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# HYPONORMAL DIFFERENTIAL OPERATORS WITH DISCRETE SPECTRUM

**Abstract.** In this work, we first describe all the maximal hyponormal extensions of a minimal operator generated by a linear differential-operator expression of the first-order in the Hilbert space of vector-functions in a finite interval. Next, we investigate the discreteness of the spectrum and the asymptotical behavior of the modules of the eigenvalues for these maximal hyponormal extensions.

**Keywords:** hyponormal operators, differential operators, minimal and maximal operators, extension of operators, compact operators, eigenvalues, asymptotes of eigenvalues.

**Mathematics Subject Classification:** 47A20.

## 1. INTRODUCTION

The general and spectral theory of linear bounded hyponormal operators in a Hilbert space was founded and developed by P.R. Halmos [8], C.R. Putnam [17], J.G. Stampfli [18,19], C.R. Williams [20], D. Xia [21].

We know that, all normal extensions and discrete spectrum of the minimal operator generated by a linear differential-operator expression for the first-order in $L^2$ has been described in terms of boundary conditions in [10–12] and [13]. We work with hyponormal operators instead of normal operators.

A densely defined closed operator $N$ in a Hilbert space $\mathcal{H}$ is called a normal operator if $D(N) = D(N^*)$ and for all $x \in D(N)$ $\|Nx\|_{\mathcal{H}} = \|N^*x\|_{\mathcal{H}}$ (cf. [3]).

A densely defined closed operator $T$ in a Hilbert space $\mathcal{H}$ is called hyponormal if $D(T) \subset D(T^*)$ and for all $x \in D(T)$, $\|T^*x\|_{\mathcal{H}} \leq \|Tx\|_{\mathcal{H}}$.

If a hyponormal operator in $\mathcal{H}$ has no non-trivial hyponormal extension, then it is called a maximal hyponormal operator. It is clear that for the hyponormality of a linear closed operator $T$ in a Hilbert space $\mathcal{H}$, it is necessary and sufficient to have $D(T) \subset D(T^*)$ and $TT^* \leq T^*T$.

This paper contains two sections. In the first section we investigate all maximal hyponormal extensions of the minimal operator in $L^2$ in terms of boundary conditions.

In the second section we investigate discreteness of the spectrum and asymptotical behavior of the modules of the eigenvalues for maximal hyponormal extensions of the minimal operator $L_0$ in $L^2$.

Let $\mathcal{H}$ be a separable Hilbert space and let $L^2 = L^2(\mathcal{H}, (a, b))$ be the Hilbert space of vector-functions from the finite interval $(a, b)$ to $\mathcal{H}$ (cf. [7, 9]).

## 1.1. DESCRIPTION OF MAXIMAL HYPONORMAL EXTENSIONS

In the space $L^2$ consider a linear differential-operator expression of first order of the form

$$l(u) = u'(t) + Au(t), \tag{1.1}$$

where $A$ is a linear maximal hyponormal operator, $A = A_R + iA_I$, $A_R$ is the real part of $A$, $A_I$ is the imaginary part of $A$ and $A_R$ is a linear lower positive definite operator in $\mathcal{H}$. For simplicity, we assume that $A_R \geq E$. $E$ denotes the identical operator in $\mathcal{H}$.

The formally adjoint expression (1.1) in the Hilbert space $L^2$ is of the form

$$l(v) = -v'(t) + A^*v(t). \tag{1.2}$$

Let us define the operator $L_0^{'}$ on the dense $L^2$ set of vector-functions $D_0^{'}$,

$$D_0' := \Big\{u(t) \in L^2 : u(t) = \sum_{k=1}^{n} \varphi_k(t) f_k, \ \ \varphi_k \in C_0^\infty(a, b), \quad k = 1, 2, \ldots, n, \ \ n \in \mathbb{N}\Big\},$$

as $L_0^{'}u = l(u)$. Since the operator $A_R \geq E$, then the $L_0^{'}$ operator is accretive, that is $Re(L_0^{'}u, u)_{L^2} \geq 0, \quad u \in D_0^{'}$. Hence the operator $L_0^{'}$ has a closure in $L^2$. The closure of $L_0^{'}$ in $L^2$ is called the minimal operator, generated by the differential-operator expression (1.1) and is denoted by $L_0$.

In a similar way we can construct the minimal operator $L_0^+$ in $L^2$ which is generated by the differential-operator expression (1.2) in $L^2$. The adjoint operator of $L_0^+$ (resp. $L_0$) in $L^2$ is called the maximal operator, generated by (1.1), (resp. (1.2)) and is denoted by $L$ (resp. $L^+$) (cf. [1, 7]).

In this section the main purpose is to describe all maximal hyponormal extensions of the minimal operator in $L^2$ in terms of boundary conditions.

Note that all normal extensions of the minimal operator generated by a linear differential-operator expression for the first-order in $L^2$ has been described in terms of boundary conditions in [12, 15].

**Lemma 1.1.** *$T$ is a hyponormal operator in a Hilbert space $\mathcal{H}$ if and only if the following two condition hold:*

(i) *$D(T) \subset D(T^*)$,*
(ii) *$Im(T_R x, T_I x) \geq 0$,*

*where $T_R = \frac{1}{2}\overline{(T + T^*)}$ and $T_I = \frac{1}{2i}\overline{(T - T^*)}$.*

*Proof.* Let us consider $T$, a hyponormal operator in $\mathcal{H}$. Hence for all $x \in D(T)$

$$\begin{aligned} Tx &= T_R x + iT_I x \quad \in \mathcal{H}, \\ T^*x &= T_R x - iT_I x \quad \in \mathcal{H}, \end{aligned}$$

then $T_R x \in \mathcal{H}$, $T_I x \in \mathcal{H}$. On the other hand, for all $x \in D(T)$,

$$\|T^*x\|_{\mathcal{H}} \leq \|Tx\|_{\mathcal{H}}.$$

From this inequality, we can easily show that

$$Im(T_R x, T_I x) \geq 0.$$

Conversely, if $D(T) \subset D(T^*)$ and $Im(T_R x, T_I x) \geq 0$ for all $x \in D(T)$, then it follows immediately from the obvious relation,

$$4Im(T_R x, T_I x)_{\mathcal{H}} = \|Tx\|^2 - \|T^*x\|^2 \text{ for all } x \in D(T).$$

