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THE UPPER EDGE GEODETIC NUMBER AND THE FORCING EDGE GEODETIC NUMBER OF A GRAPH

Abstract. An *edge geodetic set* of a connected graph G of order $p \geq 2$ is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $g_1(G)$ of G is the minimum cardinality of its edge geodetic sets and any edge geodetic set of cardinality $g_1(G)$ is a *minimum edge geodetic set* of G or an *edge geodetic basis* of G. An edge geodetic set S in a connected graph G is a *minimal edge geodetic set* if no proper subset of S is an edge geodetic set of G. The *upper edge geodetic number* $g_1^+(G)$ of G is the maximum cardinality of a minimal edge geodetic set of G. The upper edge geodetic number of certain classes of graphs are determined. It is shown that for every two integers a and b such that $2 \leq a \leq b$, there exists a connected graph G with $g_1(G) = a$ and $g_1^+(G) = b$. For an edge geodetic basis S of G, a subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique edge geodetic basis containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing edge geodetic number of* S, denoted by $f_1(S)$, is the cardinality of a minimum forcing subset of S. The *forcing edge geodetic number of* G, denoted by $f_1(G)$, is $f_1(G) = min\{f_1(S)\}$, where the minimum is taken over all edge geodetic bases S in G. Some general properties satisfied by this concept are studied. The forcing edge geodetic number of certain classes of graphs are determined. It is shown that for every pair a, b of integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph G such that $f_1(G) = a$ and $g_1(G) = b$.

Keywords: geodetic number, edge geodetic basis, edge geodetic number, upper edge geodetic number, forcing edge geodetic number.

Mathematics Subject Classification: 05C12.

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [6]. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$

path in G. A $u-v$ path of length $d(u,v)$ is called a $u-v$ *geodesic*. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* G and the maximum eccentricity is its *diameter*, *diam* G of G. Two vertices u and v of G are called *antipodal* if $d(u,v)$=*diam* G. A vertex v is a *peripheral vertex* if $e(v)$=*diam* G. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices of S. The *geodetic number* $g(G)$ of G is the minimum cardinality of its geodetic sets and any geodetic set of cardinality $g(G)$ is a *minimum geodetic set* or a *geodetic basis* or a *g-set* of G. The geodetic number of a graph was introduced in [1, 7] and further studied in [2–4]. It was shown in [7] that determining the geodetic number of a graph is an NP-hard problem. The forcing geodetic number of a graph was introduced and studied in [5] . Santhakumaran et.al studied the connected geodetic number of a graph in [9] and the upper connected geodetic number and the forcing connected geodetic number of a graph in [10].

An *edge geodetic set* of G is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices of S. The *edge geodetic number* $g_1(G)$ of G is the minimum cardinality of its edge geodetic sets and any edge geodetic set of cardinality $g_1(G)$ is a *minimum edge geodetic set* of G or an *edge geodetic basis* of G or a g_1*-set* of G. The edge geodetic number of a graph was studied by Santhakumaran and John in [8]. These concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design.

For the graph G given in Figure 1, $S = \{v_3, v_5\}$ is a minimum geodetic set of G so that $g(G) = 2$. The edge v_1v_2 does not lie on any geodesic joining a pair of vertices in S so that S is not an edge geodetic set of G. However, $S_1 = \{v_1, v_2, v_4\}$ is a minimum edge geodetic set of G so that $g_1(G) = 3$. It is proved in [8] that for any connected graph G of order p, $2 \le g_1(G) \le p$ and no cut vertex of G belongs to any edge geodetic basis of G. Further, several interesting results and realization theorems were proved in [8].

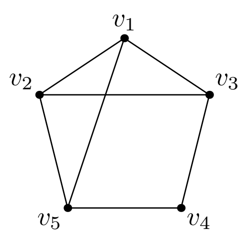

Fig. 1. Graph G

For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v to $V(H)$ is called a *branch* of G at v. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbours is complete. The following theorems will be used in the sequel.

Theorem 1.1 ([8]). *Each extreme vertex of a connected graph G belongs to every edge geodetic set of G. In particular, each end-vertex of G belongs to every edge geodetic set of G.*

Theorem 1.2 ([8]). *For any non-trivial tree T, the edge geodetic number $g_1(T)$ equals the number of end-vertices in T. In fact, the set of all end-vertices of T is the unique edge geodetic basis of T.*

Theorem 1.3 ([8]). *For the complete graph $G = K_p$ $(p \geq 2), g_1(G) = p$.*

Throughout the following G denotes a connected graph with at least two vertices.

2. THE UPPER EDGE GEODETIC NUMBER OF A GRAPH

Definition 2.1. An edge geodetic set S in a connected graph G is called a *minimal edge geodetic set* if no proper subset of S is an edge geodetic set of G. The *upper edge geodetic number* $g_1^+(G)$ of G is the maximum cardinality of a minimal edge geodetic set of G.

Example 2.2. For the graph G given in Figure 2, $S = \{v_2, v_4, v_5\}$ is an edge geodetic basis of G so that $g_1(G) = 3$. The set $S' = \{v_1, v_3, v_4, v_5\}$ is an edge geodetic set of G and it is clear that no proper subset of S' is an edge geodetic set of G and so S' is a minimal edge geodetic set of G. Since $|V(G)| = 5$, it follows that $g_1^+(G) = 4$.

