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ON SOME PROPERTIES OF THE SUPERPOSITION OPERATOR ON TOPOLOGICAL MANIFOLDS

Abstract. In this paper the superposition operator in the space of vector-valued, bounded and continuous functions on a topological manifold is considered. The acting conditions and criteria of continuity and compactness are established. As an application, an existence result for the nonlinear Hammerstein integral equation is obtained.

Keywords: superposition operator, continuous function, topological manifold, Hammerstein integral equation.

Mathematics Subject Classification: 47H30, 45G15.

1. INTRODUCTION

Let S be a given set and $f : S \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $n \geq 1$, be a given function. The operator F defined for functions $x : S \longrightarrow \mathbb{R}^n$ by the formula

$$(Fx)(s) = f(s, x(s))$$

is called the *superposition operator* (or *Nemytskij operator*) generated by the function $f(s, x)$. This operator plays an essential role in numerous mathematical investigations – e.g. in the theory of nonlinear integral equations (see [2, 8]) – and to date has been studied thoroughly. The basic facts and ideas concerning the superposition operator have been collected by J. Appell and P.P. Zabrejko in their monograph [1].

One of the most important problems considered in the theory of the superposition operator is to establish necessary and sufficient conditions guaranteeing that this operator transforms a given function space into itself (so called "acting conditions"), is continuous, compact or possesses other useful properties. Such conditions are known for many function spaces (cf. [1]). Nevertheless, for many other spaces, especially those consisting of functions defined on noncompact sets, conditions of such a type are not known.

The aim of this paper is to give acting conditions and to establish criteria of continuity and compactness for the superposition operator in the space of vector-valued functions, bounded and continuous on a given topological manifold (not necessarily compact). As an application, we give conditions guaranteeing solvability of the nonlinear Hammerstein integral equation in the above mentioned space. Thus we generalize the results obtained in [6], for example.

2. PRELIMINARIES

From now, let S be a normal topological space and let M denote a Hausdorff topological manifold, with or without a boundary, satisfying the second countability axiom.

Consider the finite dimensional space $\mathbb{R}^n, n \geq 1$, with an arbitrarily fixed norm $\|\cdot\|$, compatible with the topology. Let the constants $C_1 > 0$ and $C_2 > 0$ be such that

$$C_1^{-1}\|x\|_{l^\infty} \leq \|x\| \leq C_2\|x\|_{l^1},$$

where

$$\|x\|_{l^\infty} = \max\{|x_1|, \dots, |x_n|\} \quad \text{and} \quad \|x\|_{l^1} = |x_1| + \dots + |x_n|$$

for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

By $BC(S, \mathbb{R}^n)$ (or $BC(M, \mathbb{R}^n)$, respectively), we denote the space of bounded and continuous functions $x : S \to \mathbb{R}^n$ with the standard norm:

$$\|x\|_{BC} = \sup_{s \in S} \|x(s)\|.$$

If $n = 1$, we will shortly write $BC(S)$ instead of $BC(S, \mathbb{R})$, and then

$$\|x\|_{BC} = \sup_{s \in S} |x(s)|.$$

Further, let the symbol B_r, $r > 0$, denote the closed ball, centered at zero and of radius r, in any of the above function spaces.

In the sequel we shall need some criteria of compactness in the space $BC(M, \mathbb{R}^n)$, but first we state the following general result (see [4], Theorem IV.6.5).

Theorem 2.1. *Let S be an arbitrary normal topological space and let K be a bounded subset of the space $BC(S)$. Then K is conditionally compact if, and only if, for every $\varepsilon > 0$ there is a finite collection $\{E_1, \dots, E_m\}$ of sets with union S, and points s_i in E_i, $i = 1, \dots, m$, such that*

$$\sup_{x \in K} \sup_{s \in E_i} |x(s_i) - x(s)| < \varepsilon, \quad i = 1, \dots, m.$$

Since $BC(S, \mathbb{R}^n)$ may be considered as a topological product of n copies of $BC(S)$, and the norm $\|\,.\,\|$ is equivalent to the l^∞ norm in $\mathbb{R}^n$, using the classical Tychonoff theorem [5] we get the following result.