This completes the proof of the theorem. □

**Theorem 1.2.** *If the minimal operator $L_0$ has at least one hyponormal extension in $L^2$, then the minimal operator $L_0$ is hyponormal in $L^2$.*

*Proof.* Let $L_h$ be a hyponormal extension of, that is, $L_0 \subset L_h \subset L$, then from the condition $D(L_h) \subset D(L_h^*)$ and following relation

$$D(L_0) \subset D(L_h) \subset D(L_h^*) \subset D(L^+) = D(L_0^*)$$

we obtain $D(L_0) \subset D(L_0^*)$.

On the other hand, from the inequality $\|L_h^* x\| \leq \|L_h x\|, x \in D(L_h)$ for any $u(t) \in D(L_0)$ we have

$$\|L_h^* u\|_{L^2} = \|L_0^* u\|_{L^2} \leq \|L_h u\|_{L^2} = \|L_0 u\|_{L^2},$$

that is, $\|L_0^* u\|_{L^2} \leq \|L_0 u\|_{L^2}, u(t) \in D(L_0)$. □

**Theorem 1.3.** *Let $A$ be a linear closed densely defined operator in $\mathcal{H}$. If the minimal operator $L_0$ generated by the differential-operator expression $l(u) = u'(t) + Au(t)$ in $L^2$ is a hyponormal operator, then the operator $A$ is hyponormal in $\mathcal{H}$.*

*Proof.* Existence of the minimal operator $L_0$ in $L^2$ follows from the result in [7]. On the other hand, since $D(L_0) \subset D(L_0^*) = D(L^+)$ and for vector-functions

$$u(t) = \varphi(t)f, \quad \varphi(t) \in \overset{0}{W}{}_2^1, \quad f \in D(A),$$

$$\overset{0}{W}{}_2^1(\mathcal{H}, (a,b)) := \{u(t) : u(t) \in L^2, u'(a) = u'(b) = 0\},$$

belonging to $D(L_0)$ and $u(t) \in D(L^+)$, we have

$$L_0^+ u = -\varphi'(t)f + \varphi(t)A^* f \in L^2(\mathcal{H}, (a,b)).$$

It follows that $f \in D(A^*)$. Hence

$$D(A) \subset D(A^*). \tag{1.3}$$

From the second condition of hyponormality of the operator $L_0$, we have

$$\|L_0^* u\|_{L^2} \leq \|L_0 u\|_{L^2}, u(t) \in D(L_0).$$

For the special case of vector-valued functions

$$u(t) = \varphi(t)f, \varphi(t) \in \overset{0}{W}{}_2^1(a,b);$$

$f \in D(A)$ in $D(L_0)$ from the last inequality, we have

$$\|A^* f\|_{\mathcal{H}}^2 \int\limits_a^b |\varphi(t)|^2 \, dt \leq 2(f, A_R f)[\int\limits_a^b [\varphi'(t)\overline{\varphi(t)} + \varphi(t)\overline{\varphi'(t)}]]dt +$$

$$+ \|Af\|_{\mathcal{H}}^2 \int\limits_a^b |\varphi(t)|^2 \, dt$$

and

$$\|A^* f\|_{\mathcal{H}}^2 \int\limits_a^b |\varphi(t)|^2 \, dt \leq 2(f, A_R f) \int\limits_a^b (\varphi(t)\overline{\varphi(t)})' dt + \|Af\|_{\mathcal{H}}^2 \int\limits_a^b |\varphi(t)|^2 \, dt \leq$$

$$\leq 2(f, A_R f)(\varphi(t)\overline{\varphi(t)}) \,|_a^b + \|Af\|_{\mathcal{H}}^2 \int\limits_a^b |\varphi(t)|^2 \, dt \leq$$

$$\leq 2(f, A_R f)[|\varphi|^2 (b) - |\varphi|^2 (a)] + \|Af\|_{\mathcal{H}}^2 \int\limits_a^b |\varphi(t)|^2 \, dt$$

for $\varphi(t) \in \overset{0}{W}{}_2^1(a,b)$. Then

$$\int\limits_a^b |\varphi(t)|^2 dt \|A^* f\|_{\mathcal{H}}^2 \leq \int\limits_a^b |\varphi(t)|^2 dt \|Af(t)\|_{\mathcal{H}}^2.$$

Choosing function $\varphi(t) \in \overset{0}{W}{}_2^1(a,b)$ with property $\int_a^b |\varphi(t)|^2 dt \neq 0$, from the last relation we obtain,

$$\|A^* f\|_{\mathcal{H}} \leq \|Af\|_{\mathcal{H}}, \quad f \in D(A). \tag{1.4}$$

Hence from (1.3) and (1.4), it is established that operator $A$ is hyponormal in $\mathcal{H}$. □

**Corollary 1.4.** *If the minimal operator $L_0$ generated by the differential-operator expression $l(u) = u'(t) + Au(t)$ with a linear closed densely defined operator in $\mathcal{H}$ is a normal operator in $L^2$, then the operator $A$ is normal in $\mathcal{H}$ (see also* [10, 11, 16]*).*

Note that furthermore we will take the operator $A$ as a normal operator in $\mathcal{H}$.

In a similar manner, we can construct the minimal $M_0$ and the maximal operator $M$ corresponding to the differential-operator expression

$$m(u) = u'(t) + A_R u(t),$$

in the Hilbert space $L^2$ of vector functions.

