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DOMINATING SETS AND DOMINATION POLYNOMIALS OF CERTAIN GRAPHS, II

Abstract. The domination polynomial of a graph G of order n is the polynomial $D(G,x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i, and $\gamma(G)$ is the domination number of G. In this paper, we obtain some properties of the coefficients of $D(G,x)$. Also, by study of the dominating sets and the domination polynomials of specific graphs denoted by $G'(m)$, we obtain a relationship between the domination polynomial of graphs containing an induced path of length at least three, and the domination polynomial of related graphs obtained by replacing the path by shorter path. As examples of graphs $G'(m)$, we study the dominating sets and domination polynomials of cycles and generalized theta graphs. Finally, we show that, if $n \equiv 0, 2 (mod\, 3)$ and $D(G,x) = D(C_n,x)$, then $G = C_n$.

Keywords: domination polynomial, dominating set, cycle, theta graph.

Mathematics Subject Classification: Primary 05C69; Secondary 11B83.

1. INTRODUCTION

Let $G = (V,E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a *γ-set*, and the family of γ-sets are denoted by $\Gamma(G)$. For a detailed treatment of this parameter, the reader is referred to [9]. Let $\mathcal{D}(G,i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G,i) = |\mathcal{D}(G,i)|$. The domination polynomial $D(G,x)$ of G is defined as $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, where $\gamma(G)$ is the domination number of G ([4]).

Let P_{m+1} be a path with vertices labeled by $y_1, y_2, \ldots, y_{m+1}$, for $m \geq 1$ and let v_0 be a specific vertex of a graph G. Denote by $G_{v_0}(m)$ (or simply $G(m)$) a graph obtained from G by identifying the vertex v_0 of G with an end vertex y_1 of P_{m+1} ([3] or [6]). For graphs $G(m)$ we proved the following theorem:

Theorem 1.1 ([3, Theorem 3.2.10])**.** *For every $m \geq 3$,*

$$D(G(m), x) = x\Big[D(G(m-1), x) + D(G(m-2), x) + D(G(m-3), x)\Big].$$

A graph G is called *P_4-free*, if G does not contain an induced subgraph P_4. (For details of P_4-free graphs, see [10]).

In the next section we obtain some results of the coefficients of the domination polynomial of a graph G. We study the dominating sets of graphs with a specific construction denoted by $G'(m)$, to obtain a relationship between $D(G'(m), x)$ and $D(G'(m-1), x)$, $D(G'(m-2), x)$ and $D(G'(m-3), x)$ in Section 3. Using recursive formula for $D(G'(m), x)$ and Theorem 1.1, we give a recursive formula for the domination polynomial of non P_4-free graphs. As examples of $G'(m)$ we study the dominating sets and the domination polynomial of a cycle C_n and generalized theta graphs in Section 4. In the last section, we show that, if $n \equiv 0, 2 (mod\, 3)$ and $D(G, x) = D(C_n, x)$, then $G = C_n$.

As usual we use $\lceil x \rceil$, $\lfloor x \rfloor$ for the smallest integer greater than or equal to x and the largest integer less than or equal to x, respectively. In this article we denote the set $\{1, 2, \ldots, n\}$ simply by $[n]$.

2. SOME PROPERTIES OF COEFFICIENTS OF DOMINATION POLYNOMIALS

In this section, we obtain some properties of the coefficients of the domination polynomial of a graph G.

We recall that a subset M of $E(G)$ is called a *matching* in G if its elements are not loops and no two of them are adjacent in G; the two ends of an edge in M are said to be *matched* under M. A matching M *saturates* a vertex v, and v is said to be *M-saturated* if some edges of M are incident with v; otherwise v is *M-unsaturated*.

We need the following result to prove Theorem 2.2:

Theorem 2.1 (Hall [7, p. 72])**.** *Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if for all $S \subseteq X$, $|N(S)| \geq |S|$.*

Theorem 2.2. *Let G be a graph of order n. Then for every $0 \leq i < \frac{n}{2}$, we have $d(G, i) \leq d(G, i+1)$.*

Proof. Consider a bipartite graph with two partite sets X and Y. The vertices of X are dominating sets of G of cardinality i, and the vertices of Y are all $(i+1)$-subsets of $V(G)$. Join a vertex A of X to a vertex B of Y, if $A \subseteq B$. Clearly, the degree of each vertex in X is $n - i$. Also for any $B \in Y$, the degree of B is at most $i + 1$.

We claim that for any $S \subseteq X$, $|N(S)| \geq |S|$ and so by Theorem 2.1, the bipartite graph has a matching which saturate all vertices of X. By contradiction suppose that there exists $S \subseteq X$ such that $|N(S)| < |S|$. The number of edges incident with S is $|S|(n-i)$. Thus by a pigeon hole principle, there exists a vertex $B \in Y$ with degree more than $n-i$. This implies that $i+1 \geq n-i+1$. Hence $i \geq \frac{n}{2}$, a contradiction. Thus for every $S \subseteq X$, $|N(S)| \geq |S|$ and the claim is proved. Since for every $A \in X$, and every $v \in V(G)\backslash A$, $A \cup \{v\}$ is a dominating set of cardinality $i+1$, we conclude that $d(G, i+1) \geq d(G, i)$ and the proof is complete. □

Here, we recall the relationship between the domination polynomial of a graph G and the minimum degree of G. Also, we recall a result which gives a relationship between the domination polynomial of a graph and its regularity.

Theorem 2.3 ([1]). *Let G be a graph of order n with domination polynomial $D(G,x) = \sum_{i=1}^{n} d(G,i)x^i$. If $d(G,j) = \binom{n}{j}$ for some j, then $\delta(G) \geq n-j$. More precisely, $\delta(G) = n-l$, where $l = \min\left\{j | d(G,j) = \binom{n}{j}\right\}$, and there are at least $\binom{n}{n-\delta(G)-1} - d(G, n-\delta(G)-1)$ vertices of degree $\delta(G)$ in G. Furthermore, if for every two vertices of degree $\delta(G)$, say u and v we have $N[u] \neq N[v]$, then there are exactly $\binom{n}{n-\delta(G)-1} - d(G, n-\delta(G)-1)$ vertices of degree $\delta(G)$.*

By Theorem 2.3, we have the following theorem which gives the relationship between the domination polynomial of a graph G and the regularity of G.

Theorem 2.4. *Let H be a k-regular graph with $N[u] \neq N[v]$, for $u, v \in V(H)$. If $D(G,x) = D(H,x)$, then G is a k-regular graph.*

3. DOMINATING SETS AND DOMINATION POLYNOMIALS OF GRAPHS $G'(m)$

Let P_m be a path with vertices labeled $y_1, \ldots, y_m$ and let a, b be two specific vertices of a graph G (note that may be $a = b$). Denote by $G'_{a,b}(m)$ (or simply $G'(m)$, if there is no likelihood of confusion) a graph obtained from G by identifying the vertices a and b of G with end vertices y_1 and y_m of P_m, respectively. See Figure 1. Throughout our discussion these two vertices a and b are fixed.

We need some lemmas to obtain main results in this section. The following lemma follows from our observation.

