Denise Amar, Evelyne Flandrin, Grzegorz Gancarzewicz

CYCLABILITY IN BIPARTITE GRAPHS

Abstract. Let $G = (X, Y; \mathrm{E})$ be a balanced 2-connected bipartite graph and $S \subset \mathrm{V}(G)$. We will say that S is *cyclable* in G if all vertices of S belong to a common cycle in G. We give sufficient degree conditions in a balanced bipartite graph G and a subset $S \subset \mathrm{V}(G)$ for the cyclability of the set S.

Keywords: graphs, cycles, bipartite graphs.

Mathematics Subject Classification: 05C20, 05C35, 05C38, 05C45.

1. INTRODUCTION

We shall consider only finite graphs without loops and multiple edges.

Several authors have given results about cycles containing specific subsets of vertices, see for example [7] or [9].

The set S of vertices is called *cyclable* in G if all vertices of S belong to a common cycle in G. We also speak about *cyclability* or *noncyclability* of the vertex set S.

In a bipartite graph $G = (X, Y; \mathrm{E})$ we will call the independent sets of vertices X and Y *the partite sets.*

Let $G = (X, Y; \mathrm{E})$ be a bipartite graph and let $S \subset \mathrm{V}(G)$, then $S_X = S \cap X$ and $S_Y = S \cap Y$. We will say that S is *balanced* iff $|S_X| = |S_Y|$.

In 1992 Shi Ronghua [8] obtained the following result:

Theorem 1.1. *Let G be a 2-connected graph of order n and S a subset of $V(G)$ with $|S| \geq 3$. If for every pair of nonadjacent vertices x and y in S we have*

$$\mathrm{d}(x) + \mathrm{d}(y) \geq n,$$

then S is cyclable in G.

Note that the assumption of 2-connectivity may be omitted in Theorem 1.1. It is an easy corollary of a result of K. Ota [7].

Recently R. Čada, E. Flandrin and Z. Ryjáček [3] proved the following generalization of Theorem 1.1:

Theorem 1.2. *Let G be a 2-connected graph of order n and S a subset of $V(G)$. If for every pair of nonadjacent vertices x and y in S we have*

$$\mathrm{d}(x) + \mathrm{d}(y) \geq n - 1,$$

then either S is cyclable in G, or n is odd and G contains an independent set $S_1 \subseteq S$ such that $|S_1| = \frac{n}{2}$ and every vertex of S_1 is adjacent to all vertices in $G \setminus S_1$.

In 2002 E. Flandrin, H. Li, A. Marczyk and M. Woźniak [4] obtained the following generalization of Theorem 1.1:

Theorem 1.3. *Let G be a k-connected graph, $k \geq 2$ of order n. Denote $S_1, \dots S_k$ subsets of the vertex set $\mathrm{V}(G)$ and let $S = S_1 \cup S_2 \cup \dots \cup S_k$. If for any x, $y \in S_i$, $xy \notin E$ we have*

$$\mathrm{d}(x) + \mathrm{d}(y) \geq n,$$

then S is cyclable in G.

The notion of cyclability is a generalization of the term of hamiltonicity. If we consider $S = \mathrm{V}(G)$ then S is cyclable iff G is hamiltonian. In fact Theorem 1.1 is a generalization of the following result of O. Ore [6]:

Theorem 1.4. *Let G be a graph on $n \geq 3$ vertices. If for all nonadjacent vertices $x, y \in \mathrm{V}(G)$ we have*

$$\mathrm{d}(x) + \mathrm{d}(y) \geq n,$$

then G is hamiltonian.

A similar result for bipartite graphs was proved by J. Moon and M. Moser [5] in 1963:

Theorem 1.5. *Let $G = (X, Y; \mathrm{E})$ be a balanced bipartite graph of order $2n$. If for all nonadjacent vertices $x \in X$ and $y \in Y$ we have*

$$\mathrm{d}(x) + \mathrm{d}(y) \geq n + 1,$$

then G is hamiltonian.

Given a balanced bipartite graph and a selected subset of vertices, we are interested in properties that imply cyclability.