Corollary 2.2. *Let S be a normal topological space and let K be a bounded subset of the space $BC(S, \mathbb{R}^n)$. Then K is conditionally compact if, and only if, for every $\varepsilon > 0$ there is a finite collection $\{E_1, \ldots, E_m\}$ of sets with union S, such that*

$$\sup_{x\in K} \sup_{s,t\in E_i} \|x(s) - x(t)\| < \varepsilon, \quad i = 1, \ldots, m.$$

Using the Dieudonné theorem it can be shown, that each Hausdorff topological manifold satisfying the second countability axiom is a normal topological space (see [7]). Moreover, it is locally compact. Hence, using the Arzelà-Ascoli theorem, we may replace arbitrary sets by the open ones in the above formulation, and obtain the following criterion.

Corollary 2.3. *Let M be a Hausdorff topological manifold satisfying the second countability axiom and let K be a bounded subset of the space $BC(M, \mathbb{R}^n)$. Then K is conditionally compact if, and only if, for every $\varepsilon > 0$ there is a finite collection $\{U_1, \ldots, U_m\}$ of open sets with union M, such that*

$$\sup_{x\in K} \sup_{s,t\in U_i} \|x(s) - x(t)\| < \varepsilon, \quad i = 1, \ldots, m.$$

Now, for the sake of completeness let us recall the Tietze extension theorem (see [4], Theorem I.5.3).

Theorem 2.4. *If g is a bounded real continuous function defined on a closed subset A of a normal topological space S, then there is a continuous real function G defined on S, with $G(s) = g(s)$ for s in A, and*

$$\sup_{s\in S} |G(s)| = \sup_{s\in A} |g(s)|.$$

Finally, the Dieudonné theorem again, we may reformulate the above result for vector-valued functions defined on M.

Corollary 2.5. *Let A be a closed subset of a Hausdorff topological manifold M satisfying the second countability axiom. Then for any function $g \in BC(A, \mathbb{R}^n)$ there is a function $G \in BC(M, \mathbb{R}^n)$ such that $G(s) = g(s)$ for $s \in A$, and*

$$\sup_{s\in M} \|G(s)\| = \sup_{s\in A} \|g(s)\|.$$

3. MAIN RESULTS

As before, let M be a Hausdorff topological manifold satisfying the second countability axiom and let $f : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a given function.

In this section we give the necessary and sufficient conditions guaranteeing the continuity and compactness of the superposition operator in the space $BC(M, \mathbb{R}^n)$. These results generalize the earlier ones, e.g. theorems of M.A. Krasnoselśkii (cf. [1]). Firstly we state the acting conditions.

Theorem 3.1. *The superposition operator F generated by f maps the space $BC(M, \mathbb{R}^n)$ into itself if, and only if, the function f is continuous and bounded on each set of the form $M \times B_r$, where $r > 0$.*

Proof. The sufficient condition is obvious.

To prove the necessary condition, let us first suppose that the function f is not continuous.

If there were $u_0 \in \mathbb{R}^n$ such that the function $M \ni s \longmapsto f(s, u_0) \in \mathbb{R}^n$ was not continuous, then for the constant function $x(s) \equiv u_0$ its image Fx would not be continuous on M.

This proves that for any $u \in \mathbb{R}^n$ the function $M \ni s \longmapsto f(s, u) \in \mathbb{R}^n$ is continuous. Further, let $(s_0, u_0) \in M \times \mathbb{R}^n$ and the sequence $(s_j, u_j) \in M \times \mathbb{R}^n$, $j = 1, 2, \ldots$, be such that

$$\lim_{j\to\infty} s_j = s_0, \quad \lim_{j\to\infty} u_j = u_0 \quad \text{and} \quad \lim_{j\to\infty} f(s_j, u_j) \neq f(s_0, u_0). \tag{3.1}$$

Since $f(s, u)$ is continuous in s for each $u \in \mathbb{R}^n$ and M is locally homeomorphic to the finite dimensional space (or halfspace) , we may assume that $s_i \neq s_j$ for $i \neq j$ here.

Now, since the set $\{s_j : j = 0, 1, 2, \ldots\}$ is closed in M, we may use Corollary 2.5 to construct a function $x \in BC(M, \mathbb{R}^n)$ satisfying

$$x(s_j) = u_j \quad \text{for} \quad j = 0, 1, 2, \ldots. \tag{3.2}$$

For such x, taking into account (3.1) and (3.2), we have

$$\lim_{j\to\infty} (Fx)(s_j) = \lim_{j\to\infty} f(s_j, u_j) \neq f(s_0, u_0) = (Fx)(s_0),$$

so Fx is not continuous.

