0.003 433	0.005 406	0.008 935

Note that we have somewhat better results for the differential equation with deviated variables than for the differential integral problem. This is due to the fact that in the first example we calculate the function $T_h z(t, \cdot)$ at the points $\frac{x+y}{2}$, $\frac{x-y}{2}$ and we use interpolation on the intervals $[0, x]$ and $[0, y]$ in the second example.

Our calculations were performed on a PC computer.

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