Lemma 3.1. *If a graph G contains a simple path of length $3k-1$, then every dominating set of G must contain at least k vertices of the path.*

Lemma 3.2. (i) *$\mathcal{D}(G'(m), i) = \emptyset$ if and only if $i > |V(G'(m))|$ or $i < \gamma(G'(m))$.*
(ii) *For any $m \geq 2$, $\gamma(G'(m)) - 1 \leq \gamma(G'(m-1)) \leq \gamma(G'(m))$.*

Proof. (i) It follows from the definition of a dominating set of a graph.
(ii) Suppose that $e \in \{y_1y_2, \ldots, y_{m-1}y_m\}$, so $G'(m) * e = G'(m-1)$, where $G * e$ is the graph obtained from G by contracting the edge e. Since for every $e \in \{y_1y_2, \ldots, y_{m-1}y_m\}$, $\gamma(G'(m)) - 1 \leq \gamma(G'(m) * e) \leq \gamma(G'(m))$, then we have the result. □

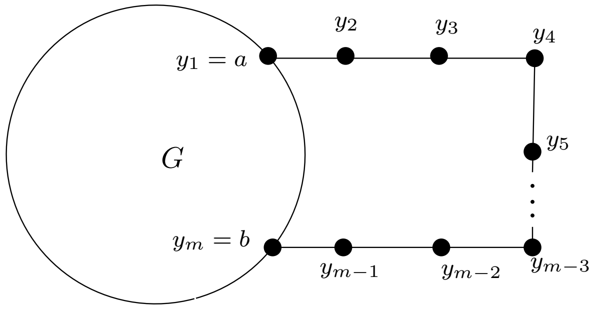

Fig. 1. The graph $G'_{a,b}(m)$, or simply $G'(m)$

To find a dominating set of $G'(m)$ with cardinality i, we do not need to consider dominating sets of $G'(m-4)$ and $G'(m-5)$ with cardinality $i-1$. We show this in Lemma 3.4. Therefore, we only need to consider dominating sets in $G'(m-1)$, $G'(m-2)$ and $G'(m-3)$ with cardinality $i-1$. The families of these dominating sets can be empty or otherwise. Thus, we have eight combinations of whether these three families are empty or not. Two of these combinations are not possible (see Lemma 3.3 (i) and (ii)). Also, the combination $\mathcal{D}(G'(m-1), i-1) = \mathcal{D}(G'(m-2), i-1) = \mathcal{D}(G'(m-3), i-1) = \emptyset$; need not to be considered because it implies $\mathcal{D}(G'(m), i) = \emptyset$; (see Lemma 3.3 (iii)). Thus, we only need to consider five combinations or cases. We consider this in Theorem 3.6.

We denote the $\mathcal{D}(G'(m), i)$ simply by $\mathcal{G'}_{m,i}$.

Lemma 3.3. *For every $m \geq 4$:*

(i) *If $\mathcal{G'}_{m-1,i-1} = \mathcal{G'}_{m-3,i-1} = \emptyset$, then $\mathcal{G'}_{m-2,i-1} = \emptyset$.*
(ii) *If $\mathcal{G'}_{m-1,i-1} \neq \emptyset, \mathcal{G'}_{m-3,i-1} \neq \emptyset$, then $\mathcal{G'}_{m-2,i-1} \neq \emptyset$.*
(iii) *If $\mathcal{G'}_{m-1,i-1} = \mathcal{G'}_{m-2,i-1} = \mathcal{G'}_{m-3,i-1} = \emptyset$, then $\mathcal{G'}_{m,i} = \emptyset$.*

Proof. (i) Since $\mathcal{G'}_{m-1,i-1} = \mathcal{G'}_{m-3,i-1} = \emptyset$, by Lemma 3.2, $i-1 > |V(G'(m-1))|$ or $i-1 < \gamma(G'(m-3))$. In either cases we have $\mathcal{G'}_{m-2,i-1} = \emptyset$.
(ii) Suppose that $\mathcal{G'}_{m-2,i-1} = \emptyset$, so by Lemma 3.2, we have $i-1 > |V(G'(m-2))|$ or $i-1 < \gamma(G'(m-2))$. If $i-1 > |V(G'(m-2))|$, then $i-1 > |V(G'(m-3))|$, and hence, $\mathcal{G'}_{m-3,i-1} = \emptyset$, a contradiction. So we have $i-1 < \gamma(G'(m-2))$, and hence, $\mathcal{G'}_{m-1,i-1} = \emptyset$, also a contradiction.
(iii) Since $\mathcal{G'}_{m-1,i-1} = \mathcal{G'}_{m-2,i-1} = \mathcal{G'}_{m-3,i-1} = \emptyset$, by Lemma 3.2, $i-1 > |V(G'(m))| - 1$ or $i-1 < \gamma(G'(m-3))$. In either case, we have $\mathcal{G'}_{m,i} = \emptyset$. □

Lemma 3.4. *If Y is in $\mathcal{G'}_{m-4,i-1}$ or $\mathcal{G'}_{m-5,i-1}$ such that $Y \cup \{x\} \in \mathcal{G'}_{m,i}$ for some $x \in \{y_1, \ldots, y_m\}$, then $Y \in \mathcal{G'}_{m-3,i-1}$.*

Proof. Let $Y \in \mathcal{G'}_{m-4,i-1}$ and $Y \cup \{x\} \in \mathcal{G'}_{m,i}$ for some $x \in \{y_1, \ldots, y_m\}$. This means, by Lemma 3.1, we only need to consider y_{m-4} or y_{m-5} as elements of Y. In each case, $Y \in \mathcal{G'}_{m-3,i-1}$. Now suppose that $Y \in \mathcal{G'}_{m-5,i-1}$ and $Y \cup \{x\} \in \mathcal{G'}_{m,i}$ for some $x \in \{y_1, \ldots, y_m\}$. This means, by Lemma 3.1, y_{m-5} must be in Y. So $Y \in \mathcal{G'}_{m-3,i-1}$. □

Now we state when five cases for the families $\mathcal{G}'_{m-1,i-1}, \mathcal{G}'_{m-2,i-1}$ and $\mathcal{G}'_{m-3,i-1}$ can occur.

Lemma 3.5. *Suppose that $\mathcal{G}'_{m,i} \neq \emptyset$, then:*

(i) $\mathcal{G}'_{m-1,i-1} = \mathcal{G}'_{m-2,i-1} = \emptyset$ *and* $\mathcal{G}'_{m-3,i-1} \neq \emptyset$ *if and only if* $\gamma(G'(m-3)) + 1 \leq i < \gamma(G'(m-2)) + 1$.

(ii) $\mathcal{G}'_{m-2,i-1} = \mathcal{G}'_{m-3,i-1} = \emptyset$ *and* $\mathcal{G}'_{m-1,i-1} \neq \emptyset$ *if and only if* $i = |V(G'(m))|$.

(iii) $\mathcal{G}'_{m-1,i-1} = \emptyset, \mathcal{G}'_{m-2,i-1} \neq \emptyset$ *and* $\mathcal{G}'_{m-3,i-1} \neq \emptyset$ *if and only if* $\gamma(G'(m-2)) + 1 \leq i < \gamma(G'(m-1)) + 1$.

(iv) $\mathcal{G}'_{m-1,i-1} \neq \emptyset, \mathcal{G}'_{m-2,i-1} \neq \emptyset$ *and* $\mathcal{G}'_{m-3,i-1} = \emptyset$ *if and only if* $i = |V(G'(m))| - 1$.

(v) $\mathcal{G}'_{m-1,i-1} \neq \emptyset, \mathcal{G}'_{m-2,i-1} \neq \emptyset$ *and* $\mathcal{G}'_{m-3,i-1} \neq \emptyset$, *if and only if* $\gamma(G'(m-1)) + 1 \leq i \leq |V(G'(m))| - 2$.