Let us introduce the following operator.

$$\begin{cases} U_t^{'}(t,s)\,f + iA_i U(t,s)\,f = 0, \\ \qquad\qquad\qquad U(s,s)\,f = f, \quad f \in D(A), \quad t,s \in [a,b]. \end{cases}$$

The operator $U(t,s)$, $t,s \in [a,b]$, is linear continuous bounded invertible unitary operator in $\mathcal{H}$ and $U^*(t,s) = U(s,t)$, $U^{-1}(t,s) = U(s,t)$ (for detailed analysis of these operators see [2] and [14]).

$$Uz(t) := U(t,a)z(t), \quad U : L^2 \to L^2.$$

In this case it is easy to see that, for the differentiable vector-function $z(t) \in L^2$ with $z(t) \in D(A)$, $t \in [a,b]$, the following relation holds:

$$\begin{aligned} l(Uz) &= (Uz)^{'}(t) + A(t)Uz(t) = U(z'(t) + A_R z(t) + (U_t^{'} + iA_I(t)U)z(t) \\ &= Um(z) \in L^2. \end{aligned}$$

From this, then we have

$$U^{-1}lU(z) = m(z).$$

It is clear that, if the operator $\widetilde{L}$ is an extension of the minimal operator $L_0$, that is, $L_0 \subset \widetilde{L} \subset L$, then

$$U^{-1}L_0U = M_0, \;\; M_0 \subset U^{-1}\widetilde{L}U = \widetilde{M} \subset M, \quad U^{-1}LU = M. \tag{1.5}$$

For example we will prove the validity of relation (1.5).

It is known that

$$D(M) = \{u(t) \in L^2 : u(t) \text{ absolutely continuous on } (a,b),\; m(u) \in L^2\}$$

and

$$D(M_0) = \{u(t) \in D(M) : u(a) = u(b) = 0\}.$$

If $u(t) \in D(M)$, then in this case $Uu(t)$ is absolutely continuous on $(a,b)$ and

$$l(Uz) = (Uz)^{'}(t) + A(t)Uz(t) = Um(z) + (U_t^{'} + iA_I(t)U)z(t) = Um(z) \in L^2, \tag{1.6}$$

that is, $Uu(t) \in D(L)$. Furthermore, from the relation (1.6) we infer that $M \subset U^{-1}LU$.

Contrary, if the vector-function $v(t) \in D(L)$, then the element $U^{-1}v(t)$ is absolutely continuous on $(a,b)$ and

$$\begin{aligned} m(U^{-1}v(t)) &= (U^{-1}v(t))^{'} + A_R(U^{-1}v(t)) = \\ &= U^{-1}[v'(t) + A_R v(t) + iA_I v(t)] = U^{-1}l(v(t)) \in L^2, \end{aligned} \tag{1.7}$$

that is, $U^{-1}v(t) \in D(M)$, and from the relation (1.7)

$$U^{-1}L \subset MU^{-1}, \quad U^{-1}LU \subset M.$$

Hence $U^{-1}LU = M$. Therefore, operator $U$ is a one to one map of $D(M)$ onto $D(L)$.

Here we define a Hilbert scale $\mathcal{H}_j(T), -\infty < j < +\infty$ of the spaces constructed via the operator $T^j$. Let $\mathcal{H} = \mathcal{H}_0$ be a Hilbert space over the field of complex numbers with inner product $(\cdot,\cdot)_{H_0}$ and norm

$$\|f\|_{\mathcal{H}_0} = (f,f)^{1/2}_{\mathcal{H}_0}, \quad f \in \mathcal{H}_0.$$

Let $T$ be a linear self-adjoint operator on the Hilbert space $\mathcal{H}$ such that

$$\|Tf\|_{\mathcal{H}_0} \geq \|f\|_{\mathcal{H}_0}.$$

The set $D(T^j),\ 0 < j < +\infty$, under an inner product

$$(f,g)_{\mathcal{H}_{+j}} := (T^j f, T^j g)_{\mathcal{H}_0}, \quad f,g \in D(T^j)$$

is a Hilbert space. We define $\mathcal{H}_{+j} := \mathcal{H}_{+j}(T),\ 0 < j < +\infty$, and it is called a positive space. In a similar way we have $\mathcal{H}_{-j} := \mathcal{H}_{-j}(T),\ 0 < j < +\infty$, and this is called a negative space. It is clear that

$$\mathcal{H}_{+\tau} \subset \mathcal{H}_{+j},\ 0 < \tau < j < \infty, \quad \mathcal{H}_{+j} \subset \mathcal{H} = \mathcal{H}_0 \subset \mathcal{H}_{-j},\ \mathcal{H}^*_{+j} = \mathcal{H}_{-j},\ 0 < j < \infty,$$

and $\mathcal{H}_{+j}, 0 < j < \infty$ is dense in $\mathcal{H}$ (for a more detailed analysis of the spaces $\mathcal{H}_j$, $-\infty < j < +\infty$, see [6] and [7]).

Let $W^1_2(\mathcal{H},(a,b))$ ($\overset{0}{W}{}^1_2(\mathcal{H},(a,b))$) be the Sobolev space of vector-functions from the finite interval $(a,b)$ into $\mathcal{H}$ (see [7]).

