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**UNIFORMLY CONTINUOUS SET-VALUED  
COMPOSITION OPERATORS IN THE SPACES  
OF FUNCTIONS OF BOUNDED VARIATION  
IN THE SENSE OF WIENER**

**Abstract.** We show that the one-sided regularizations of the generator of any uniformly continuous and convex compact valued composition operator, acting in the spaces of functions of bounded variation in the sense of Wiener, is an affine function.

**Keywords:**  $\varphi$ -variation in the sense of Wiener, set-valued functions, left and right regularizations, uniformly continuous composition (Nemytskii) operator, Jensen equation.

**Mathematics Subject Classification:** 47B33, 26B30, 26E25, 26B40.

INTRODUCTION

Let  $(X, |\cdot|)$  and  $(Y, |\cdot|)$  be two real normed spaces,  $C$  be a convex cone in  $X$  and  $I \subset \mathbb{R}$  an interval. Let  $cc(Y)$  be the family of all non-empty convex compact subsets of  $Y$ . We consider the Nemytskii operator, i.e. the composition operator defined by  $h(t, F(t))$  for  $F : I \rightarrow cc(X)$ , where  $h : I \times \mathbb{R} \rightarrow cc(Y)$  is a given set-valued function. It is shown that if the operator  $H$  maps the space  $BV_\varphi(I; C)$  of functions of bounded  $\varphi$ -variation in the sense of Wiener into the space  $BW_\psi(I; cc(Y))$  of convex compact valued functions of bounded  $\psi$ -variation in the sense of Wiener, and is uniformly continuous, then the one-sided regularizations  $h^-$  and  $h^+$  of  $h$  which respect to the first variable, are affine with respect to the second variable. In particular,

$$h^-(t, x) = A(t)x + B(t), \quad \text{for } t \in I, \quad x \in C,$$

for some function  $A : I \rightarrow \mathcal{L}(C, cc(Y))$  and  $B \in BW_\psi(I; cc(Y))$ , where  $\mathcal{L}(C, cc(Y))$  stands for the space of all linear mappings  $C$  into  $cc(Y)$ .

## 1. PRELIMINARIES

In this section we present some definitions and recall known results concerning the Wiener  $\varphi$ -variation.

Let  $\mathcal{F}$  be the set of all convex functions  $\varphi : [0, \infty) \rightarrow [0, \infty)$  such that:  $\varphi(0) = \varphi(0+) = 0$  and  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ . Then we have

**Remark 1.1.** If  $\varphi \in \mathcal{F}$ , then  $\varphi$  is continuous and strictly increasing. Indeed, the continuity of  $\varphi$  at each point  $t > 0$  follows from its convexity and continuity at 0 from the assumption  $\varphi(0) = \varphi(0+) = 0$ . Suppose that  $\varphi(t_1) \geq \varphi(t_2)$  for some  $0 < t_1 < t_2$ . Then

$$\frac{\varphi(t_1) - \varphi(0)}{t_1 - 0} = \frac{\varphi(t_1)}{t_1} > \frac{\varphi(t_2)}{t_2} = \frac{\varphi(t_2) - \varphi(0)}{t_2 - 0},$$

contradicting the convexity of  $\varphi$ .

Let  $I \subset \mathbb{R}$  be an interval. For a set  $X$  we denote by  $X^I$  the set of all functions  $f : I \rightarrow X$ .

**Definition 1.2.** Let  $\varphi \in \mathcal{F}$  and  $(X, |\cdot|)$  be a real normed space. A function  $f \in X^I$  is of bounded  $\varphi$ -variation in the sense of Wiener in  $I$ , if

$$v_\varphi(f) := \sup_\xi \sum_{i=1}^m \varphi(|f(t_i) - f(t_{i-1})|) < \infty, \quad (1.1)$$

where the supremum is taken over all finite and increasing sequences  $\xi = (t_i)_{i=0}^m$ ,  $t_i \in I$ ,  $m \in \mathbb{N}$ .