In 2000 D. Amar, M. El Kadi Abderrezzak, E. Flandrin [2] proved the following generalization of Theorem 1.1 for bipartite graphs:

Theorem 1.6. *Let $G = (X, Y; \mathrm{E})$ be a balanced 2-connected bipartite graph of order $2n$, $S \subset X$. If for every $x \in S$, $y \in Y$, $xy \notin \mathrm{E}$ we have*

$$\mathrm{d}(x) + \mathrm{d}(y) \geq n + 1,$$

then S is cyclable in G.

Note that in this case $S = S_X$ and Theorem 1.6 is also a generalization of Theorem 1.5.

The main result of the present paper, given in Section 3, is Theorem 3.1, which improves upon Theorem 1.6.

2. DEFINITIONS

Let G be a graph and H a subgraph of G.

Definition 2.1. $\mathrm{N}_G(H)$ denotes the set of all vertices of the graph G which are adjacent to a vertex of the subgraph H, i.e. $\mathrm{N}_G(H) = \{u \in \mathrm{V}(G) : \exists v \in \mathrm{V}(H)$ such that $uv \in \mathrm{E}(G)\}$.

Consider an arbitrary vertex $x \in \mathrm{V}(G)$. $\mathrm{N}(x)$ denotes the set of all neighbors of the vertex x in G, i.e. $\mathrm{N}(x) = \{u \in \mathrm{V}(G) : xu \in \mathrm{E}(G)\}$. $\mathrm{N}_H(x)$ denotes the set of all neighbors of the vertex x in the subgraph H, i.e. $\mathrm{N}_H(x) = \{u \in \mathrm{V}(H) : xu \in \mathrm{E}(G)\}$.

$\mathrm{d}_H(x)$ denotes the number of neighbors of x in the subgraph H i.e. $\mathrm{d}_H(x) = |\mathrm{N}_H(x)|$, and $\mathrm{d}_H(x)$ denotes *the degree of the vertex x in the subgraph H*.

In the proof we will only use cycles and paths with a given orientation. For a cycle $C : c_1 \dots c_k$ or a path $P : p_1 \dots p_l$ we will use implicit orientation.

Thus it makes sense to speak of a *successor* c_{i+1} and a *predecessor* c_{i-1} of a vertex c_i (addition modulo $l+1$). Denote the successor of a vertex x by x^+ and its predecessor by x^-. This notation can be extended to $A^+ = \{x^+ : x \in A\}$, and similarly, to A^- when $A \subseteq \mathrm{V}(G)$.

Let P be a path $p_1 \dots p_k$ and u, $v \in \mathrm{V}(G)$ such that up_1, $vp_k \in \mathrm{E}(G)$, then: uPv is the path $up_1 \dots p_k v$ and vPu is the path $vp_k \dots p_1 u$.

Definition 2.2. We shall call a path $P : p_1 \dots p_l$ a C-path of the cycle C iff $\mathrm{V}(P) \cap \mathrm{V}(C) = \{p_1, p_l\}$. Note that a C-path is a generalized chord of the cycle.

Definition 2.3. Let $C : c_1 \dots c_l$ be a cycle in G with the orientation, the indices $1, \dots, l$ are considered modulo l. For any pair of vertices c_i, $c_j \in \mathrm{V}(C)$ $(i \neq j)$ we define four intervals:

- $]c_i, c_j[$ is the path $c_{i+1} \dots c_{j-1}$.
- $[c_i, c_j[$ is the path $c_i \dots c_{j-1}$.
- $]c_i, c_j]$ is the path $c_{i+1} \dots c_j$.
- $[c_i, c_j]$ is the path $c_i \dots c_j$.

Note that these four intervals are subsets of the cycle C.

For notation and terminology not defined above a good reference is [1].

3. THEOREM

Theorem 3.1. *If $G = (X, Y; \mathrm{E})$ is a balanced 2-connected bipartite graph of order $2n$ and $S \subset \mathrm{V}(G)$ satisfying conditions:*

$$\textit{For every} \quad x \in S_X, \quad y \in Y, \quad xy \notin \mathrm{E} \quad \textit{we have} \quad \mathrm{d}(x) + \mathrm{d}(y) \geq n+1 \tag{3.1}$$

$$\textit{For every} \quad x \in X, \quad y \in S_Y, \quad xy \notin \mathrm{E} \quad \textit{we have} \quad \mathrm{d}(x) + \mathrm{d}(y) \geq n+1 \tag{3.2}$$

then S is cyclable in G.