Proof. (i) ($\Rightarrow$) Since $\mathcal{G}'_{m-1,i-1} = \mathcal{G}'_{m-2,i-1} = \emptyset$, by Lemma 3.2, we have $i - 1 > |V(G'(m))| - 1$ or $i - 1 < \gamma(G'(m-2))$. If $i - 1 > |V(G'(m))| - 1$, then $i > |V(G'(m))|$, and by Lemma 3.2, $\mathcal{G}'_{m,i} = \emptyset$, a contradiction. So we have $i - 1 < \gamma(G'(m-2))$, and since $\mathcal{G}'_{m-3,i-1} \neq \emptyset$, together we have $\gamma(G'(m-3)) \leq i - 1 < \gamma(G'(m-2))$. So $\gamma(G'(m-3)) + 1 \leq i < \gamma(G'(m-2)) + 1$.
($\Leftarrow$) If $\gamma(G'(m-3)) + 1 \leq i < \gamma(G'(m-2)) + 1$, then by Lemma 3.2 (i), we have $\mathcal{G}'_{m-1,i-1} = \mathcal{G}'_{m-2,i-1} = \emptyset$ and $\mathcal{G}'_{m-3,i-1} \neq \emptyset$.

(ii) ($\Rightarrow$) Since $\mathcal{G}'_{m-2,i-1} = \mathcal{G}'_{m-3,i-1} = \emptyset$, by Lemma 3.2, $i - 1 > |V(G'(m))| - 2$ or $i - 1 < \gamma(G'(m-3))$. If $i - 1 < \gamma(G'(m-3))$, then $i - 1 < \gamma(G'(m-1))$, and hence $\mathcal{G}'_{m-1,i-1} = \emptyset$, a contradiction. So we must have $i - 1 > |V(G'(m))| - 2$. Also since $\mathcal{G}'_{m-1,i-1} \neq \emptyset$, we have $i - 1 \leq |V(G'(m))| - 1$. Therefore we have $i = |V(G'(m))|$.
($\Leftarrow$) If $i = |V(G'(m))|$, then by Lemma 3.2 (i), we have $\mathcal{G}'_{m-2,i-1} = \mathcal{G}'_{m-3,i-1} = \emptyset$ and $\mathcal{G}'_{m-1,i-1} \neq \emptyset$.

(iii) ($\Rightarrow$) Since $\mathcal{G}'_{m-1,i-1} = \emptyset$, by Lemma 3.2, $i - 1 > |V(G'(m))| - 1$ or $i - 1 < \gamma(G'(m-1))$. If $i - 1 > |V(G'(m))| - 1$, then $i - 1 > |V(G'(m))| - 2$ and by lemma 3.2, $\mathcal{G}'_{m-2,i-1} = \mathcal{G}'_{m-3,i-1} = \emptyset$, a contradiction. So we must have $i-1 < \gamma(G'(m-1))$. But we also have $i-1 \geq \gamma(G'(m-2))$ because $\mathcal{G}'_{m-2,i-1} \neq \emptyset$. Hence, we have $\gamma(G'(m-2)) + 1 \leq i < \gamma(G'(m-1)) + 1$.
($\Leftarrow$) If and $\gamma(G'(m-2)) + 1 \leq i < \gamma(G'(m-1)) + 1$, then by Lemma 3.2 (i), $\mathcal{G}'_{m-1,i-1} = \emptyset, \mathcal{G}'_{m-2,i-1} \neq \emptyset$ and $\mathcal{G}'_{m-3,i-1} \neq \emptyset$.

(iv) ($\Rightarrow$) Since $\mathcal{G}'_{m-3,i-1} = \emptyset$, by Lemma 3.2, we have $i - 1 > |V(G'(m))| - 3$ or $i-1 < \gamma(G'(m-3))$. Since $\mathcal{G}'_{m-2,i-1} \neq \emptyset$, by Lemma 3.2, we have $\gamma(G'(m-2)) \leq i - 1 \leq |V(G'(m))| - 2$. Therefore $i - 1 < \gamma(G'(m-3))$ is not possible. Hence we must have $i - 1 > |V(G'(m))| - 3$. Thus $i = |V(G'(m))| - 1$ or $|V(G'(m))|$. But $i \neq |V(G'(m))|$ because $\mathcal{G}'_{m-3,i-1} = \emptyset$. So we have $i = |V(G'(m))| - 1$.
($\Leftarrow$) If $i = |V(G'(m))| - 1$, then by Lemma 3.2 (i), $\mathcal{G}'_{m-1,i-1} \neq \emptyset, \mathcal{G}'_{m-2,i-1} \neq \emptyset$ and $\mathcal{G}'_{m-3,i-1} = \emptyset$.

(v) ($\Rightarrow$) Since $\mathcal{G}'_{m-1,i-1} \neq \emptyset, \mathcal{G}'_{m-2,i-1} \neq \emptyset$ and $\mathcal{G}'_{m-3,i-1} \neq \emptyset$, then by applying Lemma 3.2, we have $\gamma(G'(m-1)) \leq i - 1 \leq |V(G'(m))| - 1, \gamma(G'(m-2)) \leq$

$i-1 \leq |V(G'(m))|-2$, and $\gamma(G'(m-3)) \leq i-1 \leq |V(G'(m))|-3$. So, by Lemma 3.2, $\gamma(G'(m-1)) \leq i-1 \leq |V(G'(m))|-3$ and hence $\gamma(G'(m-1))+1 \leq i \leq |V(G'(m))|-2$.
($\Leftarrow$) If $\gamma(G'(m-1))+1 \leq i \leq |V(G'(m))|-2$, then by Lemma 3.2 (i) we have the result. $\square$

By Lemma 3.4, for the construction of $\mathcal{G'}_{m,i}$, it suffices to consider $\mathcal{G'}_{m-1,i-1}, \mathcal{G'}_{m-2,i-1}$ and $\mathcal{G'}_{m-3,i-1}$. By Lemma 3.3, we need only consider the five cases in the following theorem:

Theorem 3.6. *For every $i \geq \gamma(G'(m))$:*

(i) *If $\mathcal{G'}_{m-1,i-1} = \mathcal{G'}_{m-2,i-1} = \emptyset$ and $\mathcal{G'}_{m-3,i-1} \neq \emptyset$, then*
$$\mathcal{G'}_{m,i} = \Big\{\{y_{m-2}\}\cup X_1, \{y_{m-1}\}\cup X_2, \{y_m\}\cup X_3 | y_{m-5}\in X_1, y_{m-4}\in X_2, y_{m-3}\in X_3, X_1, X_2, X_3 \in \mathcal{G'}_{m-3,i-1}\Big\}.$$

(ii) *If $\mathcal{G'}_{m-2,i-1} = \mathcal{G'}_{m-3,i-1} = \emptyset$ and $\mathcal{G'}_{m-1,i-1} \neq \emptyset$, then*
$$\mathcal{G'}_{m,i} = \mathcal{G'}_{m,|V(G'(m))|} = \Big\{\{y_m\}\cup X | X \in \mathcal{G'}_{m-1,i-1}\Big\}.$$

(iii) *If $\mathcal{G'}_{m-1,i-1} = \emptyset, \mathcal{G'}_{m-2,i-1} \neq \emptyset$ and $\mathcal{G'}_{m-3,i-1} \neq \emptyset$, then*
$$\mathcal{G'}_{m,i} = \Big\{\{y_m\}\cup X_1, \{y_{m-1}\}\cup X_2 | X_1, X_2 \in \mathcal{G'}_{m-2,i-1}, y_{m-4}\in X_2, \text{ and } y_{m-4}\notin X_1\Big\}\cup$$
$$\cup\Big\{X \cup \begin{cases} \{y_{m-2}\}, & \text{if} \quad y_{m-5} \text{ or } y_{m-4} \in X \\ \{y_{m-1}\}, & \text{if} \quad y_{m-3} \in X \end{cases} |X \in \mathcal{G'}_{m-3,i-1}\Big\}.$$