**Theorem 1.5.** *If the minimal operator $M_0$ is a hyponormal operator in $L^2$, then*

$$\begin{aligned} D(M_0) &\subset \overset{0}{W}{}^1_2(\mathcal{H},(a,b)), \\ A_R D(M_0) &\subset L^2(\mathcal{H},(a,b)). \end{aligned}$$

*Proof.* Indeed, in this case for any vector-functions $u(t)$ from $D(M_0)$ we have

$$\begin{aligned} u' + A_R u &\in L^2(\mathcal{H},(a,b)), \\ -u' + A_R u &\in L^2(\mathcal{H},(a,b)). \end{aligned}$$

From these relations

$$\begin{aligned} &u'(t) \in L^2(\mathcal{H},(a,b)), \\ &A_R u(t) \in L^2(\mathcal{H},(a,b)), \ \ u(t) \in D(M_0) \quad \text{and} \quad u(a) = u(b) = 0, \end{aligned}$$

we obtain

$$\begin{aligned} u(t) &\in \overset{0}{W}{}^1_2(\mathcal{H},(a,b)), \\ A_R D(M_0) &\subset L^2(\mathcal{H},(a,b)). \end{aligned}$$

□

**Theorem 1.6.** *If the minimal operator $A$ is normal in $\mathcal{H}$ and the following condition holds*

$$A_R \overset{0}{W}{}_2^1(\mathcal{H},(a,b)) \subset L^2(\mathcal{H},(a,b)),$$

*then the minimal operators $M_0$ and $L_0$ are hyponormal in $L^2$.*

*Proof.* First, we will prove hyponormality of $M_0$ in $L^2$. Under these conditions for each $u(t) \in D(M_0) \subset \overset{0}{W}{}_2^1(\mathcal{H},(a,b))$, we have

$$\begin{aligned} M_0^* u &= -u'(t) + A_R u(t) = \\ &= -(u'(t) + A_R u(t)) + 2A_R u(t) \in L^2(\mathcal{H},(a,b)). \end{aligned}$$

That is, $D(M_0) \subset D(M_0^*)$.

On the other hand, for each $u(t) \in D(M_0)$, we have

$$\begin{aligned} \|M_0 u\|_{L^2}^2 - \|M_0^* u\|_{L^2}^2 &= (u'(t) + A_R u, u'(t) + A_R u)_{L^2} - \\ &\quad - (-u'(t) + A_R u, -u'(t) + A_R u)_{L^2} = \\ &= \|u'\|_{L^2}^2 + (u', A_R u)_{L^2} + (A_R u, u')_{L^2} + \|A_R u\|_{L^2}^2 - \|u'\|_{L^2}^2 + \\ &\quad + (u', A_R u)_{L^2} + (A_R u, u')_{L^2} - \|A_R u\|_{L^2}^2 = \\ &= 2[(u', A_R u)_{L^2} + (A_R u, u')_{L^2}] = 2(u, A_R u)_{\mathcal{H}} \mathrm{I}_a^b = 0, \end{aligned}$$

that is, $\|M_0^* u\| \le \|M_0 u\|$ for each $u(t) \in D(M_0)$. Hence, the operator $M_0$ is hyponormal in $L^2$. Now we will prove hyponormality of $L_0$ in $L^2$. From the following properties

$$L_0 = U M_0 U^{-1}, \quad L_0^* = U M_0^* U^{-1}$$

and

$$D(L_0) = D(U M_0 U^{-1}) \subset D(U M_0^* U^{-1}) = D(L_0^*)$$

we have $D(L_0) \subset D(L_0^*)$. Furthermore, for each $u(t) \in D(L_0)$

$$\begin{aligned} \|L_0^* u\|_{L^2}^2 &= \|U M_0^* U^{-1}\|_{L^2}^2 = (U M_0^* U^{-1} u, U M_0^* U^{-1} u)_{L^2} = \\ &= (M_0^* U^{-1} u, U^* U M_0^* U^{-1} u)_{L^2} = \\ &= (M_0^*(U^{-1}u), M_0^*(U^{-1}u))_{L^2} = \|M_0^*(U^{-1}u)\|_{L^2}^2 \le \\ &\le \|M_0^*(U^{-1}u)\|_{L^2}^2 = (M(U^{-1}u), M_0(U^{-1}u))_{L^2} = \\ &= (U^* U M_0 U^{-1} u, M_0 U^{-1} u)_{L^2} = (U M_0 U^{-1} u, U M_0 U^{-1} u)_{L^2} \le \|L_0 u\|_{L^2}^2. \end{aligned}$$

Therefore, operator $L_0$ is hyponormal. □

**Theorem 1.7.** *Let $A_R^{1/2}[D(L) \cap D(L^+)] \subset W_2^1(\mathcal{H},(a,b))$. Each hyponormal extension $L_h$ of the minimal operator $L_0$ in $L^2$ is generated by the differential-operator expression* (1.1) *with the following boundary condition,*

$$u(a) = V U(a,b) u(b), \tag{1.8}$$

*where $V$ is isometric and $A_R^{1/2}VA_R^{-1/2}$ is also a contraction operators in $\mathcal{H}$. The isometric operator $V$ is determined uniquely by the extensions $L_h$, i.e. $L_h = L_V$.*

*Contrary, the restriction of the maximal operator $L$ to the manifold of vector-functions $u(t) \in D(L) \cap D(L^+)$ that satisfies condition (1.8) for some isometric operator $V$, where $A_R^{1/2}VA_R^{-1/2}$ is also contraction operator in $\mathcal{H}$, is a maximal hyponormal extension of the minimal operator $L_0$ in the space $L^2$.*

*Proof.* First, we will describe all maximal hyponormal extensions $M_h$ of the minimal operator $M_0$ in $L^2$ in terms of boundary values.

Let $M_h$ be a maximal hyponormal extension of $M_0$. In this case, for every $u(t) \in D(M_h)$, we have

$$\begin{aligned} M_h u &= \quad u'(t) + A_R u(t) \in L^2, \\ M_h^* u &= -u'(t) + A_R u(t) \in L^2. \end{aligned}$$

From this relation we find that $u'(t) \in L^2$ and $A_R u(t) \in L^2$. In other words, $D(M_h) \subset W_2^1(\mathcal{H},(a,b))$ and $A_R D(M_h) \subset L^2$.