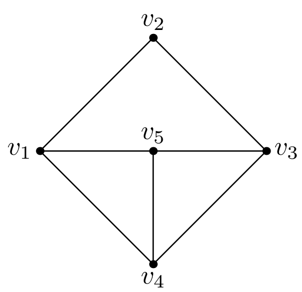

Fig. 2. Graph G

Remark 2.3. Every minimum edge geodetic set of G is a minimal edge geodetic set of G and the converse is not true. For the graph G given in Figure 2, $S' = \{v_1, v_3, v_4, v_5\}$ is a minimal edge geodetic set but not a minimum edge geodetic set of G.

Theorem 2.4. *Each extreme vertex of a connected graph G belongs to every minimal edge geodetic set of G.*

Proof. This follows from Theorem 1.1. □

Theorem 2.5. *For a connected graph G, $2 \leq g_1(G) \leq g_1^+(G) \leq p$.*

Proof. Any edge geodetic set needs at least two vertices and so $g_1(G) \geq 2$. Since every minimal edge geodetic set is an edge geodetic set, $g_1(G) \leq g_1^+(G)$. Also, since $V(G)$ is an edge geodetic set of G, it is clear that $g_1^+(G) \leq p$. Thus $2 \leq g_1(G) \leq g_1^+(G) \leq p$. □

Remark 2.6. The bounds in Theorem 2.5 are sharp. For any non-trivial path P, $g_1(P) = 2$. It follows from Theorem 1.2 and Theorem 1.3 that $g_1(T) = g_1^+(T)$ for any tree T and $g_1^+(K_p) = p$ $(p \geq 2)$ respectively. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 2, $g_1(G) = 3$, $g_1^+(G) = 4$ and $p = 5$.

Theorem 2.7. *For a connected graph G, $g_1(G) = p$ if and only if $g_1^+(G) = p$.*

Proof. Let $g_1^+(G) = p$. Then $S = V(G)$ is the unique minimal edge geodetic set of G. Since no proper subset of S is an edge geodetic set, it is clear that S is the unique minimum edge geodetic set of G and so $g_1(G) = p$. The converse follows from Theorem 2.5. □

Corollary 2.8. *For the complete graph $G = K_p$ $(p \geq 2)$, $g_1^+(G) = p$.*

Proof. This follows from Theorem 1.3 and Theorem 2.7. □

Theorem 2.9. *If G is a connected graph of order p with $g_1(G) = p-1$, then $g_1^+(G) = p-1$.*

Proof. Since $g_1(G) = p - 1$, it follows from Theorem 2.5 that $g_1^+(G) = p$ or $p - 1$. If $g_1^+(G) = p$, then by Theorem 2.7, $g_1(G) = p$, which is a contradiction. Hence $g_1^+(G) = p - 1$. □

Remark 2.10. The converse of the Theorem 2.9 is false. For the graph G given in Figure 2, $g_1^+(G) = 4 = p - 1$ and $g_1(G) = 3 = p - 2$.

Theorem 2.11. *Let G be a connected graph with cut-vertices and let S be minimal edge geodetic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S*

Proof. Suppose that there is a component B of $G - v$ such that B contains no vertex of S. By Theorem 1.1, B does not contain any end-vertex of G. Hence B contains at least one edge say uw. Since S is an edge geodetic set, there exist vertices $x, y \in S$ such that uw lies on some $x - y$ geodesic $P : x = u_0, u_1, u_2, \ldots, u, w, \ldots, u_t = y$ in G. Let P_1 be the $x - u$ subpath of P and P_2 be the $u - y$ subpath of P. Since v is a cut-vertex of G, both P_1 and P_2 contain v so that P is not a path, which is a contradiction. Thus every component of $G - v$ contains an element of S. □

Corollary 2.12. *Let G be a connected graph with cut-vertices and let S be a minimal edge geodetic set of G. Then every branch of G contains an element of S.*

Theorem 2.13. *No cut-vertex of a connected graph G belongs to any minimal edge geodetic set of G.*

Proof. Let S be any minimal edge geodetic set of G and let $v \in S$ be any vertex. We claim that v is not a cut vertex of G. Suppose that v is a cut vertex of G. Let $G_1, G_2, \ldots, G_r$ $(r \geq 2)$ be the components of $G - v$. Then v is adjacent to at least one vertex of G_i for every i $(1 \leq i \leq r)$. Let $S' = S - \{v\}$. Let uw be an edge of G which lies on a geodesic P joining a pair of vertices, say x and v of S. Assume without loss of generality that $x \in G_1$. Since v is adjacent to atleast one vertex of each G_i $(1 \leq i \leq r)$, assume that v is adjacent to a vertex y in G_k $(k \neq 1)$. Since S is an edge geodetic set, vy lies on a geodesic Q joining v and a vertex z of S such that z (possibly y itself) must necessarily belong to G_k. Thus $z \neq v$. Now, since v is a cut vertex of G, the union $P \cup Q$ of the two geodesics P and Q is obviously a geodesic in G joining x and z in S and thus the edge uw lies on this geodesic joining the two vertices x and z of S'. Thus we have proved that every edge that lies on a geodesic joining a pair of vertices x and v of S also lies on a geodesic joining two vertices of S'. Hence it follows that every edge of G lies on a geodesic joining two vertices of S', which shows that S' is an edge geodetic set of G. Since $|S'| = |S| - 1$, this contradicts the fact that S is a minimal edge geodetic set of G. Hence $v \notin S$. Thus no cut vertex of G belongs to any minimal edge geodetic set of G. □