(iv) *If $\mathcal{G'}_{m-1,i-1} \neq \emptyset, \mathcal{G'}_{m-2,i-1} \neq \emptyset$ and $\mathcal{G'}_{m-3,i-1} = \emptyset$, then*
$$\mathcal{G'}_{m,i} = \Big\{\{y_m\}\cup X_1, \{y_{m-1}\}\cup X_2 \mid X_1 \in \mathcal{G'}_{m-1,i-1}, X_2 \in \mathcal{G'}_{m-2,i-1}\Big\}.$$

(v) *If $\mathcal{G'}_{m-1,i-1} \neq \emptyset, \mathcal{G'}_{m-2,i-1} \neq \emptyset$ and $\mathcal{G'}_{m-3,i-1} \neq \emptyset$, then*
$$\mathcal{G'}_{m,i} = \Big\{\{y_m\}\cup X | X \in \mathcal{G'}_{m-1,i-1}\Big\}\cup$$
$$\cup\Big\{X_1\cup \begin{cases} \{y_m\}, & \text{if} \quad y_{m-2} \text{ for } y_{m-3}\in X_1, \text{ for } X_1 \in \mathcal{G'}_{m-2,i-1}\setminus \mathcal{G'}_{m-1,i-1} \\ \{y_{m-1}\}, & \text{if} \quad y_{m-2}\notin X_1, y_{m-3}\notin X_1 \text{ or } X_1 \in \mathcal{G'}_{m-1,i-1}\cap \mathcal{G'}_{m-2,i-1} \end{cases}\Big\}\cup$$
$$\cup\Big\{X_2\cup \begin{cases} \{y_{m-2}\}, & \text{if} \quad y_{m-5}\in X_2 \in \mathcal{G'}_{m-3,i-1} \text{ for } X_2 \in \mathcal{G'}_{m-3,i-1}\cap \mathcal{G'}_{m-2,i-1} \\ \{y_{m-1}\}, & \text{if} \quad y_{m-3} \text{ or } y_{m-4}\in X_2, \text{ for } X_2 \in \mathcal{G'}_{m-3,i-1}\setminus \mathcal{G'}_{m-2,i-1} \end{cases}\Big\}.$$

Proof. (i) $\mathcal{G'}_{m-1,i-1} = \mathcal{G'}_{m-2,i-1} = \emptyset$ and $\mathcal{G'}_{m-3,i-1} \neq \emptyset$. Suppose that $X \in \mathcal{G'}_{m-3,i-1}$. By Lemma 3.1, X contains at least one of vertices labeled y_{m-3}, y_{m-4} or y_{m-5}. It's obvious that
$$\Big\{\{y_{m-2}\}\cup X_1, \{y_{m-1}\}\cup X_2, \{y_m\}\cup X_3 | y_{m-5}\in X_1, y_{m-4}\in X_2, y_{m-3}\in X_3, X_1, X_2, X_3 \in \mathcal{G'}_{m-3,i-1}\Big\} \subseteq \mathcal{G'}_{m,i}.$$

Now, let $Y \in \mathcal{G'}_{m,i}$, then at least one of the vertices labeled y_m, y_{m-1} or y_{m-2} are in Y. Suppose that $y_m \in Y$. By Lemma 3.1 at least one of the vertices

y_{m-1}, y_{m-2} or y_{m-3} are in Y. If $y_{m-1} \in Y$, then $Y - \{y_m\} \in \mathcal{G}'_{m-1,i-1}$, a contradiction. If $y_{m-2} \in Y$ and $y_{m-1} \notin Y$, then $Y - \{y_m\} \in \mathcal{G}'_{m-2,i-1}$, a contradiction. If $y_{m-3} \in Y, y_{m-2} \notin Y$ and $y_{m-1} \notin Y$, then $Y - \{y_m\} \in \mathcal{G}'_{m-3,i-1}$. So $Y = X \cup \{y_m\}$ where $X \in \mathcal{G}'_{m-3,i-1}$ and $y_{m-3} \in X$. If y_{m-1} or y_{m-2} is in Y, we also have the result by a similar argument to that above. Hence in this case

$$\Big\{X_1 \cup \{y_{m-2}\}, X_2 \cup \{y_{m-1}\}, X_3 \cup \{y_m\} | y_{m-5} \in X_1, y_{m-4} \in X_2, y_{m-3} \in X_3, X_1, X_2, X_3 \in \mathcal{G}'_{m-3,i-1}\Big\} = \mathcal{G}'_{m,i}.$$

(ii) $\mathcal{G}'_{m-2,i-1} = \mathcal{G}'_{m-3,i-1} = \emptyset$ and $\mathcal{G}'_{m-1,i-1} \neq \emptyset$. By Lemma 3.5 (ii), $i = |V(G'(m))|$. Suppose that $G'(m)$ is labeled with numbers in $[|V(G'(m))|]$, then

$$\mathcal{G}'_{m,i} = \mathcal{G}'_{|V(G')|,|V(G')|} = \Big\{[|V(G')|]\Big\} = \Big\{\{y_m\} \cup X | X \in \mathcal{G}'_{m-1,i-1}\Big\}.$$

(iii) $\mathcal{G}'_{m-1,i-1} = \emptyset, \mathcal{G}'_{m-2,i-1} \neq \emptyset$, and $\mathcal{G}'_{m-3,i-1} \neq \emptyset$. We denote the families

$$\Big\{ \{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 | X_1, X_2 \in \mathcal{G}'_{m-2,i-1}, y_{m-4} \in X_2, y_{m-3} \text{ or } y_{m-2} \in X_1\Big\},$$

and

$$\Big\{X \cup \begin{cases} \{y_{m-2}\}, & \text{if} \quad y_{m-5} \text{ or } y_{m-4} \in X \\ \{y_{m-1}\}, & \text{if} \quad y_{m-3} \in X \end{cases} | X \in \mathcal{G}'_{m-3,i-1}\Big\}$$

by Y_1 and Y_2, respectively.

We shall prove that $\mathcal{G}'_{m,i} = Y_1 \cup Y_2$. Obviously $Y_1 \subseteq \mathcal{G}'_{m,i}$. Let $X \in \mathcal{G}'_{m-3,i-1}$, if y_{m-4} or y_{m-5} are in X, then $X \cup \{y_{m-2}\} \in \mathcal{G}'_{m,i}$. If $y_{m-3} \in X$, then $X \cup \{y_{m-1}\} \in \mathcal{G}'_{m,i}$. Therefore, $Y_2 \subseteq \mathcal{G}'_{m,i}$, and hence $Y_1 \cup Y_2 \subseteq \mathcal{G}'_{m,i}$.

Now let $Y \in \mathcal{G}'_{m,i}$, then by Lemma 3.1, at least one of the vertices labeled y_m, y_{m-1} or y_{m-2} is in Y. Suppose that $y_m \in Y$, then by Lemma 3.1, at least one of the vertices labeled y_{m-1}, y_{m-2} or y_{m-3} are in Y. If y_{m-1} or y_{m-2} are in Y, then $Y - \{y_m\} \in \mathcal{G}'_{m-1,i-1}$, a contradiction. If $y_{m-3} \in Y$, then $Y - \{y_m\} \in \mathcal{G}'_{m-2,i-1} \cap \mathcal{G}'_{m-3,i-1}$, that is $Y \in Y_1 \cup Y_2$.