On the other hand, if $u(t) \in D(M_h) \subset D(M_h^*)$, then we have representations,

$$u(t) = e^{-A_R(t-a)}f + \int_a^t e^{-A_R(t-s)}(M_h u)(s)ds,$$

$$u(t) = e^{A_R(t-b)}g + \int_t^b e^{A_R(t-s)}(M_h^* u)(s)ds,$$

where $f, g \in \mathcal{H}_{-1/2}(A_R)$. Hence every $u(t) \in D(M_h)$ has the property $u(t) \in C(\mathcal{H}_{+1/2},[a,b])$ (see [7]). Furthermore, from the relation

$$(M_h u, v)_{L^2} = (u(b), v(b))_{\mathcal{H}} - (u(a), v(a))_{\mathcal{H}} + (u, M_h^* v)_{L^2},$$

which holds for every $u(t) \in D(M_h)$ and $v(t) \subset D(M_h^*)$, we have

$$\|u(b)\|_{\mathcal{H}} = \|u(a)\|_{\mathcal{H}}.$$

Then there exists an isometric operator $V$ in $\mathcal{H}$ such that

$$u(a) = Vu(b). \tag{1.9}$$

On the other hand, for any $u(t) \in D(M_h)$ from the second condition of hyponormality of the extensions $M_h$ we have

$$\begin{aligned} \|M_h^* u\|_{L^2}^2 - \|M_h u\|_{L^2}^2 &= (-u' + A_R u, -u' + A_R u)_{L^2} - (u' + A_R u, u' + A_R u)_{L^2} = \\ &= -2[(u', A_R u)_{L^2} + (A_R u, u')_{L^2}] = -2(u, A_R u)_{\mathcal{H}} \mathrm{I}_a^b = \\ &= -2[(u(b), A_R u(b))_{\mathcal{H}} - (u(a), A_R u(a))_{\mathcal{H}}] = \\ &= 2[\|A_R^{1/2} u(a)\|_{\mathcal{H}}^2 - \|A_R^{1/2} u(b)\|_{\mathcal{H}}^2] \leq 0, \end{aligned}$$

that is,

$$\|A_R^{1/2}u(a)\|_{\mathcal{H}}^2 \le \|A_R^{1/2}u(b)\|_{\mathcal{H}}^2, \qquad u(t) \in D(M_h).$$

Hence there exists a contraction operator $K$ in $\mathcal{H}$ such that

$$A_R^{1/2}u(a) = KA_R^{1/2}u(b), \;\; u(t) \in D(M_h). \tag{1.10}$$

Now we will prove that, if a hyponormal extension $\widetilde{M}$, $M_0 \subset \widetilde{M} \subset M$ is maximal, then

$$\mathcal{H}_b(\widetilde{M}) := \{u(b) \in \mathcal{H} : u(t) \in D(\widetilde{M})\} = \mathcal{H}_{+1/2}(A_R).$$

To prove this, we assume that there exists $f \in \mathcal{H}_{+1/2}(A_R)$ such that for each vector-function $u(t) \in D(M)$, $u(b) \neq f$. Now we will look at the vector-function $u_*(t) = f, \;\; a \le t \le b$.

It is clear that

$$f \in D(M) \cap D(M^+), A_R f \in L^2, \; f \notin \mathcal{H}_b(\widetilde{M})$$

and

$$\|u_*(a)\|_{\mathcal{H}} = \|u_*(b)\|_{\mathcal{H}}, \qquad \|A_R^{1/2}u_*(a)\|_{\mathcal{H}} \le \|A_R^{1/2}u_*(b)\|_{\mathcal{H}}.$$

Now we consider an extension $\widetilde{M_*}$ , $\widetilde{M_*} \subset M$ of the operator $\widetilde{M}$ to the linear manifold

$$D(\widetilde{M_*}) = \operatorname{span}\{D(\widetilde{M}), u_*\}.$$

On the other hand, if we denote by

$$x = \begin{cases} Vx, & x \in D(V) \quad or \quad V_*(\lambda x + f) := \lambda V x + f, \\ f & x = f \qquad\qquad\;\; x \in D(V), \; \lambda \in \mathbb{C}, \end{cases}$$

then $V_* : D(V_*) \to \mathcal{H}$, $V \subset V_*$ and an operator $V_*$ is an isometric operator in $\mathcal{H}$. For the vector-functions $z(t)$ of the manifold $D(\widetilde{M_*})$ holds. That is, there exists a hyponormal extension of the operator $\widetilde{M}$ to $u_*(t)$. This cannot happen since the extension $\widetilde{M}$ is maximal. Furthermore, from the relation (1.9), (1.10) and $\overline{\mathcal{H}_{+1/2}(A_R)} = \mathcal{H}$ we have

$$V = A_R^{-1/2}KA_R^{1/2}, \text{ that is, } K = A_R^{1/2}VA_R^{-1/2}.$$

It is clear that, an isometric operator $V$ is determined uniquely by the extension of $M_h$

Now let $L_h$ be a maximal hyponormal extension of the minimal operator $L_0$ in $L^2$. It is clear that $M_h = U^{-1}L_hU$, $\;M_0 \subset M_h \subset M$, is a maximal hyponormal extension of $M_0$. Then in the first part of the proof $M_h$ is described by the differential-operator expression $m(u)$ and boundary condition (1.9) with some isometric operator $V$ in $\mathcal{H}$ i.e.

$$v(a) = Vv(b), \quad v(t) \in D(M_h), \tag{1.11}$$

where the operator $K = A_R^{\frac{1}{2}}VA_R^{\frac{-1}{2}}$ is also a contraction operator in $\mathcal{H}$. Since

$$v(t) = U(a,t)u(t), \quad v(t) \in D(M_h),$$

then the boundary condition (1.11) will be of the form

$$u(a) = VU(a,b)u(b), \quad u(t) \in D(L_h).$$

Now let $L_V$ be an operator generated by the differential-operator expression $l(u)$ with boundary condition (1.8) in $L^2$, that is,

$$\begin{aligned} &L_V u = l(u), \\ &u(a) = VU(a,b)u(b), \quad u(t) \in D(L_V), \end{aligned}$$

where $V$ and $K = A_R^{\frac{1}{2}} V A_R^{\frac{-1}{2}}$ are isometric and contraction operators in $\mathcal{H}$ respectively.