Theorem 2.14. *For any tree T with k end-vertices, $g_1(T) = g_1^+(T) = k$.*

Proof. This follows from Theorem 1.2 and Theorem 2.13. □

Theorem 2.15. *For the complete bipartite graph $G = K_{m,n}$,*

(i) $g_1^+(G) = 2$ *if* $m = n = 1$.
(ii) $g_1^+(G) = n$ *if* $m = 1, n \geq 2$.
(iii) $g_1^+(G) = max\{m, n\}$ if $m, n \geq 2$.

Proof. (i) and (ii) follow from Theorem 2.14.
(iii) Let $m, n \geq 2$. Assume without loss of generality that $m \leq n$. First assume that $m < n$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be a bipartition of G. Let $S = Y$. We prove that S is a minimal edge geodetic set of G. Any edge $y_i x_j$ $(1 \leq i \leq n$ and $1 \leq j \leq m)$ lies on the geodesic $y_i x_j y_k$ for $k \neq i$ so that S is an edge geodetic set of G. Let $S' \subsetneq S$. Then there exists a vertex $y_j \in S$ such that $y_j \notin S'$. Then the edge $y_j x_i$ $(1 \leq i \leq m)$ does not lie on a geodesic joining a pair of vertices in S'. Thus S' is not an edge geodetic set of G. This shows that S is a minimal edge geodetic set of G. Hence $g_1^+(G) \geq n$.

Let S_1 be any minimal edge geodetic set of G such that $|S_1| \geq n+1$. Since any edge $x_i y_j$ $(1 \leq i \leq m$ and $1 \leq j \leq n)$ lies on the geodesic $x_i y_j x_k$ for any $k \neq i$, it follows that X is an edge geodetic set of G. Hence S_1 cannot contain X. Similarly, since Y is a minimal edge geodetic set of G, S_1 cannot contain Y also. Hence $S_1 \subsetneq X' \cup Y'$, where $X' \subsetneq X$ and $Y' \subsetneq Y$. Hence there exists a vertex $x_i \in X$ $(1 \leq i \leq m)$ and a vertex $y_j \in Y$ $(1 \leq j \leq n)$ such that $x_i, y_j \notin S_1$. Hence the edge $x_i y_j$ does not lie on a geodesic joining a pair of vertices in S_1. It follows that S_1 is not an edge geodetic set of G, which is a contradiction. Thus any minimal edge geodetic set of G contains at most n elements so that $g_1^+(G) \leq n$. Hence $g_1^+(G) = n$. Similarly, if $m = n$, $g_1^+(G) = m = n$. □

In view of Theorem 2.5, the following theorem gives a realization result.

Theorem 2.16. *For every two positive integers a and b, where $2 \le a \le b$, there exists a connected graph G with $g_1(G) = a$ and $g_1^+(G) = b$.*

Proof. If $a = b$, let $G = K_{1,a}$. Then by Theorem 2.14, $g_1(G) = g_1^+(G) = a$. So, let $2 \le a < b$. Let $V(K_2) = \{x, y\}$ and $V\ (K_{b-a+1}) = \{v_1, v_2, \ldots, v_{b-a+1}\}$. Let $H = \overline{K}_{b-a+1} + \overline{K}_2$. Let G be the graph in Figure 3 obtained from H by adding $a-1$ new vertices $u_1, u_2, \ldots, u_{a-1}$ and joining each vertex u_i $(1 \le i \le a-1)$ with y. Let $S = \{u_1, u_2, \ldots, u_{a-1}\}$. It is clear that S is not an edge geodetic set of G. Let $S' = S \cup \{x\}$. Then S' is an edge geodetic set of G and so by Theorem 1.1, S' is an edge geodetic basis of G. Hence $g_1(G) = a$.

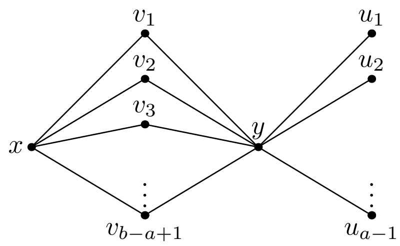

Fig. 3. Graph G

Now, $T = S \cup \{v_1, v_2, \ldots, v_{b-a+1}\}$ is an edge geodetic set of G. We show that T is a minimal edge geodetic set of G. Let W be any proper subset of T. Then there exists at least one vertex say $v \in T$ such that $v \notin W$. Assume first that $v = u_i$ for some i $(1 \le i \le a-1)$. Then the edge yu_i does not lie on any geodesic joining a pair of vertices in W and so W is not an edge geodetic set of G. Now, assume that $v = v_j$ for some j $(1 \le j \le b-a+1)$. Then the edges xv_j and yv_j do not lie on a geodesic joining any pair of vertices in W and so W is not an edge geodetic set of G. Hence T is a minimal edge geodetic set of G so that $g_1^+(G) \ge b$.