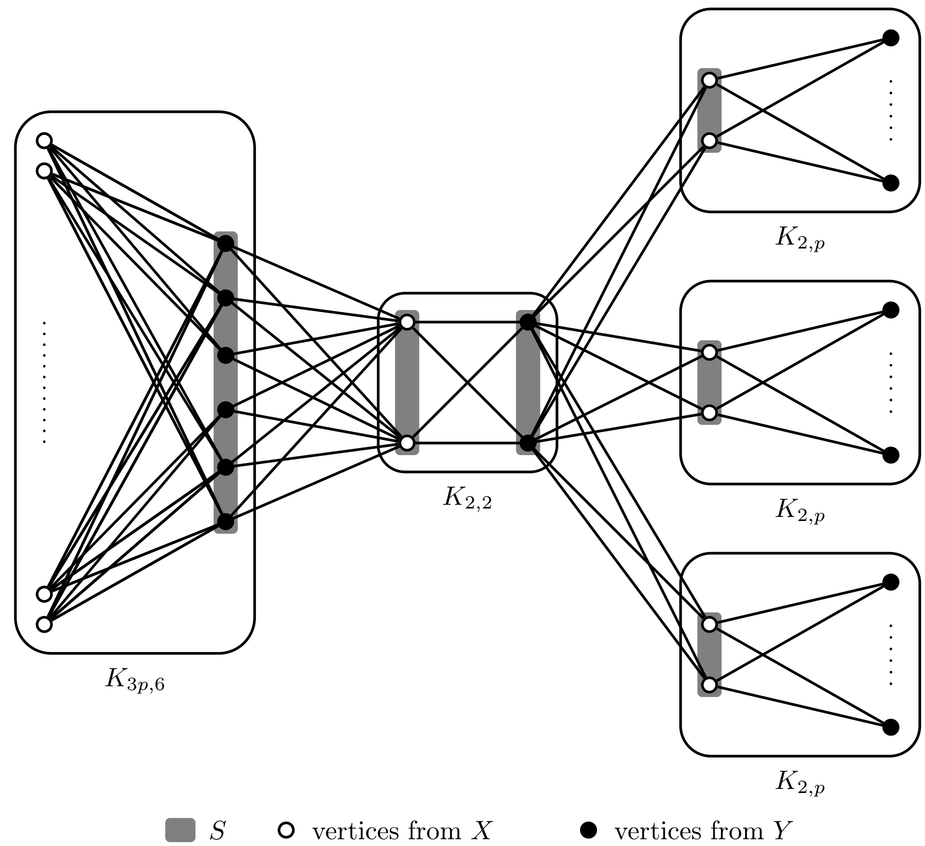

Fig. 1. $G_{p,3}$

Theorem 3.1 is obviously a generalization of Theorem 1.6.

We first tried to find a generalization satisfying two conditions:

– The vertices of S are in both partite sets X and Y.
– The degree sum condition holds only for vertices from S.

However, even if we assume that S is balanced (i.e. $|S_X| = |S_Y|$), such a result is not true.

For every $k \geq 1$ we will give an example of a 2-connected, balanced bipartite graph $G = (X, Y; \mathrm{E})$ and a balanced set $S \subset \mathrm{V}(G)$, satisfying the following condition:

$$\text{For every } x \in S_X,\ y \in S_Y \text{ if } xy \notin \mathrm{E} \quad \text{then} \quad \mathrm{d}(x) + \mathrm{d}(y) \geq n + k, \tag{3.3}$$

such that S is not cyclable in G.

Let $k \geq 1$, $p \geq k + 2$ and $2 \leq r \leq 1 + \dfrac{p-k}{2}$.

First consider bipartite graphs $K_{pr,2r}$, $K_{2,2}$ and r copies of $K_{2,p}$. In $K_{2,2}$ we have two partite sets say X_2 and Y_2. The graph $G = (X, Y; \mathrm{E})$ is obtained out of $K_{pr,2r}$, $K_{2,2}$ and the r copies of $K_{2,p}$ by joining every vertex of degree pr from $K_{pr,2r}$ with all vertices from X_2 and every vertex of degree p from the r copies of $K_{2,p}$ with all vertices from Y_2.

Let S_1 be the set of all vertices from $K_{pr,2r}$ of degree pr in $K_{pr,2r}$.