In this case the adjoint operator $L_V^*$ is generated by the differential-operator expression $l^*(v)$ with the boundary condition

$$v(b) = U(b,a)V^* v(a), \quad v(t) \in D(L_V^*).$$

It is easy to see that $D(L_V) \subset D(L_V^*)$ and the second condition of the hyponormality extension in $L^2$ holds. □

**Proposition 1.8.** *In order for a densely defined closed operator $T$ to be hyponormal in $\mathcal{H}$, the necessary and sufficient condition is the hyponormality of $T + \lambda E$, $\lambda \in \mathbb{C}$, in $\mathcal{H}$.*

*Proof.* It is clear that for any $\lambda \in \mathbb{C}$

$$\begin{aligned} &D(T + \lambda E) = D(T), \\ &D(T^* + \overline{\lambda} E) = D(T^*). \end{aligned}$$

In addition, it can be verified that for $x \in D(T)$

$$((T+\lambda E)(T^*+\overline{\lambda}E)x, x)_{\mathcal{H}} - ((T^*+\overline{\lambda}E)(T+\lambda E)x, x)_{\mathcal{H}} = (TT^*x, x)_{\mathcal{H}} - (T^*Tx, x)_{\mathcal{H}}. \ \square$$

**Remark 1.9.** *If in* (1.1) $A_R \geq 0$*, then writing* (1.1) *in the form*

$$\begin{aligned} l(u) &= u'(t) + Au(t) = u'(t) + (A+E)u(t) - u(t) = \\ &= [u'(t) + (A_R + E)u(t) + iA_I(t)u(t)] - u(t), \end{aligned}$$

*using Proposition 1.8 and Theorem 1.7 we may describe all maximal hyponormal extension of minimal operator $L_0$ in $L^2$ generated by* (1.1) *and boundary condition* (1.8)*, where $V$ is an isometric and*

$$(A_R + E)^{1/2} V (A_R + E)^{-1/2}$$

*is a contraction operators in $\mathcal{H}$.*

## 2. ASYMPTOTICAL BEHAVIOR OF THE MODULES OF THE EIGENVALUES FOR MAXIMAL HYPONORMAL EXTENSIONS

In this section we will investigate discreteness of the spectrum and asymptotical behavior of the modules of the eigenvalues for maximal hyponormal extensions of minimal operator $L_0$ in $L^2$. For the convenience of the reader we give all the proofs which are similar to those used in [13].

First of all it is easy to see that the following result holds.

**Theorem 2.1.** *If $L_h$ is a maximal hyponormal extension of the minimal operator $L_0$ and $M_h = U^{-1}L_hU$ is the maximal hyponormal extension of the minimal operator $M_0$ corresponding to $L_h$, then on the spectrum of these extensions in $L^2$, we have $\sigma(L_h) = \sigma(M_h)$. We denote by $C_p(\mathcal{H}), p \geq 1$, the Schatten – von Neumann class of operators in the Hilbert space $\mathcal{H}$ (see [5]) and $B(\mathcal{H})$ is a class of linear bounded operators in $\mathcal{H}$ (see [4]).*

Now we prove the following theorem about the spectrum of maximal hyponormal extensions.

**Theorem 2.2.** *The spectrum of maximal hyponormal extensions $L_V$ has the form*

$$\sigma(L_V) = \Big\{\lambda \in \mathbb{C} : \lambda = \lambda_0 + \frac{2k\pi i}{b-a}, \text{ where } \lambda_0 \text{ is a set}$$

$$\text{of solutions on } \lambda \text{ for the equation } e^{-\lambda}(b-a) - \mu = 0, \mu \in \sigma(Ve^{-A_R(b-a)}), k \in \mathbb{Z}\Big\}.$$

*Proof.* Since $\sigma(L_V) = \sigma(M_V)$, $M_V = U^{-1}L_VU$, then we investigate the spectrum of maximal hyponormal extension $M_V$ in $L^2$. Now let us consider a problem for the spectrum of maximal hyponormal extension $M_V$,

$$u'(t) + A_Ru(t) = \lambda u(t) + f(t),$$
$$u(a) = Vu(b),$$

where $\lambda \in \mathbb{C}, f(t) \in L^2$, $V$ is an isometric operator and $A_R^{1/2}VA_R^{-1/2}$ is a contraction operator in $\mathcal{H}$. It is clear that a general solution of a differential equation in $L^2$ has the form

$$u_\lambda(t) = e^{-(A_R-\lambda)(t-a)}f + \int_a^t e^{-(A_R-\lambda)(t-s)}f(s)ds, \quad f \in \mathcal{H}_{-1/2}(A_R).$$

In this case from the boundary condition, we get the following relation

$$(Ve^{-A_R(b-a)} - e^{-\lambda(b-a)})f = -V\int_a^b e^{-A_R(b-s)}f(s)ds.$$

From this we see that, $\lambda \in \mathbb{C}$ has a point of spectrum of extension $M_V$ it is necessary and sufficient for the following relation to hold:

$$e^{-\lambda(b-a)} = \mu \in \sigma(Ve^{-A_R(b-a)}).$$

Therefore, $\lambda = \lambda_0 + \frac{2k\pi i}{b-a}$, where $\lambda_0 \in \sigma\left(Ve^{-A_R(b-a)}\right)$ and $k \in \mathbb{Z}$. □

**Theorem 2.3.** *Since $Ve^{-A_R(b-a)} \in B(\mathcal{H})$, then $\sigma(L_V) \neq \emptyset$ and is infinite.*

It is easy to see that following result holds.