Now suppose that $y_{m-1} \in Y$. By Lemma 3.1, at least one of the vertices labeled y_{m-2}, y_{m-3} or y_{m-4} are in Y. If $y_{m-2} \in Y$, then $Y - \{y_{m-1}\} \in \mathcal{G}'_{m-2,i-1}$, that is $Y \in Y_1$. If $y_{m-3} \in Y$, then $Y - \{y_{m-1}\} \in \mathcal{G}'_{m-2,i-1} \cap \mathcal{G}'_{m-3,i-1}$, that is $Y \in Y_1 \cup Y_2$. If $y_{m-4} \in Y$, then $Y - \{y_{m-1}\} \in \mathcal{G}'_{m-2,i-1}$, that is $Y \in Y_2$.

Finally suppose that $y_{m-2} \in Y$. By Lemma 3.1, at least one of the vertices labeled y_{m-3}, y_{m-4} or y_{m-5} are in Y. In each case $Y - \{y_{m-2}\} \in \mathcal{G}'_{m-3,i-1}$, that is $Y \in Y_2$. Therefore, we have $Y \subseteq Y_1 \cup Y_2$.

(iv) $\mathcal{G}'_{m-3,i-1} = \emptyset$, $\mathcal{G}'_{m-2,i-1} \neq \emptyset, \mathcal{G}'_{m-1,i-1} \neq \emptyset$. By Lemma 3.5 (iv), $i = |V(G'(m)| - 1$. Suppose that $G'(m)$ is labeled with numbers in $[|V(G'(m))|]$, then

$$\mathcal{G}'_{m,i} = \mathcal{G}'_{|V(G'(m))|,|V(G'(m))|-1} = \Big\{[|V(G'(m))|] - \{x\} | x \in [|V(G'(m))|]\Big\} = \\ = \Big\{ \{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \mid X_1 \in \mathcal{G}'_{m-1,i-1}, X_2 \in \mathcal{G}'_{m-2,i-1}\Big\}.$$

(v) ${\mathcal{G}'}_{m-1,i-1} \neq \emptyset, {\mathcal{G}'}_{m-2,i-1} \neq \emptyset$ and ${\mathcal{G}'}_{m-3,i-1} \neq \emptyset$. First, suppose that $X \in {\mathcal{G}'}_{m-1,i-1}$, then $X \cup \{y_m\} \in {\mathcal{G}'}_{m,i}$. So $Y_1 = \Big\{ \{y_m\} \cup X \mid X \in {\mathcal{G}'}_{m-1,i-1} \Big\} \subseteq {\mathcal{G}'}_{m,i}$. Now suppose that ${\mathcal{G}'}_{m-2,i-1} \neq \emptyset$. Let $X \in {\mathcal{G}'}_{m-2,i-1}$.
We denote

$$\Big\{ X \cup \begin{cases} \{y_m\}, & \text{if} \quad y_{m-2} \text{ or } y_{m-3} \in X, \text{ for } X \in {\mathcal{G}'}_{m-2,i-1} \setminus {\mathcal{G}'}_{m-1,i-1} \\ \{y_{m-1}\}, & \text{if} \quad y_{m-2} \notin X, y_{m-3} \notin X \text{ or } X \in {\mathcal{G}'}_{m-2,i-1} \cap {\mathcal{G}'}_{m-1,i-1} \end{cases} \Big\},$$

simply by Y_2. By Lemma 3.1, at least one of the vertices labeled y_{m-2}, y_{m-3} or y_{m-4} is in X. If y_{m-2} or y_{m-3} is in X, then $X \cup \{y_m\} \in {\mathcal{G}'}_{m,i}$, otherwise $X \cup \{y_{m-1}\} \in {\mathcal{G}'}_{m,i}$. Hence $Y_2 \subseteq {\mathcal{G}'}_{m,i}$. Here we shall consider ${\mathcal{G}'}_{m-3,i-1} \neq \emptyset$. Let $X \in {\mathcal{G}'}_{m-3,i-1}$. We denote

$$\Big\{ X \cup \begin{cases} \{y_{m-2}\}, & \text{if} \quad y_{m-5} \in X \text{ for } X \in {\mathcal{G}'}_{m-3,i-1} \text{ or} \\ & \qquad X \in {\mathcal{G}'}_{m-3,i-1} \cap {\mathcal{G}'}_{m-2,i-1} \\ \{y_{m-1}\}, & \text{if} \quad y_{m-3} \text{ or } y_{m-4} \in X \text{ for } X \in {\mathcal{G}'}_{m-3,i-1} \setminus {\mathcal{G}'}_{m-2,i-1} \end{cases} \Big\},$$

simply by Y_3. If y_{m-3} or y_{m-4} is in X, then $X \cup \{y_{m-1}\} \in {\mathcal{G}'}_{m,i}$, otherwise $X \cup \{y_{m-2}\} \in {\mathcal{G}'}_{m,i}$. Hence $Y_3 \subseteq Y$. Therefore we have proved that $Y_1 \cup Y_2 \cup Y_3 \subseteq {\mathcal{G}'}_{m,i}$.
Now suppose that $Y \in {\mathcal{G}'}_{m,i}$, so by Lemma 3.1, Y contains at least one of the vertices labeled y_m, y_{m-1} or y_{m-2}. If $y_m \in Y$, so again by Lemma 3.1 at least one of the vertices labeled y_{m-1}, y_{m-2} or y_{m-3} are in Y. In all cases $Y - \{y_m\} \in {\mathcal{G}'}_{m-1,i-1}$, so $Y \in Y_1$.
If $y_{m-1} \in Y$ and $y_m \notin Y$, by Lemma 3.1 at least one of the vertices labeled y_{m-2}, y_{m-3} or y_{m-4} are in Y. If $y_{m-2} \in Y$, then $Y - \{y_{m-1}\} \in {\mathcal{G}'}_{m-1,i-1} \cap {\mathcal{G}'}_{m-2,i-1}$, that is $Y \in Y_2$. If $y_{m-3} \in Y$ or $y_{m-4} \in Y$, then $Y = X \cup \{y_{m-1}\}$ for $X \in {\mathcal{G}'}_{m-3,i-1} \setminus {\mathcal{G}'}_{m-2,i-1}$, that is $Y \in Y_3$.
If $y_{m-2} \in Y, y_{m-1} \notin Y$, and $y_m \notin Y$. By Lemma 3.1 at least one of the vertices labeled y_{m-3}, y_{m-4} or y_{m-5} are in Y. In each case, $Y - \{y_{m-2}\} \in {\mathcal{G}'}_{m-3,i-1}$, that is $Y \in Y_3$. So we've proved that ${\mathcal{G}'}_{m,i} \subseteq Y_1 \cup Y_2 \cup Y_3$. □

Theorem 3.7. *If ${\mathcal{G}'}_{m,i}$ is the family of dominating sets of $G'(m)$ with cardinality i, then for every $m \geq 5$,*

$$|{\mathcal{G}'}_{m,i}| = |{\mathcal{G}'}_{m-1,i-1}| + |{\mathcal{G}'}_{m-2,i-1}| + |{\mathcal{G}'}_{m-3,i-1}|.$$

Proof. We consider five cases in Theorem 3.6:

(i) In this case $|{\mathcal{G}'}_{m,i}| = |{\mathcal{G}'}_{m-3,i-1}|$. Since $|{\mathcal{G}'}_{m-1,i-1}| = |{\mathcal{G}'}_{m-2,i-1}| = 0$, we have the result.

(ii) In this case $|{\mathcal{G}'}_{m,i}| = |{\mathcal{G}'}_{m-1,i-1}|$. Since $|{\mathcal{G}'}_{m-2,i-1}| = |{\mathcal{G}'}_{m-3,i-1}| = 0$, we have the result.