Denote by  $BV_\varphi(I, X)$  the set of all functions  $f \in X^I$  such that  $v_\varphi(\lambda f) < \infty$  for some  $\lambda > 0$ .  $BV_\varphi(I, X)$  is a normed space endowed with the norm

$$\|f\|_\varphi := |f(a)| + p_\varphi(f), \quad f \in BV_\varphi(I, X), \quad (1.2)$$

where  $I = [a, b]$  and

$$p_\varphi(f) = \inf \left\{ \epsilon > 0 : v_\varphi(f/\epsilon) \leq 1 \right\}. \quad (1.3)$$

Let  $cc(X)$  be the family of all non-empty convex compact subsets of  $X$ , and let  $D$  be the Pompeiu-Hausdorff metric in  $cc(X)$ , i.e.

$$D(A, B) := \max \left\{ e(A, B), e(B, A) \right\}, \quad A, B \in cc(X), \quad (1.4)$$

where

$$e(A, B) = \sup \left\{ d(x, B) : x \in A \right\}, \quad d(x, B) = \inf \left\{ d(x, y) : y \in B \right\}. \quad (1.5)$$

The Pompeiu-Hausdorff metric  $D$  is translation invariant on  $cc(X)$  in the sense that (see [13, Lemma 3]):

$$D(A, B) = D(A + Q, B + Q) \quad (1.6)$$

for all  $A, B \in cc(X)$  and bounded nonempty set  $Q$  of  $X$ .

**Definition 1.3.** Let  $\varphi \in \mathcal{F}$  and  $F : I \longrightarrow cc(X)$ . We say that  $F$  has bounded  $\varphi$ -variation in the Wiener sense if

$$w_\varphi(F) := \sup_\xi \sum_{i=1}^m \varphi(D(F(t_i), F(t_{i-1}))) < \infty, \quad (1.7)$$

where the supremum is taken over all finite and increasing sequences  $\xi = (t_i)_{i=0}^m$ ,  $t_i \in I$ ,  $m \in \mathbb{N}$ .

Let

$$BW_\varphi(I, cc(X)) := \left\{ F \in cc(X)^I : w_\varphi(\lambda F) < \infty \text{ for some } \lambda > 0 \right\}. \quad (1.8)$$

For  $F_1, F_2 \in BW_\varphi(I, cc(X))$  put

$$D_\varphi(F_1, F_2) := D(F_1(a), F_2(a)) + p_\varphi(F_1, F_2), \quad (1.9)$$

where

$$p_\varphi(F_1, F_2) := \inf \left\{ \epsilon > 0 : W_\epsilon(F_1, F_2) \leq 1 \right\} \quad (1.10)$$

and

$$W_\epsilon(F_1, F_2) := \sup_\xi \sum_{i=1}^m \varphi\left(\frac{1}{\epsilon} D(F_1(t_i) + F_2(t_{i-1}); F_2(t_i) + F_1(t_{i-1}))\right), \quad (1.11)$$

where the supremum is taken over all finite and increasing sequences  $\xi = (t_i)_{i=0}^m$ ,  $t_i \in I$ ,  $m \in \mathbb{N}$ .

**Lemma 1.4** (cf. M. Castillo [2, p. 107]). *The set  $BW_\varphi(I, cc(X))$  endowed with  $D_\varphi$  is a metric space.*

**Lemma 1.5** (cf. V.V. Chistyakov [3, Lemma 5.2]). *Let  $F_1, F_2 \in BW_\varphi(I, cc(X))$  and  $\varphi \in \mathcal{F}$ . Then, for  $\lambda > 0$ ,*

$$W_\lambda(F_1, F_2) \leq 1 \quad \text{if and only if} \quad p_\varphi(F_1, F_2) \leq \lambda.$$

Now, let  $(X, |\cdot|)$ ,  $(Y, |\cdot|)$  be two real normed spaces and  $C$  be a convex cone in  $X$ . Given a set-valued function  $h : I \times C \longrightarrow cc(Y)$  we consider the composition operator  $H : C^I \longrightarrow Y^I$  generated by  $h$ , i.e.

$$(Hf)(t) := h(t, f(t)), \quad f \in C^I, \quad t \in I. \quad (1.12)$$

Denote by  $\mathcal{A}(C, cc(Y))$  the space of all additive functions and by  $\mathcal{L}(C, cc(Y))$  the space of all set-valued linear functions, i.e., all set-valued functions  $A \in \mathcal{A}(C, cc(Y))$  which are positively homogeneous.