Now, we show that there is no minimal edge geodetic set X of G with $|X| \ge b+1$. Suppose that there exists a minimal edge geodetic set X of G such that $|X| \ge b+1$. Since $|V(G)| = b+2$ and since S' is an edge geodetic set of G, it follows that $|X| = b+1$. Now, by Theorem 2.13, $y \notin X$ and so $X = V(G) - \{y\}$. Since S' is an edge geodetic set of G, it follows that X is not a minimal edge geodetic set of G, which is a contradiction. Thus $g_1^+(G) = b$. □

Remark 2.17. Let $b - a \ge 2$ in Theorem 2.16. Suppose that there exists a minimal edge geodetic set M such that $a < |M| < b$. By Theorem 1.1, $S \subseteq M$ and so there exists at least one vertex v_i $(1 \le i \le b-a+1)$ such that $v_i \notin M$. Then the edges xv_i and yv_i do not lie on a geodesic joining any pair of vertices of M, which is a contradiction. Hence it follows that if k is an integer such that $a < k < b$, then there need not be a graph G with $g_1(G) = a$ and $g_1^+(G) = b$ containing a minimal edge geodetic set of cardinality k, that is, a graph G need not contain an "intermediate" minimal edge geodetic set.

3. THE FORCING EDGE GEODETIC NUMBER OF A GRAPH

The concept of "forcing subsets" was introduced and studied by Chartrand and Zhang in [7]. For each edge geodetic basis S in a connected graph G, there is always some subset T of S that uniquely determines S as the edge geodetic basis containing T. Such "forcing subsets" will be considered in this section.

Definition 3.1. Let G be a connected graph and S an edge geodetic basis of G. A subset $T \subseteq S$ is called a *forcing subset for* S if S is the unique edge geodetic basis containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset of* S. The *forcing edge geodetic number of* S, denoted by $f_1(S)$, is the cardinality of a minimum forcing subset of S. The *forcing edge geodetic number of* G, denoted by $f_1(G)$, is $f_1(G) = min\{f_1(S)\}$, where the minimum is taken over all edge geodetic bases S in G.

Example 3.2. For the graph G given in Figure 2, $S = \{v_2, v_4, v_5\}$ is the unique edge geodetic basis of G so that $f_1(G) = 0$ and for the graph G given in Figure 4, $S_1 = \{v_1, v_5, v_7\}$ and $S_2 = \{v_1, v_5, v_6\}$ are the only two edge geodetic bases of G. It is clear that $f_1(S_1) = f_1(S_2) = 1$ so that $f_1(G) = 1$.

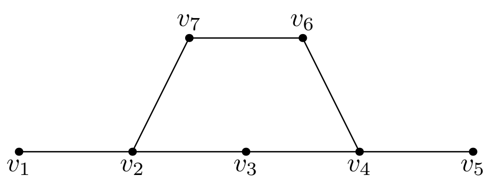

Fig. 4. Graph G

The next theorem follows immediately from the definition of the edge geodetic number and the forcing edge geodetic number of a connected graph G.

Theorem 3.3. *For every connected graph G, $0 \leq f_1(G) \leq g_1(G) \leq p$.*

Remark 3.4. The bounds in Theorem 3.3 are sharp. For the graph G given in Figure 2, $f_1(G) = 0$ and for the complete graph K_p $(p \geq 2)$, $g_1(K_p) = p$. Also, for the graph G given in Figure 4, $g_1(G) = 3$ and $f_1(G) = 1$. Thus $0 < f_1(G) < g_1(G)$.

The following theorem is an easy consequence of the definitions of the edge geodetic number, the forcing edge geodetic number and Theorem 2.5. In fact, the theorem characterizes graphs G for which the lower bound in Theorem 3.3 is attained and also graphs G for which $f_1(G) = 1$ and $f_1(G) = g_1(G)$.

Theorem 3.5. *Let G be a connected graph. Then*

(a) $f_1(G) = 0$ *if and only if G has a unique edge geodetic basis.*
(b) $f_1(G) = 1$ *if and only if G has at least two edge geodetic bases, one of which is a unique edge geodetic basis containing one of its elements, and*

(c) *$f_1(G) = g_1(G)$ if and only if no edge geodetic basis of G is the unique edge geodetic basis containing any of its proper subsets.*

Definition 3.6. A vertex v of a connected graph G is said to be an *edge geodetic vertex* of G if v belongs to every edge geodetic basis of G.

Example 3.7. For the graph G given in Figure 5, $S_1 = \{v_1, v_3, v_4\}$ and $S_2 = \{v_1, v_3, v_5\}$ are the only edge geodetic bases so that v_1 and v_3 are the edge geodetic vertices of G.

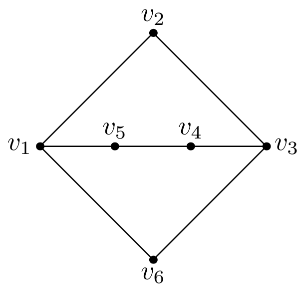

Fig. 5. Graph G

The following theorem and corollary follow immediately from the definitions of an edge geodetic vertex and a forcing subset.