(iii) In this case ${\mathcal{G}'}_{m,i} = \mathcal{A}_1 \cup \mathcal{A}_2$, where $|\mathcal{A}_1| = |{\mathcal{G}'}_{m-2,i-1}|$ and $|\mathcal{A}_2| = |{\mathcal{G}'}_{m-3,i-1}|$. Suppose that $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. If $y_{m-4} \in A_1$, then $y_{m-1} \in A_1$. On the other hand for A_2, if $y_{m-4} \in A_2$, then $y_{m-2} \in A_2$ and $y_{m-1} \notin A_2$, so $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$, also if $y_{m-4} \notin A_2$ and $y_{m-3} \in A_2$, then $y_{m-1} \in A_2$, so again $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$, because y_{m-4}, y_{m-1} are in A_1 together. Finally, if $y_{m-4} \notin A_2$ and $y_{m-3} \notin A_2$, we have $y_{m-2} \in A_2$ and so $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$, because every $A_1 \in \mathcal{A}_1$ contains y_m or y_{m-1}. Therefore we have the result.

(iv) In this case $\mathcal{G'}_{m,i} = \mathcal{A}_1 \cup \mathcal{A}_2$, where $|\mathcal{A}_1| = |\mathcal{G'}_{m-1,i-1}|$ and $|\mathcal{A}_2| = |\mathcal{G'}_{m-2,i-1}|$. Since for every $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, $y_m \in A_1$ and $y_m \notin A_2$, then $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. Therefore we have the result.

(v) Here $\mathcal{G'}_{m,i} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where $|\mathcal{A}_1| = |\mathcal{G'}_{m-1,i-1}|$, $|\mathcal{A}_2| = |\mathcal{G'}_{m-2,i-1}|$ and $|\mathcal{G'}_{m-3,i-1}| = |\mathcal{A}_3|$. In a similar to argument the previous cases, we have for every $1 \leq i, j \leq 3$, $i \neq j$, $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$. Therefore,

$$\big|\mathcal{G'}_{m,i}\big| = \big|\mathcal{A}_1\big| + \big|\mathcal{A}_2\big| + \big|\mathcal{A}_3\big| = \big|\mathcal{G'}_{m-1,i-1}\big| + \big|\mathcal{G'}_{m-2,i-1}\big| + \big|\mathcal{G'}_{m-3,i-1}\big|. \qquad \square$$

Theorem 3.8. *For every $m \geq 5$,*

$$D(G'(m), x) = x\Big[D(G'(m-1), x) + D(G'(m-2), x) + D(G'(m-3), x)\Big].$$

Proof. It follows from Theorem 3.7 and the definition of the domination polynomial. $\square$

By Theorems 1.1 and 3.8 we have the following theorem, which expresses a recursive formula for the domination polynomial of non P_4-free graphs.

Theorem 3.9. *Let G be a graph which contains a simple path of length at least three. Then*

$$D(G, x) = x\Big[D(G * e_1, x) + D(G * e_1 * e_2, x) + D(G * e_1 * e_2 * e_3, x)\Big],$$

where e_1, e_2 and e_3 are three edges of the simple path, $G * e$ is the graph obtained from G by contracting the edge e, and $G * e_1 * e_2 = (G * e_1) * e_2$ and $G * e_1 * e_2 * e_3 = \big((G * e_1) * e_2\big) * e_3$.

4. DOMINATION POLYNOMIAL OF CYCLES AND GENERALIZED THETA GRAPHS

Let $C_n, n \geq 3$, be the cycle with n vertices $V(C_n) = [n]$ and $E(C_n) = \{(1,2), (2,3), \ldots, (n-1,n), (n,1)\}$ (see Fig. 2).

If G is P_2 with two vertices a and b, then $G'_{a,b}(n) = C_n$. So we use the results for $G'(m)$, to investigate the dominating sets and the domination polynomial of the cycle. Suppose that $\mathcal{D}(C_n, i)$ or simply $\mathcal{C}_n^i$ denote the family of dominating set of C_n with cardinality i.

Lemma 4.1 ([8, p.364])**.** $\gamma(C_n) = \lceil \frac{n}{3} \rceil$.

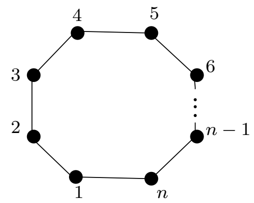

Fig. 2. The cycle C_n with vertices labeled $[n]$

Lemma 4.2. *Suppose that $C_n^i \neq \emptyset$, then:*

(i) $\mathcal{C}_{n-1}^{i-1} = \mathcal{C}_{n-2}^{i-1} = \emptyset$ *and* $\mathcal{C}_{n-3}^{i-1} \neq \emptyset$, *if and only if* $n = 3k$, $i = k$ *for some* $k \in \mathbb{N}$.
(ii) $\mathcal{C}_{n-2}^{i-1} = \mathcal{C}_{n-3}^{i-1} = \emptyset$ *and* $\mathcal{C}_{n-1}^{i-1} \neq \emptyset$, *if and only if* $i = n$.
(iii) $\mathcal{C}_{n-1}^{i-1} = \emptyset, \mathcal{C}_{n-2}^{i-1} \neq \emptyset$ *and* $\mathcal{C}_{n-3}^{i-1} \neq \emptyset$, *if and only if* $n = 3k+2$, $i = \lceil \frac{3k+2}{3} \rceil$ *for some* $k \in \mathbb{N}$.
(iv) $\mathcal{C}_{n-1}^{i-1} \neq \emptyset, \mathcal{C}_{n-2}^{i-1} \neq \emptyset$ *and* $\mathcal{C}_{n-3}^{i-1} = \emptyset$, *if and only if* $i = n-1$.
(v) $\mathcal{C}_{n-1}^{i-1} \neq \emptyset, \mathcal{C}_{n-2}^{i-1} \neq \emptyset$ *and* $\mathcal{C}_{n-3}^{i-1} \neq \emptyset$, *if and only if* $\lceil \frac{n-1}{3} \rceil + 1 \leq i \leq n-2$.

Proof. (i) ($\Rightarrow$) By Lemmas 4.1 and 3.5 (i), we have $\lceil \frac{n}{3} \rceil \leq i < \lceil \frac{n-2}{3} \rceil + 1$, which give us $n = 3k$ and $i = k$ for some $k \in \mathbb{N}$.
($\Leftarrow$) It follows from Lemmas 4.1 and 3.5 (i).
(ii) It follows from Lemma 3.5 (ii).
(iii) ($\Rightarrow$) By Lemmas 4.1 and 3.5 (iii), we have $\lceil \frac{n-2}{3} \rceil + 1 \leq i < \lceil \frac{n-1}{3} \rceil + 1$, which give us $n = 3k+2$ and $i = k+1 = \lceil \frac{3k+2}{3} \rceil$ for some $k \in \mathbb{N}$.
($\Leftarrow$) It follows from Lemmas 4.1 and 3.5 (iii).
(iv) It follows from Lemma 3.5 (iv).
(v) It follows from Lemmas 4.1 and 3.5 (v). □

By Theorem 3.6 we can now construct $\mathcal{C}_n^i$ from $\mathcal{C}_{n-1}^{i-1}, \mathcal{C}_{n-2}^{i-1}$ and $\mathcal{C}_{n-3}^{i-1}$.