**Lemma 1.6** (cf. K. Nikodem [12, Th. 5.6]). *Let  $(X, |\cdot|)$ ,  $(Y, |\cdot|)$  be normed spaces and  $C$  a convex cone in  $X$ . A set-valued function  $F : C \longrightarrow cc(Y)$  satisfies the Jensen equation*

$$F\left(\frac{x+y}{2}\right) = \frac{1}{2}(F(x) + F(y)), \quad x, y \in C, \quad (1.13)$$

*if and only if, there exist an additive set-valued function  $A : C \longrightarrow cc(Y)$  and a set  $B \in cc(Y)$  such that  $F(x) = A(x) + B$  for all  $x \in C$ .*

## 2. THE COMPOSITION OPERATOR

Our main result reads as follows:

**Theorem 2.1.** *Let  $(X, |\cdot|)$  be a real normed space,  $(Y, |\cdot|)$  a real Banach space,  $C$  a convex cone in  $X$  and suppose that  $\varphi, \psi \in \mathcal{F}$ . If the composition operator  $H$  generated by a set-valued function  $h : I \times C \rightarrow cc(Y)$  maps  $BV_\varphi(I, C)$  into  $BW_\psi(I, cc(Y))$ , and is uniformly continuous, then the left regularization of  $h$ , i.e. the function  $h^- : I \times X \rightarrow Y$  defined by*

$$h^-(t, x) := \lim_{s \uparrow t} h(s, x), \quad t \in I, \quad x \in C,$$

exists and

$$h^-(t, x) = A(t)x + B(t), \quad t \in I, \quad x \in C,$$

for some  $A : I \rightarrow \mathcal{A}(X, cc(Y))$  and  $B : I \rightarrow cc(Y)$ . Moreover, if  $0 \in C$ , then  $B \in BW_\psi(I, cc(Y))$  and, for any  $t \in I$ , the linear set-valued function  $A(t)$  is continuous.

*Proof.* For every  $x \in C$ , the constant function  $I \ni t \rightarrow x$  belongs to  $BV_\varphi(I, C)$ . Since  $H$  maps  $BV_\varphi(I, C)$  into  $BW_\psi(I, cc(Y))$ , for every  $x \in C$ , the function  $I \ni t \rightarrow h(t, x)$  belongs to  $BW_\psi(I, cc(Y))$ . Now the completeness of  $cc(Y)$  with respect to the Pompeiu-Hausdorff metric (see [17, Th. A]), implies the existence of the left regularization  $h^-$  of  $h$ .

By assumption,  $H$  is uniformly continuous on  $BV_\varphi(I, C)$ . Let  $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be the modulus of continuity of  $H$ , that is

$$\omega(\rho) := \sup \left\{ D_\psi(H(f_1), H(f_2)) : \|f_1 - f_2\|_\varphi \leq \rho; f_1, f_2 \in BV_\varphi(I, C) \right\}, \quad \rho > 0.$$

Hence we get

$$D_\psi(H(f_1), H(f_2)) \leq \omega(\|f_1 - f_2\|_\varphi), \quad \text{for } f_1, f_2 \in BV_\varphi(I, C). \quad (2.1)$$

From the definition of the metric  $D_\psi$  and (2.1), we obtain

$$p_\psi(H(f_1); H(f_2)) \leq \omega(\|f_1 - f_2\|_\varphi), \quad \text{for } f_1, f_2 \in BV_\varphi(I, C). \quad (2.2)$$

From Lemma 1.5, if  $\omega(\|f_1 - f_2\|_\varphi) > 0$ , the inequality (2.2) is equivalent to

$$W_{\omega(\|f_1 - f_2\|_\varphi)}(H(f_1), H(f_2)) \leq 1, \quad f_1, f_2 \in BV_\varphi(I, C). \quad (2.3)$$

Therefore, for any  $a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_m < \beta_m = b$ ,  $\alpha_i, \beta_i \in I$ ,  $i \in \{1, 2, \dots, m\}$ ,  $m \in \mathbb{N}$ , the definitions of the operator  $H$  and the functional  $W_\epsilon$ , imply

$$\sum_{i=1}^m \psi \left( \frac{D(h(\beta_i, f_1(\beta_i)) + h(\alpha_i, f_2(\alpha_i)); h(\beta_i, f_2(\beta_i)) + h(\alpha_i, f_1(\alpha_i)))}{\omega(\|f_1 - f_2\|_\varphi)} \right) \leq 1. \quad (2.4)$$

For  $\alpha, \beta \in \mathbb{R}$ ,  $\alpha < \beta$ , we define functions  $\eta_{\alpha, \beta} : \mathbb{R} \rightarrow [0, 1]$  by

$$\eta_{\alpha, \beta}(t) := \begin{cases} 0 & \text{if } t \leq \alpha, \\ \frac{t - \alpha}{\beta - \alpha} & \text{if } \alpha \leq t \leq \beta, \\ 1 & \text{if } \beta \leq t. \end{cases} \quad (2.5)$$