Theorem 3.8. *Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective edge geodetic bases in G. Then $\bigcap_{F \in \Im} F$ is the set of edge geodetic vertices of G.*

Corollary 3.9. *Let G be a connected graph and S an edge geodetic basis of G. Then no edge geodetic vertex of G belongs to any minimum forcing set of S.*

Theorem 3.10. *Let G be a connected graph and W be the set of all edge geodetic vertices of G. Then $f_1(G) \leq g_1(G) - |W|$.*

Proof. Let S be any edge geodetic basis of G. Then $g_1(G) = |S|$, $W \subseteq S$ and S is the unique edge geodetic basis containing $S - W$. Thus $f_1(G) \leq |S - W| = |S| - |W| = g_1(G) - |W|$. □

Corollary 3.11. *If G is a connected graph with k extreme vertices, then $f_1(G) \leq g_1(G) - k$.*

Proof. This follows from Theorem 1.1 and Theorem 3.10. □

Remark 3.12. The bound in Theorem 3.10 is sharp. For the graph G given in Figure 5, $S_1 = \{v_1, v_3, v_5\}$, $S_2 = \{v_1, v_3, v_4\}$ are the only two g_1-sets so that $g_1(G) = 3$ and $f_1(G) = 1$. Also, $W = \{v_1, v_3\}$ is the set of all edge geodetic vertices of G and so $f_1(G) = g_1(G) - |W|$. Also, the inequality in Theorem 3.10 can be strict. For the graph G given in Figure 6, $S_1 = \{v_1, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_6\}$ and $S_3 = \{v_1, v_3, v_5\}$

are the only three g_1-sets of G so that $g_1(G) = 3$ and $f_1(G) = 1$. Now, v_1 is the only edge geodetic vertex of G and so $f_1(G) < g_1(G) - |W|$.

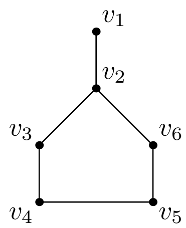

Fig. 6. Graph G

Now, we proceed to determine the forcing edge geodetic numbers of certain classes of graphs.

Theorem 3.13. *For any even cycle $G = C_p$ $(p \geq 4)$, a set $S \subseteq V(G)$ is an edge geodetic basis if and only if S consists of two antipodal vertices.*

Proof. If S consists of two antipodal vertices, then it is clear that S is an edge geodetic basis of C_p. Conversely, let S be any edge geodetic basis of C_p. Then $g_1(C_p) = |S|$. Let S' be any set of two antipodal vertices of C_p. Then, as in the first part of this theorem, S' is an edge-geodetic basis of C_p. Hence $|S'| = |S|$. Thus S consists of two vertices, say $S = \{u, v\}$. If u and v are not antipodal, then any edge that is not on the $u - v$ geodesic does not lie on the $u - v$ geodesic. Thus S is not an edge geodetic basis, which is a contradiction. □

Corollary 3.14. *For an even cycle C_p $(p \geq 4)$, $g_1(C_p) = 2$.*

Proof. This follows from Theorem 3.13. □

Theorem 3.15. *For any cycle C_p $(p \geq 4)$,*

$$f_1(C_p) = \begin{cases} 1 & \textit{if } p \textit{ is even,} \\ 2 & \textit{if } p \textit{ is odd.} \end{cases}$$

Proof. If p is even, then by Theorem 3.13, every g_1-set of C_p consists of a pair of antipodal vertices. Hence C_p has $\frac{p}{2}$ g_1-sets and it is clear that each singleton set is the minimum forcing set for exactly one g_1-set of C_p. Hence it follows from Theorem 3.5 (a) and (b) that $f_1(C_p) = 1$.

Let p be odd. Let $p = 2n + 1$. Let the cycle be $C : v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n+1}, v_1$. If $S = \{u, v\}$ is any set of two vertices of C_p, then no edge of the $u - v$ longest path lies on the $u - v$ geodesic in C_p and so no two element subset of C_p is an edge geodetic set of C_p. Now, it is clear that the sets $S_1 = \{v_1, v_{n+1}, v_{n+2}\}, S_2 = \{v_2, v_{n+2}, v_{n+3}\}, \ldots, S_{n+2} = \{v_{n+2}, v_1, v_2\}, \ldots, S_{2n+1} = \{v_{2n+1}, v_n, v_{n+1}\}$ are g_1-sets of C_p. (Note that there are more g_1-sets of C_p, for example, $S' = \{v_1, v_{n+1}, v_{n+3}\}$ is a g_1-set different from these). It is clear from the g_1-sets S_i $(1 \leq i \leq 2n + 1)$

that each $\{v_i\}$ $(1 \leq i \leq 2n+1)$ is a subset of more than one g_1-set S_i. Hence it follows from Theorem 3.5 (a) and (b) that $f_1(C_p) \geq 2$. Now, since v_{n+1} and v_{n+2} are antipodal to v_1, it is clear that S_1 is the unique g_1-set containing $\{v_{n+1}, v_{n+2}\}$ and so $f_1(C_p) = 2$. □

Theorem 3.16. *For any complete graph $G = K_p$ $(p \geq 2)$ or any non-trivial tree $G = T$, $f_1(G) = 0$*