**Theorem 2.4.** *If $\dim \mathcal{H} < +\infty$, then each maximal hyponormal extension $L_V$ has a pure point spectrum and the modules of the eigenvalues of extensions $L_V$ have the same asymptotics $|\lambda_n(L_V)| \sim \frac{2\pi n}{b-a}$, as $n \to \infty$.*

**Theorem 2.5.** *If $A_R^{-1} \in C_\infty(\mathcal{H})$ and the operator $L_V$ is any maximal hyponormal extension of the minimal operator $L_0$, then $L_V^{-1} \in C_\infty(L^2)$.*

*Proof.* Let $L_V$ be any maximal hyponormal extension of the operator $L_0$ and $M_V$ be a maximal hyponormal extension of the minimal operator $M_0$ corresponding to $L^2$, that is,

$$M_V = U^{-1} L_V U.$$

It can be verified that, for $f(t) \in L^2$

$$M_V^{-1} f(t) = e^{-A_R(t-a)}(E - Ve^{-A_R(b-a)})^{-1} V \int_a^b e^{-A_R(b-s)} f(s)ds + \\ + \int_a^t e^{-A_R(t-s)} f(s)ds.$$

Now we prove that, if $A_R^{-1} \in C_\infty(\mathcal{H})$, then

$$Kf(t) := \int_a^t e^{-A_R(t-s)} f(s)ds \in C_\infty(L^2).$$

In order to prove this, for $\varepsilon > 0$ we define a new operator $K_\varepsilon : L^2 \to L^2$ of the form

$$K_\varepsilon f(t) := \int_a^{t-\varepsilon} e^{-A_R(t-s)} f(s)ds, \quad f(t) \in L^2, \varepsilon > 0.$$

For each $\varepsilon > 0$, the operator $K_\varepsilon$ can be represented in the form

$$K_\varepsilon f(t) := \int_a^b K_\varepsilon(t,s) f(s)ds,$$

where $f(t) \in L^2$ and for each $(t,s) \in [a,b] \times [a,b]$,

$$K_\varepsilon(t,s) = \begin{cases} e^{-A_R(t-s)}, & \text{if} \quad a \leq s < t - \varepsilon, \\ 0, & \text{if} \quad t - \varepsilon \leq s \leq b. \end{cases}$$

Since for each pair $(t,s) \in [a,b] \times [a,b]$, $a \leq s < t - \varepsilon$, satisfies the following property

$$A_R e^{-A_R(t-s)} \in B(\mathcal{H}), e^{-A_R(t-s)} = \left[A_R e^{-A_R(t-s)}\right] A_R^{-1} \in C_\infty(\mathcal{H}),$$

then $K_\varepsilon \in C_\infty\left(L^2\right), \varepsilon > 0$. On the other hand, the following estimate holds:

$$\begin{aligned}\|(K_\varepsilon - K) f\|_{L^2} &= \left\| \int\limits_{t-\varepsilon}^{t} e^{-A_R(t-s)} f(s) ds \right\|_{L^2} \leq \int\limits_{t-\varepsilon}^{t} \left\| e^{-A_R(t-s)} \right\| \cdot \|f(s)\|_{\mathcal{H}} \, ds \leq \\ &\leq \int\limits_{t-\varepsilon}^{t} \|f(s)\|_{\mathcal{H}} \, ds \leq \left( \int\limits_{t-\varepsilon}^{t} \|f(s)\|_{\mathcal{H}}^2 \, ds \right)^{1/2} \left( \int\limits_{t-\varepsilon}^{t} 1^2 ds \right)^{1/2} \leq \\ &\leq \left( \int\limits_{a}^{b} \|f(s)\|_{\mathcal{H}}^2 \, ds \right)^{1/2} \varepsilon^{1/2} = \varepsilon^{1/2} \|f\|_{L^2}, \quad f(t) \in L^2,\end{aligned}$$

that is,

$$\|K_\varepsilon - K\| \leq \varepsilon^{1/2}.$$

Therefore, $K_\varepsilon \to K$, as $\varepsilon \to 0$.

Hence from the important theorem [5], we have $K \in C_\infty(L^2)$. Thus the representation of $M_V$ implies that $M_V^{-1} \in C_\infty(L^2)$. Hence $L_V^{-1} \in C_\infty(L^2)$. □

**Corollary 2.6.** *Let $L_V$ be any maximal hyponormal extension of the minimal operator $L_0$ and $\lambda \in \rho(L_V)$. Then $R_\lambda(L_V) \in C_\infty(L^2)$.*

This result follows from the relation

$$R_\lambda(L_V) = L_V^{-1} - \lambda R_\lambda(L_V) L_V^{-1}.$$

Using the method in the proof of Theorem 2.5 the following result can be proved.

**Corollary 2.7.** *If $A_R^{-1} \in C_p(\mathcal{H}), p \geq 1$ and $L_V$ is any maximal hyponormal extension of $L_0$, then $L_V^{-1} \in C_p(L^2)$.*

Furthermore, from the representation of resolvent $R_\lambda(L_V), \lambda \in \rho(L_V)$, of the operator $L_V$ we have the following corollary.