Theorem 4.3. (i) *If* $\mathcal{C}_{n-1}^{i-1} = \mathcal{C}_{n-2}^{i-1} = \emptyset$ *and* $\mathcal{C}_{n-3}^{i-1} \neq \emptyset$, *then* $\mathcal{C}_n^i = \mathcal{C}_n^{\frac{n}{3}} = \Big\{\{1,4,\ldots,n-2\},\{2,5,\ldots,n-1\},\{3,6,\ldots,n\}\Big\}$.
(ii) *If* $\mathcal{C}_{n-2}^{i-1} = \mathcal{C}_{n-3}^{i-1} = \emptyset$ *and* $\mathcal{C}_{n-1}^{i-1} \neq \emptyset$, *then* $\mathcal{C}_n^i = \mathcal{C}_n^n = \big\{[n]\big\}$.
(iii) *If* $\mathcal{C}_{n-1}^{i-1} = \emptyset$, $\mathcal{C}_{n-2}^{i-1} \neq \emptyset$ *and* $\mathcal{C}_{n-3}^{i-1} \neq \emptyset$, *then*

$$\mathcal{C}_n^i = \Big\{\{1,4,\ldots,n-4,n-1\},\{2,5,\ldots,n-3,n\},\{3,6,\ldots,n-2,n\}\Big\} \cup$$

$$\left\{X \cup \begin{cases} \{n-2\}, & if \quad 1 \in X \\ \{n-1\}, & if \quad 1 \notin X, 2 \in X \\ \{n\}, & otherwise \end{cases} \;\middle|\; X \in \mathcal{C}_{n-3}^{i-1}\right\}.$$

(iv) *If* $\mathcal{C}_{n-3}^{i-1} = \emptyset, \mathcal{C}_{n-2}^{i-1} \neq \emptyset$ *and* $\mathcal{C}_{n-1}^{i-1} \neq \emptyset$, *then* $\mathcal{C}_n^i = \mathcal{C}_n^{n-1} = \Big\{[n] - \{x\} | x \in [n]\Big\}$.

(v) *If* $\mathcal{C}^{i-1}_{n-1} \neq \emptyset, \mathcal{C}^{i-1}_{n-2} \neq \emptyset$ *and* $\mathcal{C}^{i-1}_{n-3} \neq \emptyset$, *then* $\mathcal{C}^{i}_{n} = \Big\{ \{n\} \cup X \mid X \in \mathcal{C}^{i-1}_{n-1} \Big\} \cup$

$$\Big\{ X_1 \cup \begin{cases} \{n\}, & \text{if } n-2 \text{ or } n-3 \in X_1, \text{ for } X_1 \in \mathcal{C}^{i-1}_{n-2} \setminus \mathcal{C}^{i-1}_{n-1} \\ \{n-1\}, & \text{if } n-2 \notin X_1, n-3 \notin X_1 \text{ for } X_1 \in \mathcal{C}^{i-1}_{n-1} \cap \mathcal{C}^{i-1}_{n-2} \end{cases} \Big\} \cup$$

$$\Big\{ X_2 \cup \begin{cases} \{n-2\}, & \text{if } n-5 \in X_2, \text{ for } X_2 \in \mathcal{C}^{i-1}_{n-3} \text{ or } X_2 \in \mathcal{C}^{i-1}_{n-3} \cap \mathcal{C}^{i-1}_{n-2} \\ \{n-1\}, & \text{if } n-3 \in X_2 \text{ or } n-4 \in X_2, \text{ for } X_2 \in \mathcal{C}^{i-1}_{n-3} \setminus \mathcal{C}^{i-1}_{n-2} \end{cases} \Big\}.$$

Proof. (i) $\mathcal{C}^{i-1}_{n-1} = \mathcal{C}^{i-1}_{n-2} = \emptyset$ and $\mathcal{C}^{i-1}_{n-3} \neq \emptyset$. By Lemma 4.2 (i), $n = 3k, i = k$ for some $k \in \mathbb{N}$. Therefore
$\mathcal{C}^{i}_{n} = \mathcal{C}^{\frac{n}{3}}_{n} = \Big\{ \{1,4,7,\ldots,n-2\}, \{2,5,8,\ldots,n-1\}, \{3,6,9,\ldots,n\} \Big\}$.

(ii) $\mathcal{C}^{i-1}_{n-2} = \mathcal{C}^{i-1}_{n-3} = \emptyset$ and $\mathcal{C}^{i-1}_{n-1} \neq \emptyset$. By Lemma 4.2 (ii), $i = n$. Therefore
$\mathcal{C}^{i}_{n} = \mathcal{C}^{n}_{n} = \big\{ [n] \big\}$.

(iii) $\mathcal{C}^{i-1}_{n-1} = \emptyset, \mathcal{C}^{i-1}_{n-2} \neq \emptyset$, and $\mathcal{C}^{i-1}_{n-3} \neq \emptyset$. By Lemma 4.2 (iii), $n = 3k+2, i = \lceil \frac{3k+2}{3} \rceil = k+1$ for some $k \in \mathbb{N}$. Since
$\mathcal{C}^{i-1}_{n-2} = \mathcal{C}^{k}_{3k} = \Big\{ \{1,4,7,\ldots,3k-2\}, \{2,5,8,\ldots,3k-1\}, \{3,6,9,\ldots,3k\} \Big\}$,
then by Theorem 3.6 (iii),

$$\Big\{ \{n\} \cup X_1, \{n-1\} \cup X_2 | X_1, X_2 \in \mathcal{C}^{i-1}_{n-2}, n-4 \in X_2, n-4 \notin X_1 \Big\} =$$
$$= \Big\{ \{1,4,\ldots,3k-2,3k+1\}, \{2,5,\ldots,3k-1,3k+2\}, \{3,6,\ldots,3k,3k+2\} \Big\}.$$

Since the condition $1 \in X$ implies $n-4$ or $n-5$ are in X, therefore we have the result by Theorem 3.6 (iii) (note that since for cycle C_n, vertices labeled 1 and n are adjacent, so we may have $1 \notin X$ and $2 \notin X$, in this case $Y = X \cup \{n\}$ for some $X \in \mathcal{C}^{i-1}_{n-3}$).

(iv) $\mathcal{C}^{i-1}_{n-3} = \emptyset$, $\mathcal{C}^{i-1}_{n-1} \neq \emptyset, \mathcal{C}^{i-1}_{n-2} \neq \emptyset$. By Lemma 4.2 (iv), $i = n-1$. Therefore
$\mathcal{C}^{i}_{n} = \mathcal{C}^{n-1}_{n} = \Big\{ [n] - \{x\} | x \in [n] \Big\}$.

(v) It follows from Theorem 3.6 (v). □

Theorem 4.4. *If* $\mathcal{C}^{i}_{n}$ *is the family of the dominating set of* C_n *with cardinality* i, *then*

$$|\mathcal{C}^{i}_{n}| = |\mathcal{C}^{i-1}_{n-1}| + |\mathcal{C}^{i-1}_{n-2}| + |\mathcal{C}^{i-1}_{n-3}|.$$

Proof. It follows from Theorem 4.3. □

By definition of the domination polynomial and Theorem 4.4, we have the following theorem.

Theorem 4.5. *For every* $n \geq 4$,

$$D(C_n, x) = x\Big[D(C_{n-1}, x) + D(C_{n-2}, x) + D(C_{n-3}, x) \Big],$$

with the initial values $D(C_1, x) = x, D(C_2, x) = x^2 + 2x, D(C_3, x) = x^3 + 3x^2 + 3x$.

Remark 4.6. We have obtained Theorems 4.3, 4.4 and 4.5 in [5] with a different approach.