Proof. For $G = K_p$, it follows from Theorem 1.1 that the set of all vertices of G is the unique edge geodetic basis. Now, it follows from Theorem 3.5 (a) that $f_1(G) = 0$. If G is a non-trivial tree, then by Theorem 1.2, the set of all end-vertices of G is the unique edge geodetic basis of G and so $f_1(G) = 0$ by Theorem 3.5 (a). □

Theorem 3.17. *For the complete bipartite graph $G = K_{m,n}$ $(m, n \geq 2)$,*

$$f_1(G) = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

Proof. Without loss of generality, assume that $m \leq n$. First assume that $m < n$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G. Let $S = U$. We prove that S is an edge geodetic basis of G. Any edge u_iw_j $(1 \leq i \leq m, 1 \leq j \leq n)$ lies on the geodesic $u_iw_ju_k$ for any $k \neq i$ so that S is an edge geodetic set of G. Let T be any set of vertices such that $|T| < |S|$. If $T \subsetneqq U$, then there exists a vertex $u_i \in U$ such that $u_i \notin T$. Then for any edge u_iw_j $(1 \leq j \leq n)$, the only geodesics containing u_iw_j are $u_iw_ju_k$ $(k \neq i)$ and $w_ju_iw_l$ $(l \neq j)$ and so u_iw_j cannot lie on a geodesic joining two vertices of T. Thus T is not an edge geodetic set of G. If $T \subsetneqq W$, again T is not an edge geodetic set of G by a similar argument. If $T \subsetneqq U \cup W$ such that T contains at least one vertex from each of U and W, then, since $|T| < |S|$, there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin T$ and $w_j \notin T$. Then, clearly the edge u_iw_j does not lie on a geodesic connecting two vertices of T so that T is not an edge geodetic set. Thus in any case, T is not an edge geodetic set of G. Hence S is an edge geodetic basis so that $g_1(K_{m,n}) = |S| = m$. Now, let S_1 be a set of vertices such that $|S_1| = m$. If S_1 is a subset of W, then since $m < n$, there exists a vertex $w_j \in W$ such that $w_j \notin S_1$. Then the edge u_iw_j $(1 \leq i \leq m)$ does not lie on a geodesic joining a pair vertices in S_1. If $S_1 \subsetneqq U \cup W$ such that S_1 contains at least one vertex from each of U and W, then since $S_1 \neq U$, there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin S_1$ and $w_j \notin S_1$. Then, clearly the edge u_iw_j does not lie on a geodesic joining two vertices of S_1 so that S_1 is not an edge geodetic set of G. It follows that U is the unique edge geodetic basis of G. Hence it follows from Theorem 3.5 (a) that $f_1(G) = 0$.

Now, let $m = n$. Then, as in the first part of this theorem, both U and W are edge geodetic bases of G. Now, let S' be any set of vertices such that $|S'| = m$ and $S' \neq U, W$. Then there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin S'$ and $w_j \notin S'$. Then, as earlier, S' is not an edge geodetic set of G. Hence it follows that U and W are the only two edge geodetic bases of G. Since U is the unique edge geodetic basis containing $\{u_i\}$, it follows that $f_1(G) = 1$. □

Theorem 3.18. *If $S = \{u, v\}$ is an edge geodetic basis of a connected graph G, then u and v are two antipodal vertices of G.*

Proof. Let $S = \{u, v\}$ be an edge geodetic basis for G. Then every edge of G lies on a geodesic joining u and v. Hence every vertex of G also lies on a geodesic joining u and v. Let $d(G)$ denote the diameter of G. We claim that $d(u, v) = d(G)$. If $d(u, v) < d(G)$, then let x and y be two vertices of G such that $d(x, y) = d(G)$. Now, it follows that x and y lie on distinct geodesics joining u and v. Hence

$$d(u, v) = d(u, x) + d(x, v) \tag{3.1}$$

and

$$d(u, v) = d(u, y) + d(y, v). \tag{3.2}$$

By the triangle inequality,

$$d(x, y) \leq d(x, u) + d(u, y). \tag{3.3}$$

Since $d(u, v) < d(x, y)$, (3.3) becomes

$$d(u, v) < d(x, u) + d(u, y). \tag{3.4}$$

Using (3.4) in (3.1), we get $d(x, v) < d(x, u) + d(u, y) - d(u, x) = d(u, y)$. Thus,

$$d(x, v) < d(u, y). \tag{3.5}$$

Also, by triangle inequality, we have

$$d(x, y) \leq d(x, v) + d(v, y). \tag{3.6}$$

Now, using (3.5) and (3.2), (3.6) becomes $d(x, y) < d(u, y) + d(v, y) = d(u, v)$. Thus, $d(G) < d(u, v)$, which is a contradiction. Hence $d(u, v) = d(G)$ so that u and v are antipodal vertices. □

Theorem 3.19. *If G is a connected graph with $g_1(G) = 2$, then $f_1(G) \leq 1$.*

Proof. Let $S = \{u, v\}$ be any edge geodetic basis of G. Then by Theorem 3.18, u and v are antipodal vertices of G. Suppose that $f_1(G) = 2$. Then $f_1(S) = 2$. Hence it follows that S is not the unique g_1-set containing u. Then there exists $x \neq u$ such that $S' = \{u, x\}$ is also a g_1-set of G. By Theorem 3.18, u and x are two antipodal vertices of G. Hence v is an internal vertex of some $u - x$ geodesic in G. Therefore, $d(u, v) < d(u, x)$, which is a contradiction. □

Next we show that every pair a, b of integers with $0 \leq a < b$ and $b \geq 2$ can be realized as the forcing edge geodetic number and the edge geodetic number respectively of some graph.