Thus, if f is not continuous, the superposition operator F generated by f does not act in the space $BC(M, \mathbb{R}^n)$.

Suppose now, that the function f is continuous on M, but is not bounded on the set $M \times B_r$ for some $r > 0$.

Let $\{(s_j, u_j)\}$ be a sequence in $M \times B_r$ such that

$$\lim_{j\to\infty} |f(s_j, u_j)| = \infty. \tag{3.3}$$

Similarly as above, we may assume that $s_i \neq s_j$ for $i \neq j$ here.

Since f is continuous and (3.3) holds, the sequence $\{s_j\}$ has no accumulation points in M. Hence the set of its values is closed in M and we can use Corollary 2.5 once again to get the function $x \in BC(M, \mathbb{R}^n)$, such that $x(s_j) = u_j$ for $j = 1, 2, \ldots$.

For such an x its image Fx is not bounded on M, so the superposition operator F does not act in $BC(M, \mathbb{R}^n)$. □

Now, let us state the result concerning the continuity of the superposition operator in the space $BC(M, \mathbb{R}^n)$.

Theorem 3.2. *If the superposition operator F generated by f maps the space $BC(M, \mathbb{R}^n)$ into itself, then it is continuous if, and only if, for each $r > 0$*

$$\lim_{\varepsilon \to 0} \sup_{s \in M} (\sup\{\|f(s,u) - f(s,w)\| : u, w \in B_r, \|u - w\| \le \varepsilon\}) = 0. \tag{3.4}$$

Proof. Suppose that the above condition is fulfilled. Let $\{x_j\}$, $j = 1, 2, \dots$, be a sequence converging to x_0 in the space $BC(M, \mathbb{R}^n)$. Further, let $r > 0$ be large enough to satisfy $\|x_j\| \le r$ for all j.

To show that the sequence $\{Fx_j\}$ converges to Fx_0 in $BC(M, \mathbb{R}^n)$ let us fix $\delta > 0$ and, by (3.4), choose $\varepsilon > 0$ such that

$$\sup_{s \in M} (\sup\{\|f(s,u) - f(s,w)\| : u, w \in B_r, \|u - w\| \le \varepsilon\}) < \delta. \tag{3.5}$$

Then there exists $j_0 \in \mathbb{N}$ such that for $j \ge j_0$ we have:

$$\|x_j - x_0\|_{BC} = \sup_{s \in M} \|x_j(s) - x_0(s)\| < \varepsilon. \tag{3.6}$$

For any such j, by (3.5) and (3.6), we get:

$$\|Fx_j - Fx_0\|_{BC} = \sup_{s \in M} \|f(s, x_j(s)) - f(s, x_0(s))\| < \delta,$$

which proves the continuity of the operator F.

Now, let us assume that the condition (3.4) is not fulfilled, i.e. there exist $r > 0$, $\delta > 0$ and a sequence $\{\varepsilon_j\}$ such that:

a) all $\varepsilon_j > 0$ and $\lim_{j \to \infty} \varepsilon_j = 0$,

b) $\sup_{s \in M} (\sup\{\|f(s,u) - f(s,w)\| : u, w \in B_r, \|u - w\| \le \varepsilon_j\}) > \delta$ for $j = 1, 2, \dots$

The last condition means, that for each j there exist $s_j \in M$ and $u_j, w_j \in B_r$ such that $\|u_j - w_j\| \le \varepsilon_j$ and $\|f(s_j, u_j) - f(s_j, w_j)\| > \delta$.

We claim that the sequence $\{s_j\}$ has no accumulation points in M.

Suppose contrary. Let $s_0 \in M$ be such a point. By taking a subsequence, we may assume that the sequence $\{s_j\}$ converges to s_0.

The sequence $\{u_j\}$ is contained in the compact set B_r, so choosing a subsequence once again we may assume that it is convergent to some $u_0 \in B_r$. Since $\lim_{j \to \infty} \varepsilon_j = 0$, the sequence $\{w_j\}$ must be convergent to the same limit. Hence, taking into account that f is continuous for every $\eta > 0$ and for sufficiently large j, $\|f(s_j, u_j) - f(s_j, w_j)\| < \eta$ – we have a contradiction.