In each copy of $K_{2,p}$ we take the two vertices of degree p in $K_{2,p}$. In this way we will get $2r$ vertices and we define the set S_2 as the set containing these $2r$ vertices.

We can define now the set S. Let $S = S_1 \cup S_2 \cup \mathrm{V}(K_{2,2})$.

For $k \geq 1$, $p \geq k+2$ and $2 \leq r \leq 1 + \dfrac{p-k}{2}$ we have obtained a balanced, 2-connected bipartite graph $G_{p,r} = (X, Y; \mathrm{E})$ of order $2n$ with $n = pr + 2r + 2$ and a balanced set S which is not cyclable, but satisfies (3.3).

We can find an example of the graph $G_{p,3}$ on the Figure 1.

This example shows that it is not enough to assume that the degree sum condition holds only for the vertices from S in a bipartite graph. Even increasing the connectivity will not be sufficient, as we can see in the following example.

For every $k \geq 1$ and $l \geq 2$ we will give an example of an l-connected, balanced bipartite graph $G' = (X, Y; \mathrm{E})$ and a balanced set $S' \subset \mathrm{V}(G')$, satisfying (3.3), such that S' is not cyclable in G'.

Let $k \geq 1$, $l \geq 2$, $p \geq l^2 - l + k$ and $l \leq r < 1 + \dfrac{p-k}{l}$.

First consider bipartite graphs $K_{pr,lr}$, $K_{l,l}$ and r copies of $K_{l,p}$. In $K_{l,l}$ we have two partite sets say X_l and Y_l. The graph $G' = (X, Y; \mathrm{E})$ is obtained out of $K_{pr,lr}$, $K_{l,l}$ and the r copies of $K_{l,p}$ by joining every vertex of degree pr from $K_{pr,lr}$ with all vertices from X_l and every vertex of degree p from the r copies of $K_{l,p}$ with all vertices from Y_l.

Let S_1' be the set of all vertices from $K_{pr,lr}$ of degree pr in $K_{pr,lr}$.

In each copy of $K_{l,p}$ we take the l vertices of degree p in $K_{l,p}$. In this way we will get lr vertices and we define the set S_2' as the set containing these lr vertices.

We can define now the set S'. Let $S' = S_1' \cup S_2' \cup \mathrm{V}(K_{l,l})$.

For $k \geq 1$, $l \geq 2$, $p \geq l^2 - l + k$ and $l \leq r < 1 + \dfrac{p-k}{l}$ we have obtained a balanced, l-connected bipartite graph $G'_{p,r,l} = (X, Y; \mathrm{E})$ of order $2n$ with $n = pr + lr + l$ and a balanced set S' which is not cyclable in G', but satisfies (3.3).

4. PROOF OF THEOREM 3.1

4.1. PRELIMINARY NOTATIONS

Let $G = (X, Y; \mathrm{E})$ be a bipartite graph and let C be a cycle in G.

In this chapter for a given cycle C and a vertex $x \in \mathrm{V}(G \setminus C)$, a C-path Q through x will be denoted $Q : uQ_1xQ_2u'$, where Q_1 and Q_2 are two vertex disjoint paths. The end vertices of the C-path Q : u and u' and the vertex x do not belong to Q_1 nor Q_2.

Note that the path Q_1 may be empty or in other words $\mathrm{V}(Q_1) = \emptyset$ and in this case $xu \in \mathrm{E}$. Similarly for Q_2.

An example of a C-path $P : uP_1xP_2u'$ through a vertex x can be found on Figure 2.

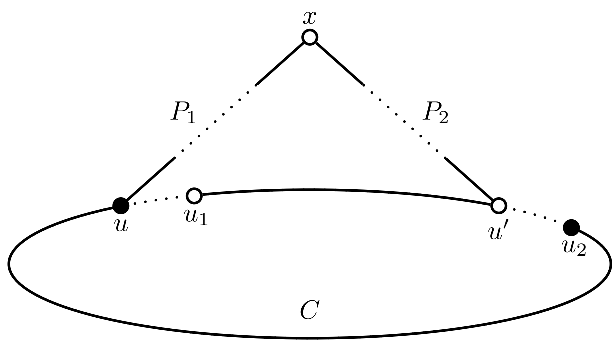

Fig. 2. An example of a cycle C and a C-path P with x, u', $u_1 \in X$ and u, $u_2 \in Y$

Remark 4.1. Given a 2-connected graph G, a nonhamiltonian cycle C and a vertex $x \in \mathrm{V}(G \setminus C)$, G contains necessarily a C-path through x.