Theorem 3.20. *For every pair a, b of integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph G such that $f_1(G) = a$ and $g_1(G) = b$.*

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 3.16, $f_1(G) = 0$ and by Theorem 1.3, $g_1(G) = b$. Thus, we assume that $0 < a < b$. We consider four cases.
Case 1. $a = 1$. If $b = 2$, then for any even cycle G, $g_1(G) = b$ by Corollary 3.14 and $f_1(G) = a$ by Theorem 3.15. So, we assume that $b \geq 3$. Let G be the graph in Figure 7 obtained from the cycle $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ by first adding the $b - 2$ new vertices $u_1, u_2, \ldots, u_{b-2}$ and the $b - 2$ edges v_1u_i $(1 \leq i \leq b - 2)$. Let $U = \{u_1, u_2, \ldots, u_{b-2}\}$ be the set of all end-vertices of G. Then U is not an edge geodetic set of G. Hence it follows from Theorem 1.1 that $S_1 = U \cup \{v_3, v_4\}, S_2 = U \cup \{v_3, v_5\}$ and $S_3 = U \cup \{v_2, v_4\}$ are the only three g_1-sets of G. Thus $g_1(G) = b$. Moreover, since S_2 is the unique g_1-set containing $\{v_5\}$, it follows that $f_1(S_2) = 1$ and so $f_1(G) = 1$.

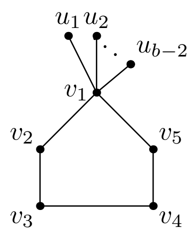

Fig. 7. Graph G

Case 2. $a = 2$. If $b = 3$, then for any odd cycle G of order at least 5, $g_1(G) = 3 = b$, as in the proof of Theorem 3.15 and $f_1(G) = a$ by Theorem 3.15. Now, let $b \geq 4$. Let H be the graph obtained from the cycle $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ by first adding a new vertex x and joining the edges xv_1 and xv_4. Now, let G be the graph in Figure 8 obtained from H by adding $(b-3)$ new vertices $u_1, u_2, \ldots, u_{b-3}$ and joining the edges xu_i $(1 \leq i \leq b - 3)$. Let $U = \{u_1, u_2, \ldots, u_{b-3}\}$ be the set of all end-vertices of G. Then U is not an edge geodetic set of G. Hence it follow from Theorem 1.1 that $S_1 = U \cup \{v_1, v_2, v_4\}$, $S_2 = U \cup \{v_1, v_3, v_4\}$, $S_3 = U \cup \{v_2, v_3, v_5\}$, $S_4 = U \cup \{v_1, v_3, v_5\}$ and $S_5 = U \cup \{v_2, v_4, v_5\}$ are the only five g_1-sets of G. Thus $g_1(G) = b$. It is clear that no singleton subset of any S_i is a forcing subset of S_i. Moreover, since S_1 is the unique g_1-set containing $\{v_1, v_2\}$, it follows that $f_1(S_1) = 2$ and so $f_1(G) = 2 = a$.

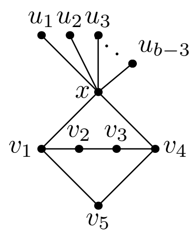

Fig. 8. Graph G

Case 3. $a \geq 3$ and $b = a + 1$. For each integer i with $0 \leq i \leq b$, let $F_i : u_i, v_i$ be a path of order 2. Then the graph G given in Figure 9 is obtained from the graph F_i

by adding the $2b$ edges u_0u_j, v_0v_j for all j with $1 \leq j \leq b$. First we show that $g_1(G) = b$. Let $U = \{u_1, u_2, \ldots, u_b\}$ and $W = \{v_1, v_2, \ldots, v_b\}$. We observe that a set S of vertices of G is a g_1-set if and only if S has the following two properties (3.1) S contains exactly one vertex from each set $\{u_j, v_j\}$ $(1 \leq j \leq b)$ and (3.2) $S \cap U \neq \phi$ and $S \cap W \neq \phi$. Then (3.1) implies that $g_1(G) \geq b$. Since $S' = \{u_1, u_2, v_3, v_4, \ldots, v_b\}$ is an edge geodetic set of G with $|S'| = b$, it follows that $g_1(G) = b = a + 1$.

Now, we prove that $f_1(G) = a$. First assume that a g_1-set, say S_1 contains exactly one vertex from U or W. Without loss of generality, let $S_1 = \{u_1, v_2, v_3, v_4, \ldots, v_b\}$ be a g_1-set of G. We claim that $f_1(G) = b - 1$. Let T be a subset of S_1 such that $|T| \leq b - 2$. Then there exist at least two vertices say $x, y \in S_1$ such that $x, y \notin T$. Suppose that $x = u_1$ and $y = v_j$ for some j $(2 \leq j \leq b)$. Now, $S_2 = (S_1 - \{v_j\}) \cup \{u_j\}$ satisfies (3.1) and (3.2) and so S_2 is a g_1-set such that $T \subseteq S_2$. Therefore S_1 is not the unique g_1-set containing T and so T is not a forcing subset of S_1. Suppose that $x = v_i$ for some i $(2 \leq i \leq b)$ and $y = v_j$ for some j $(2 \leq j \leq b)$ and $i \neq j$. Now, $S_3 = (S_1 - \{v_i, v_j\}) \cup \{u_i, u_j\}$ satisfies (3.1) and (3.2) and so S_3 is a g_1-set containing T. Hence T is not a forcing subset of S_1 and so $f_1(S_1) \geq b - 1$. Now, it is clear that S_1 is the unique g_1-set containing $\{v_2, v_3, v_4, \ldots, v_b\}$ so that $f_1(S_1) = b - 1$.