Let us fix  $t \in I$ . For an arbitrary finite sequence  $a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_m < \beta_m < t$  and  $x_1, x_2 \in C$ ,  $x_1 \neq x_2$ , the functions  $f_1, f_2 : I \rightarrow X$  defined by

$$f_\ell(\tau) := \frac{1}{2} [\eta_{\alpha_i, \beta_i}(\tau)(x_1 - x_2) + x_\ell + x_2], \quad \tau \in I, \ell = 1, 2, \quad (2.6)$$

belong to the space  $BV_\varphi(I, C)$ . From (2.6) we have

$$f_1(\tau) - f_2(\tau) = \frac{x_1 - x_2}{2}, \quad \tau \in I,$$

therefore

$$\|f_1 - f_2\|_\varphi = \left| \frac{x_1 - x_2}{2} \right|;$$

moreover

$$f_1(\beta_i) = x_1; \quad f_2(\beta_i) = \frac{x_1 + x_2}{2}; \quad f_1(\alpha_i) = \frac{x_1 + x_2}{2}; \quad f_2(\alpha_i) = x_2.$$

Using (2.4), we hence get

$$\sum_{i=1}^m \psi \left( \frac{D \left( h(\beta_i, x_1) + h(\alpha_i, x_2); h \left( \alpha_i, \frac{x_1 + x_2}{2} \right) + h \left( \beta_i, \frac{x_1 + x_2}{2} \right) \right)}{\omega \left( \left| \frac{x_1 - x_2}{2} \right| \right)} \right) \leq 1. \quad (2.7)$$

Since, for any  $x \in C$ , the constant function  $I \ni \tau \rightarrow x$  belongs to space  $BV_\varphi(I, C)$  and  $H$  maps  $BV_\varphi(I, C)$  into  $BW_\psi(I, cc(Y))$ , the function  $I \ni \tau \rightarrow h(\tau, x)$  belongs to  $BW_\psi(I, cc(Y))$  for any  $x \in C$ . From the continuity of  $\psi$  and the definition of  $h^-$ , letting  $\alpha_1 \uparrow t$  in (2.7), we obtain

$$\sum_{i=1}^m \psi \left( \frac{D \left( h^-(t, x_1) + h^-(t, x_2); 2h^-\left(t, \frac{x_1 + x_2}{2}\right) \right)}{\omega \left( \left| \frac{x_1 - x_2}{2} \right| \right)} \right) \leq 1, \quad (2.8)$$

that is

$$\psi \left( \frac{D \left( h^-(t, x_1) + h^-(t, x_2); 2h^-\left(t, \frac{x_1 + x_2}{2}\right) \right)}{\omega \left( \left| \frac{x_1 - x_2}{2} \right| \right)} \right) \leq \frac{1}{m}.$$

Hence, since  $m \in \mathbb{N}$  is arbitrary,

$$\psi \left( \frac{D \left( h^-(t, x_1) + h^-(t, x_2); 2h^-\left(t, \frac{x_1 + x_2}{2}\right) \right)}{\omega \left( \left| \frac{x_1 - x_2}{2} \right| \right)} \right) = 0,$$

and, as  $\psi(r) = 0$  only if  $r = 0$ , we obtain

$$D\left(h^-(t, x_1) + h^-(t, x_2); 2h^-\left(t, \frac{x_1 + x_2}{2}\right)\right) = 0.$$

Therefore

$$h^-\left(t, \frac{x_1 + x_2}{2}\right) = \frac{h^-(t, x_1) + h^-(t, x_2)}{2}$$

for all  $t \in I$  and all  $x_1, x_2 \in C$ .

Thus, for each  $t \in I$ , the function  $h^-(t, \cdot)$  satisfies the Jensen functional equation in  $C$ . Consequently, by Lemma 1.6, for every  $t \in I$  there exist an additive set valued function  $A(t) : C \rightarrow cc(Y)$  and a set  $B(t) \in cc(Y)$  such that

$$h^-(t, x) = A(t)x + B(t) \quad \text{for } x \in C, \quad t \in I, \quad (2.9)$$

which proves the first part of our result.