Since the sequence $\{s_j\}$ has no accumulation points, the set of its values is infinite and closed in M. Without loss of generality we may assume that $s_i \ne s_j$ for $i \ne j$, $i, j = 1, 2, \dots$. Thus, in view of Corollary 2.5 we can construct a function $x_0 \in BC(M, \mathbb{R}^n)$ such that

$$x_0(s_j) = u_j \quad \text{for} \quad j = 1, 2, \dots.$$

Now, let us define the sequence $x_j \in BC(M, \mathbb{R}^n)$, $j = 1, 2, \dots$, by the formula

$$x_j(s) = x_0(s) + (w_j - u_j) \quad \text{for} \quad s \in M.$$

We have

$$\sup_{s\in M}\|x_j(s)-x_0(s)\| = \|w_j-u_j\|,$$

hence

$$\lim_{j\to\infty}\|x_j-x_0\|_{BC}=0.$$

On the other hand

$$\|Fx_j-Fx_0\|_{BC} = \sup_{s\in M}\|f(s,x_j(s))-f(s,x_0(s))\| \geq$$

$$\geq \|f(s_j,x_j(s_j))-f(s_j,x_0(s_j))\| = \|f(s_j,w_j)-f(s_j,u_j)\| > \delta$$

for $j=1,2,\ldots$

This implies that if f does not satisfy condition (3.4), then the superposition operator F generated by f is not continuous. □

At the end of this section we shall state the result concerning compactness of the superposition operator in the space $BC(M,\mathbb{R}^n)$, which is, in fact, a slight generalization of the corresponding result of M.A. Krasnoselśkii for a compact interval. Nevertheless, we give the proof for the sake of completeness.

Theorem 3.3. *If the superposition operator F generated by f maps the space $BC(M,\mathbb{R}^n)$ into itself then it is compact if and only if it is constant, i.e., if the function f does not depend on the second argument.*

Proof. Suppose that F is not constant, i.e. there is $s_0\in M$ and $u,v\in\mathbb{R}^n$ such that $f(s_0,u)\neq f(s_0,v)$. Let $r_0=\max\{\|u\|,\|v\|\}$. Consider any $\varepsilon>0$ satisfying

$$\|f(s_0,u)-f(s_0,v)\|>2\varepsilon,$$

and any finite collection $\{U_1,\ldots,U_m\}$ of open sets with union M. If U_k, $1\leq k\leq m$, contains s_0 then there exists $s_1\in U_k$, $s_1\neq s_0$, such that

$$\|f(s_0,v)-f(s_1,v)\|<\varepsilon.$$

Thus we can construct (using Corollary 2.5, e.g.) a function $x\in BC(M,\mathbb{R}^n)$ such that

$$\|x\|_{BC}\leq r_0,\quad x(s_0)=u\quad\text{and}\quad x(s_1)=v.$$

For such an x we have

$$\|(Fx)(s_0)-(Fx)(s_1)\| = \|f(s_0,u)-f(s_1,v)\|>\varepsilon.$$

Hence, by Corollary 2.3, the set $F(B_{r_0})$ is not compact and the operator F is not compact, too. □

Remark 3.4. Let us mention that the results of this sections do not hold in the space $BC(S,\mathbb{R}^n)$ for any normal topological space S.

As an example, let us consider a space $S_{\mathcal{D}}$ with the discrete topology. This is a normal (even metrizable) space but the superposition operator F, generated by any $f: S_{\mathcal{D}}\longrightarrow\mathbb{R}^n$ bounded on the sets $M\times B_r$, $r>0$, and not necessarily continuous, acts on $BC(S_{\mathcal{D}},\mathbb{R}^n)$.

4. AN APPLICATION

In this section we prove an existence result for the nonlinear Hammerstein integral equation, which generalizes the results of [3] and [6].

Let M be a Hausdorff topological manifold satisfying the second countability axiom and additionally equipped with a positive Borel measure μ.

We want to solve the following functional equation

$$x = \varphi + Hx \tag{4.1}$$

where $x \in BC(M, \mathbb{R}^n)$ is the unknown, $\varphi \in BC(M, \mathbb{R}^n)$ is given, and H is the Hammerstein integral operator

$$(Hx)(t) = \int_M k(t,s) f(s, x(s)) d\mu(s), \quad t \in M.$$

We shall consider the equation (4.1) under the following assumptions:

i) the function $f : \; M \times \mathbb{R}^n \to \mathbb{R}^n$, and the superposition operator F generated by f maps the space $BC(M, \mathbb{R}^n)$ into itself and is continuous (see Theorems 3.1 and 3.2),