Next assume that any g_1-set contains at least two vertices from each U and W (This is possible since $b \geq 4$). Without loss of generality, let $S = \{u_1, u_2, v_3, v_4, \ldots, v_b\}$ be a g_1-set of G. Let T be any proper subset of S and $x \in S - T$. If $x = u_i$ for $i = 1, 2$, then $S' = (S - \{u_i\}) \cup \{v_i\}$ satisfies properties (3.1) and (3.2). Thus S' is a g_1-set of G such that $T \subsetneqq S'$. Since $S' \neq S$ and $T \subsetneqq S$, it follows that S is not the unique g_1-set containing T. Similarly, if $x = v_i$ for some i $(3 \leq i \leq b)$, then $S^* = (S - \{v_i\}) \cup \{u_i\}$ is a g_1-set distinct from S and $T \subsetneqq S^*$. Thus S is not the unique g_1-set containing T and so $f_1(S) = b = a + 1$. Hence it follows that $f_1(G) = b - 1 = a$.

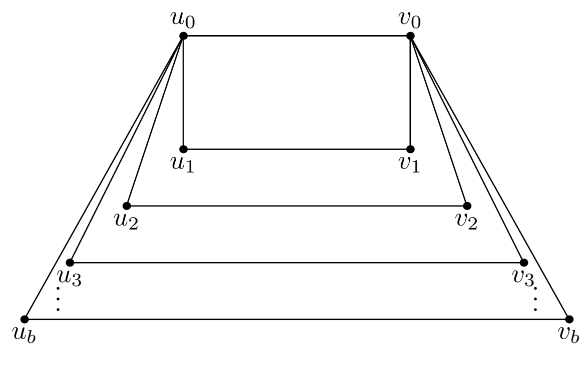

Fig. 9. Graph G

Case 4. $a \geq 3$ and $b \neq a + 1$. Let $F_i : u_i, v_i, w_i, x_i, u_i (1 \leq i \leq a)$ be a copy of C_4. Let G be the graph obtained from $F_i (1 \leq i \leq a)$ by first identifying the vertices x_{i-1} of F_{i-1} and u_i of F_i $(2 \leq i \leq a)$ and then adding $b - a$ new vertices $z_1, z_2, \ldots, z_{b-a-1}, u$ and joining the $b - a$ edges u_1z_i $(1 \leq i \leq b - a - 1)$ and x_au. The graph G is given in Figure 10. Let $Z = \{z_1, z_2, \dots, z_{b-a-1}, u\}$ be the set of end-vertices of G. Let $H_i = \{v_i, w_i\}$ $(1 \le i \le a)$.

First we show that $g_1(G) = b$. Since none of the edges u_iv_i, v_iw_i and w_ix_i of F_i $(1 \le i \le a)$ lies on a geodesic joining a pair of vertices of Z, Z is not an edge geodetic set of G. We observe that every edge geodetic set of G must contain at least one vertex from H_i $(1 \le i \le a)$. Thus $g_1(G) \ge b - a + a = b$. On the other hand, since the set $S_1 = Z \cup \{v_1, v_2, \dots, v_a\}$ is an edge geodetic set of G, it follows that $g_1(G) \le |S_1| = b$. Thus $g_1(G) = b$.

Next we show that $f_1(G) = a$. Since every g_1-set of G contains Z, it follows from Theorem 3.10 that $f_1(G) \le g_1(G) - |Z| = b - (b - a) = a$. Now, since $g_1(G) = b$ and every edge geodetic basis of G contains Z, it is easily seen that every edge geodetic basis S is of the form $Z \cup \{c_1, c_2, \dots, c_a\}$, where $c_i \in H_i (1 \le i \le a)$. Let T be any proper subset of S with $|T| < a$. Then there is a vertex c_j $(1 \le j \le a)$ such that $c_j \notin T$. Let d_j be a vertex of H_j distinct from c_j. Then $S_2 = (S - \{c_j\}) \cup \{d_j\}$ is a g_1-set properly containing T. Thus S is not the unique g_1-set containing T and so T is not a forcing subset of S. This is true for all edge geodetic bases of G and so it follows that $f_1(G) = a$. □

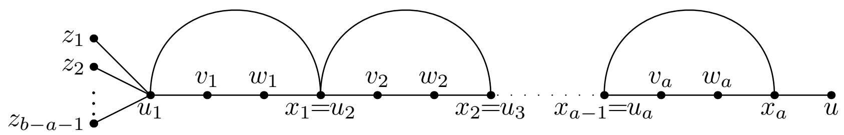

Fig. 10. Graph G

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