**Corollary 2.8.** *Let $L_{V_1}, L_{V_2}$ be two maximal hyponormal extensions of the minimal operator $L_0$ in $L^2$ and $\lambda \in \rho(L_{V_1}) \cap \rho(L_{V_2})$. Then we have*

$$R_\lambda(L_{V_1}) - R_\lambda(L_{V_2}) \in C_p(L^2), \quad 1 \leq p,$$

*if and only if*

$$V_1 - V_2 \in C_p(\mathcal{H}), \quad p \geq 1.$$

*Now we prove a result on the structure of the spectrum of the maximal extension of the minimal operator $L_0$.*

**Theorem 2.9.** *If $A_R^{-1} \in C_\infty(\mathcal{H})$ and $L_V$ is any maximal hyponormal extension of the minimal operator $L_0$ in $L^2$, then the spectrum of $L_V$ has the form*

$$\sigma(L_V) = \Big\{ \lambda_n(A_R) + \frac{i}{a-b} (\arg \lambda_n(V e^{-A_R(b-a)}) + 2k\pi i), n \in \mathbb{N}, k \in \mathbb{Z} \Big\}.$$

*Proof.* Since $\sigma(L_V) = \sigma(M_V) = \sigma_p(M_V)$, then we will investigate the structure of the spectrum of $M_V$. From Theorem 2.2 we obtain

$$\sigma(L_V) = \Big\{\lambda \in \mathbb{C} : \lambda = \frac{1}{a-b}(\ln|\mu| + i\arg\mu + 2k\pi i), \mu \in \sigma(Ve^{-A_R(b-a)}), k \in \mathbb{Z}\Big\}.$$

Since $A_R^{-1} \in C_\infty(\mathcal{H})$, then $Ve^{-A_R(b-a)} = V(A_R e^{-A_R(b-a)})A_R^{-1} \in C_\infty(\mathcal{H})$. For any eigenvector $x_\lambda \in \mathcal{H}$ corresponding to the eigenvalue $\lambda \in \sigma_p(Ve^{-A_R(b-a)})$, we have $Ve^{-A_R(b-a)}x_\lambda = \lambda(Ve^{-A_R(b-a)})x_\lambda$. This implies that

$$\begin{aligned} e^{-A_R(b-a)}V^*Ve^{-A_R(b-a)}x_\lambda &= \lambda(Ve^{-A_R(b-a)})e^{-A_R(b-a)}V^*x = \\ &= \lambda(Ve^{-A_R(b-a)})\overline{\lambda(Ve^{-A_R(b-a)})}x_\lambda, \end{aligned}$$

that is,

$$e^{-2A_R(b-a)}x_\lambda = |\lambda(Ve^{-A_R(b-a)})|^2 x_\lambda.$$

Hence $|\lambda(Ve^{-A_R(b-a)})|^2 = \lambda(e^{-2A_R(b-a)}) = e^{-2\lambda A_R(b-a)}$, that is,

$$|\mu| = |\lambda(Ve^{-A_R(b-a)})| = e^{-\lambda(A_R)(b-a)}.$$

From this relation we have $\ln|\mu| = \lambda(A_R)(a-b)$. Thus

$$\sigma(L_V) = \Big\{\lambda \in \mathbb{C} : \lambda = \lambda_n(A_R) + \frac{i}{a-b}(\arg\lambda_n(Ve^{-A_R(b-a)}) + 2k\pi), n \in \mathbb{N}, k \in \mathbb{Z}\Big\}. \quad \square$$

Now we can prove the main theorem of this section.

**Theorem 2.10.** *If* $A_R^{-1} \in C_\infty(\mathcal{H})$, $\lambda_n(A_R) \sim cn^\alpha$, $0 < c, \alpha < \infty$, as $n \to \infty$, *then* $L_V^{-1} \in C_\infty(L^2)$ *and*

$$|\lambda_n(L_V)| \sim dn^\beta, \ \ 0 < d < \infty, \beta = \frac{\alpha}{1+\alpha}, \ \ \textit{as } n \to \infty.$$

*Proof.* Since $A_R^{-1} \in C_\infty(\mathcal{H})$, then $M_V^{-1}, L_V^{-1} = U^{-1}M_V^{-1}U \in C_\infty(L^2)$ and $\lambda_n(L_V) = \lambda_n(M_V), n \in \mathbb{N}$. It is clear that

$$\begin{aligned} |\lambda_m(L_V)| &= |\lambda_n(A_R) + \frac{i}{a-b}(\arg\lambda_n(Ve^{-A_R(b-a)}) + 2k\pi)| = \\ &= |\lambda_n(A_R) + \frac{i}{a-b}(\delta_n + 2k\pi)| = \\ &= \Big[c^2n^{2\alpha} + \frac{1}{(b-a)^2}(\delta_n + 2k\pi)^2\Big]^{1/2}, \end{aligned}$$

where $m = m(n,k) \in \mathbb{N}, n \in \mathbb{N}, k \in \mathbb{Z}, \delta_n = \arg\lambda_n\left(Ve^{-A_R(b-a)}\right)$. Since $0 \le \delta_n \le 2\pi$ for each $n \in \mathbb{N}$, then from the last equality we have

$$\left[c^2n^{2\alpha} + \frac{4\pi^2}{(b-a)^2}k^2\right]^{1/2} \le |\lambda(L_V)| \le \left[c^2n^{2\alpha} + \frac{4\pi^2}{(b-a)^2}(k+1)^2\right]^{1/2}, n \in \mathbb{N}, k \in \mathbb{Z}.$$

Therefore, $|\lambda(L_V)| \sim \sqrt{c^2n^{2\alpha}+h^2k^2}$, $n\in\mathbb{N}, k\in\mathbb{Z}$, where $h=\frac{4\pi}{b-a}$. On the other hand, we note that $(c^2n^{2\alpha}+h^2k^2)^{1/2}$, $n\in\mathbb{N}, k\in\mathbb{Z}$, are modules of eigenvalues of the periodical boundary condition (for the Dirichlet problem), i.e.

$$|\lambda(L_E)| = (c^2n^{2\alpha}+h^2k^2), \quad n\in\mathbb{N}, k\in\mathbb{Z}.$$

Therefore, asymptotical behavior of the modules of eigenvalues of each maximal hyponormal extension $L_V$ and Dirichlet extension are the same, that is,

$$|\lambda_m(L_V)| \sim |\lambda_m(L_E)|, \text{ as } m\to\infty.$$

Using the method established in [6, 7] (in our case $k\in\mathbb{Z}$). It can be found that

$$|\lambda_m(L_E)| \sim dm^{\frac{\alpha}{1+\alpha}}\ , \ 0<d<\infty,\ \ m\to\infty,$$

which completes the proof. □

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