The uniform continuity of the operator  $H : BV_\varphi(I, C) \rightarrow BW_\psi(I, cc(Y))$  implies the continuity of the function  $A(t)$ , so that  $A(t) \in \mathcal{L}(C, cc(Y))$  (see [12, Th. 5.3]). Putting  $x = 0$  in (2.9), and taking into account that  $A(t)0 = \{0\}$  for  $t \in I$ , we get

$$h^-(t, 0) = B(t), \quad t \in I,$$

which implies that  $B \in BW_\psi(I, cc(Y))$ . □

**Remark 2.2.** The counterpart of Theorem 2.1 for the right regularization  $h^+$  of  $h$  defined by

$$h^+(t, x) := \lim_{s \downarrow t} h(s, x); \quad t \in I,$$

is also true.

**Remark 2.3.** Taking  $\varphi(t) = \psi(t) = t^p$  ( $t \geq 0$ ) for all  $1 < p < \infty$  we obtain the main result of [4].

**Remark 2.4.** The condition  $\text{int}C \neq \emptyset$  is assumed to guarantee the continuity of the linear functions  $A(t)$ .

**Remark 2.5.** The uniformly continuous composition operators in the single-valued case for some function spaces were considered in [7, 8] and [9]. The globally Lipschitzian composition operators in some special function spaces were considered in [6, 10, 11] (cf. also [1] for other references). The first paper in which the set-valued Lipschitzian composition operators were considered is due to A. Smajdor and W. Smajdor [14]. Note also that G. Zawadzka [17] considered the set-valued Lipschitzian composition operators in the space of set-valued functions of bounded variation.

## REFERENCES

- [1] J. Appell, P.P. Zabrejko, *Nonlinear Superposition Operator*, Cambridge University Press, New York, 1990.
- [2] M. Castillo, *Funciones conjunto-valuadas de  $\varphi$ -variación acotada en el sentido de Wiener y el Operador de Composición*, Tesis de Maestría. Facultad de Ciencia, Universidad Central de Venezuela, Caracas, Venezuela, 1999.
- [3] V.V. Chistyakov, *Generalized variation of mappings with applications to composition operators and multifunctions*, Positivity **5** (2001) 4, 323–358.
- [4] J.A. Guerrero, J. Matkowski, N. Merentes J.L. Sánchez, *Uniformly continuous set-valued composition operators in the space of functions of the Wiener bounded  $p$ -variation*, accepted for J. Math. Appl.
- [5] J. Matkowski, *Functional equations and Nemytskij operators*, Funkc. Ekvacioj Ser. Int. **25** (1982), 127–132.
- [6] J. Matkowski, *Lipschitzian composition operators in some function spaces*, Nonlinear Anal. **3** (1997), 719–726.
- [7] J. Matkowski, *Uniformly continuous superposition operators in the space of differentiable function and absolutely continuous functions*, Internat. Ser. Numer. Math. **157** (2008), 155–166.
- [8] J. Matkowski, *Uniformly continuous superposition operators in the space of Hölder functions*, J. Math. Anal. Appl. **359** (2009), 56–61.
- [9] J. Matkowski, *Uniformly continuous superposition operators in the space of bounded variation functions*, accepted for Math. Nach.
- [10] J. Matkowski, J. Miś, *On a characterization of Lipschitzian operators of substitution in the space  $BV\langle a, b \rangle$* , Math. Nachr. **117** (1984), 155–159.
- [11] N. Merentes, *Composition of functions of bounded  $\varphi$ -variation*, P.U.M.A., Ser. 1, (1991), 39–45.
- [12] K. Nikodem,  *$K$ -convex and  $K$ -concave set-valued functions*, Politech. Lodz. Zeszyty Nauk. **559**, (1989).
- [13] H. Rådström, *An embedding theorem for space of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165–169.
- [14] A. Smajdor, W. Smajdor, *Jensen equation and Nemytskii operator for set-valued functions*, Rad. Mat. **5** (1989), 311–320.
- [15] N. Wiener, *The quadratic variation of function and its Fourier coefficients*, Massachusetts J. Math. **3** (1924), 72–94.
- [16] L.C. Young, *Sur une généralisation de la notion de variation de puissance  $p$ -ieme bornée au sens de N. Wiener, et sur la convergence des séries de Fourier*, C. R. Acad. Sci., **204** (1937) 7, 470–472.
- [17] G. Zawadzka, *On Lipschitzian operators of substitution in the space of set-valued functions of bounded variation*, Radovi Matematički **6** (1990), 279–293.

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