ii) the function $k(t,s)$ maps $M \times M$ into the space of $n \times n$ real matrices; each entry $k_{ij}(t,s)$ $(i,j = 1, \ldots, n)$ is continuous in t for almost all s, and summable in s for all t; moreover, the function $t \to \|k_{ij}(t, \cdot)\|_{L^1}$ is bounded on M for each pair (i,j), where

$$\|k_{ij}(t, \cdot)\|_{L^1} = \int_M |k_{ij}(t,s)| d\mu(s).$$

The Hammerstein integral operator may be written as a product

$$H = KF$$

of the superposition operator F generated by f and the linear integral operator

$$(Kx)(t) = \int_M k(t,s) x(s) d\mu(s), \quad t \in M.$$

From the assumption ii) it follows that K maps the space $BC(M, \mathbb{R}^n)$ into itself and is continuous.

Let us denote by $\|K\|$ its norm in the space of continuous linear operators acting on $BC(M, \mathbb{R}^n)$.

Now we can formulate the main result of this section.

Theorem 4.1. *Suppose assumptions* i) *and* ii) *are satisfied. For $r > 0$ let*

$$\beta(r) = \sup\{\|f(t,u)\| : \; t \in M, \; u \in B_r\} \; .$$

If

$$\liminf_{r\to\infty} \frac{\beta(r)}{r} < \|K\|^{-1} \tag{4.2}$$

and the set K_{ij} *consisting of all functions of the form* $s \to k_{ij}(t,s)$, *where* $t \in M$, *is conditionally compact in* $L^1(M)$ *for each pair* (i,j), *then, for each* $\varphi \in BC(M,\mathbb{R}^n)$, *equation* (4.1) *has at least one solution* $x \in BC(M,\mathbb{R}^n)$.

If $\varphi \equiv 0$, *instead of* (4.2) *it suffices to assume, that*

$$\inf_{r>0} \frac{\beta(r)}{r} < \|K\|^{-1} \ . \tag{4.3}$$

Proof. Denote by P the operator defined by the right hand side of the equation (4.1):

$$Px = \varphi + Hx \quad \text{for} \quad x \in BC(M,\mathbb{R}^n) \ .$$

The assumptions i) and ii) imply that P is continuous. To prove that it is compact it suffices to show that the operator K is compact (F is bounded and continuous due to i)).

So let us fix arbitrarily $\varepsilon > 0$.

Since each of the sets K_{ij} is compact in $L^1(M)$, there exists a finite collection of sets $\{E_1, \ldots, E_m\}$ with union M such that for any l, $1 \le l \le m$, any $t_1, t_2 \in E_l$ and any pair (i,j), $1 \le i,j \le n$, we have

$$\|k_{ij}(t_1,\cdot) - k_{ij}(t_2,\cdot)\|_{L^1} = \int_M |k_{ij}(t_1,s) - k_{ij}(t_2,s)| d\mu(s) < \varepsilon.$$

Hence for any $x \in BC(M,\mathbb{R}^n)$ we get

$$\begin{aligned}
\|(Kx)(t_1) - (Kx)(t_2)\| &= \| \int_M [k(t_1,s) - k(t_2,s)] x(s) d\mu(s)\| \le \\
&\le C_2 \sum_{i=1}^n | \int_M \sum_{j=1}^n [k_{ij}(t_1,s) - k_{ij}(t_2,s)] x_j(s) d\mu(s)| \le \\
&\le C_1 C_2 \|x\| \sum_{i=1}^n \sum_{j=1}^n \int_M |k_{ij}(t_1,s) - k_{ij}(t_2,s)| d\mu(s) < \\
&< C_1 C_2 \|x\| n^2 \varepsilon,
\end{aligned}$$

which, together with Corollary 2.2, implies that the operator K is compact.

Now observe, that using (4.2) (or (4.3) in the case $\varphi \equiv 0$) we can find $r > 0$ such that

$$\frac{\|\varphi\|_{BC} \|K\|^{-1}}{r} + \frac{\beta(r)}{r} < \|K\|^{-1} \ .$$

For such r, and all $x \in B_r$, we have

$$\|Px\|_{BC} \le \|\varphi\|_{BC} + \|KFx\|_{BC} \le \|\varphi\|_{BC} + \|K\| \beta(r) \le r \ ,$$

so P maps the ball B_r into itself. Thus, we can use Schauder's fixed point theorem to finish the proof. □

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