Table 1. $d(C_n, j)$ The number of dominating sets of C_n with cardinality j

n \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1													
2	2	1												
3	3	3	1											
4	0	6	4	1										
5	0	5	10	5	1									
6	0	3	14	15	6	1								
7	0	0	14	28	21	7	1							
8	0	0	8	38	48	28	8	1						
9	0	0	3	36	81	75	36	9	1					
10	0	0	0	25	102	150	110	45	10	1				
11	0	0	0	11	99	231	253	154	55	11	1			
12	0	0	0	3	72	282	456	399	208	66	12	1		
13	0	0	0	0	39	273	663	819	598	273	78	13	1	
14	0	0	0	0	14	210	786	1372	1372	861	350	91	14	1

Using Theorem 4.4, we obtain the coefficients of $D(C_n, x)$ for $1 \leq n \leq 14$ in Table 1. There are interesting relationships between the numbers $d(C_n, j)(\frac{n}{3} \leq j \leq n)$ in Table 1. In the following theorem, we state some properties of $d(C_n, j)$.

Theorem 4.7 ([5])**.**

(i) *$d(C_{3n}, n) = 3$, for each $n \in \mathbb{N}$. More precisely, if $n \equiv 0 (mod\, 3)$, then $d(C_n, \frac{n}{3}) = 3$.*
(ii) *$d(C_n, j) = d(C_{n-1}, j-1) + d(C_{n-2}, j-1) + d(C_{n-3}, j-1)$, for each $n \geq 4$, $j \geq \lceil \frac{n}{3} \rceil$.*
(iii) *$d(C_{3n+2}, n+1) = 3n+2$, for each $n \in \mathbb{N}$. More precisely, if $n \equiv 2 (mod\, 3)$, then $d(C_n, \frac{n+1}{3}) = n$.*
(iv) *$d(C_{3n+1}, n+1) = \frac{n(3n+7)+2}{2}$, for each $n \in \mathbb{N}$. More precisely, if $n \equiv 1 (mod\, 3)$, then $d(C_n, \frac{n+2}{3}) = \frac{n^2+5n}{6}$.*
(v) *$d(C_n, n) = 1$, for each $n \geq 3$.*
(vi) *$d(C_n, n-1) = n$, for each $n \geq 3$.*
(vii) *$d(C_n, n-2) = \frac{(n-1)n}{2}$, for each $n \geq 3$.*
(viii) *$d(C_n, n-3) = \frac{(n-4)(n)(n+1)}{6}$, for each $n \geq 4$.*
(ix) *$\sum_{i=j}^{3j} d(C_i, j) = 3 \sum_{i=j-1}^{3j-3} d(C_i, j-1)$, for every $j \geq 4$.*
(x) *$1 = d(C_n, n) < d(C_{n+1}, n) < d(C_{n+2}, n) < \ldots < d(C_{2n-1}, n) < d(C_{2n}, n) > d(C_{2n+1}, n) > \ldots > d(C_{3n-1}, n) > d(C_{3n}, n) = 3$, for each $n \geq 3$.*
(xi) *If $S_n = \sum_{j=\lceil \frac{n}{3} \rceil}^{n} |\mathcal{C}_n^j|$, then for every $n \geq 4$, $S_n = S_{n-1} + S_{n-2} + S_{n-3}$ with initial values $S_1 = 1, S_2 = 3$ and $S_3 = 7$.*

As another example of graphs $G'(m)$, we consider graph $C'_n(m)$, as show in Figure 3 (here we suppose that $a \neq b$). The following theorem states the recursive formula for the domination polynomial of $C'_n(m)$.

Theorem 4.8. *For every $n \geq 3$ and $m \geq 5$,*

$$D(C'_n(m), x) = x\Big[D(C'_n(m-1), x) + D(C'_n(m-2), x) + D(C'_n(m-3), x)\Big].$$

Proof. It follows from Theorem 3.7. □

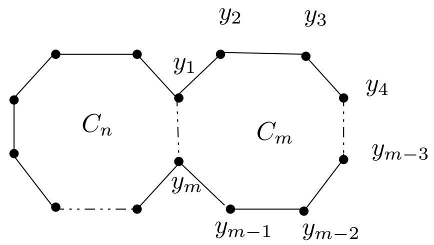

Fig. 3. The graph $C'_n(m)$

The generalized theta graphs $\Theta_{s_1,\ldots,s_k}$, are formed by taking a pair of vertices u, v (called the end vertices) and joining them by k internally disjoint paths of lengths $s_1, \ldots, s_k \geq 1$. (A generalized theta graph with three paths is traditionally called a theta graph without adjectives).

Obviously, the graph $C'_n(m)$ is a generalized theta graph. Therefore we have the following theorem for the domination polynomial of generalized theta graphs:

Theorem 4.9. *If there exist $1 \leq j \leq k$, such that $s_j \geq 3$, then*

$$\begin{aligned} D(\Theta_{s_1,\ldots,s_j,\ldots,s_k}, x) = x\Big[&D(\Theta s_1, \ldots, s_j - 1, \ldots, s_k, x) + \\ &+ D(\Theta s_1, \ldots, s_j - 2, \ldots, s_k, x) + \\ &+ D(\Theta s_1, \ldots, s_j - 3, \ldots, s_k, x)\Big]. \end{aligned}$$

5. ON THE GRAPHS G WITH $D(G, x) = D(C_n, x)$

We begin this section with the following theorem, which follows from the definitions of isomorphic graphs and the domination polynomial.

Theorem 5.1. *Let G and H be isomorphic. Then $D(G, x) = D(H, x)$.*

Here we want to characterize graphs G with $D(G, x) = D(C_n, x)$.

Theorem 5.2. *Let n be a natural number, and $n \equiv 0, 2\,(mod\,3)$. If $D(G, x) = D(C_n, x)$, then $G = C_n$.*

Proof. Assume that G is a graph such that $D(G,x) = D(C_n,x)$. By Theorem 2.4, G is 2-regular graph. Hence G is a union of finitely many disjoint cycles. Let G be the disjoint union of cycles $C_{m_1},\ldots,C_{m_r}$. First suppose that $n \equiv 0(mod\,3)$. Since $D(G,x) = \prod_{i=1}^{r} D(C_{m_i},x) = D(C_n,x)$ and $d(C_n,\frac{n}{3}) = 3$, by Theorem 4.7, we conclude that $r = 1$ and $G = C_n$.

Now, suppose that $n \equiv 2(mod\,3)$. If there exists some j, such that $m_j \equiv 0(mod\,3)$, noting that $d(C_n,\lceil\frac{n}{3}\rceil) = n$ and $d(C_{m_j},\frac{m_j}{3}) = 3$, we have $n \equiv 0(mod\,3)$, a contradiction. Thus for each i, $1 \le i \le k$, $m_i \equiv 1$ or $2(mod\,3)$. Clearly, for every natural number n, $n \le \frac{n^2+5n}{6}$. This implies that the coefficient of $x^{\lceil\frac{n}{3}\rceil}$ in the polynomial $D(G,x) = \prod_{i=1}^{r} D(m_i,x)$ is at least $\prod_{i=1}^{r} m_i$. By induction on r it is not hard to see that, if $r \ge 2$ and $m_1,\ldots,m_r$ are natural numbers more than 2, then $\prod_{i=1}^{r} m_i > \sum_{i=1}^{r} m_i = n = d(G,\lceil\frac{n}{3}\rceil)$, a contradiction. Therefore $r = 1$ and $G = C_n$. The proof is complete. □

Remark 5.3. Recently S. Akbari and M. R. Oboudi proved that the above theorem is also true for $n \equiv 1(mod\,3)$ ([2]).

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