In the remaining part of Section 4 we will always consider a 2-connected bipartite graph G and a subset $S \subset \mathrm{V}(G)$ not cyclable in G. Given a cycle C, a vertex $x \in \mathrm{V}(G \setminus C) \cap S$ such that C contains $S \setminus \{x\}$ but does not contain S, we will denote by P a C-path through x. We will always assume that the cycle C and the C-path P are chosen such that P is shortest possible among all C-paths through x for all cycles C containing $S \setminus \{x\}$, i.e. for any cycle C' containing $S \setminus \{x\}$, and for any C'-path P' containing the vertex x we have $|\mathrm{V}(P)| \leq |\mathrm{V}(P')|$. We will denote this C-path $P : uP_1xP_2u'$ (note that P_1 and/or P_2 may be empty).

We will denote by u_1 the first vertex on the cycle C from S after u (u_1 exists since S is not cyclable). Similarly u_2 is the first vertex on the cycle C from S after u'.

R is the subgraph induced in G by $\mathrm{V}(G) \setminus \mathrm{V}(C)$.

All the intervals of type $[a, b]$, $[a, b[$, $]a, b]$ and $]a, b[$ are intervals on the cycle C and we sometimes identify the vertex set of an interval with the corresponding interval.

Remark 4.2. The C-path $P : uP_1xP_2u'$ has the following properties:

– If $\mathrm{V}(P_1) \neq \emptyset$ or $\mathrm{V}(P_2) \neq \emptyset$ then $\mathrm{d}_C(x) \leq 1$. (4.1)

– If $\mathrm{V}(P_1) \neq \emptyset$ and $\mathrm{V}(P_2) \neq \emptyset$ then $\mathrm{d}_C(x) = 0$. (4.2)

Remark 4.2 is an immediate consequence of the choice of the cycle C and the C-path P.

4.2. FORMULATION AND PROOF OF LEMMA 4.3

In the proof of Theorem 3.1 we shall use the following lemma. Notations G, S, C, R, x and C-path $P : uP_1xP_2u'$, u_1 and u_2 are defined in Section 4.1 but we recall them for completeness. We denote by C a cycle containing $S \setminus \{x\}$. Let P be a C-path through x. The cycle C and the C-path P are chosen such that P is shortest possible among all C-paths through x for all cycles C containing $S \setminus \{x\}$. Let u_1 be the first vertex from S after u on the cycle C, and let u_2 be the first vertex from S after u' on the cycle C. The subgraph of G induced by $\mathrm{V}(G) \setminus \mathrm{V}(C)$, will be denoted R.

Lemma 4.3. *Let $G = (X, Y; \mathrm{E})$ be a 2-connected bipartite graph and let C, P, R and S be as above. Then we have:*

- *For every $C - path$ $Q : aQ_1xQ_2a'$ through x we have:*
  $\mathrm{V}(]a, a'[) \cap S \neq \emptyset$ *and* $\mathrm{V}(]a', a[) \cap S \neq \emptyset$. (4.3)
- *For any $b \in \mathrm{V}(]u, u_1])$ and $c \in \mathrm{V}(]u', u_2])$ we have:*
  $\mathrm{N}_{P_1xP_2}(b) = \mathrm{N}_{P_1xP_2}(c) = \emptyset$. (4.4)
- *If $z \in \mathrm{V}(P) \setminus \{u, u'\}$, then* $\mathrm{N}_R(z) \cap (\mathrm{N}(]u, u_1]) \cup \mathrm{N}(]u, u_2])) = \emptyset$. (4.5)
- $\mathrm{N}_R(]u, u_1]) \cap \mathrm{N}_R(]u', u_2]) = \emptyset$. (4.6)
- *For any $y \in \mathrm{N}_R(x)$ we have* $\mathrm{N}_{]u, u_1]}(y) = \mathrm{N}_{]u', u_2]}(y) = \emptyset$. (4.7)

Proof of Lemma 4.3. Suppose that $\mathrm{V}(]a, a'[) \cap S = \emptyset$, then the cycle:

$$C' : \quad aQ_1xQ_2a'a'^{+} \dots a, \tag{4.8}$$

is a cycle containing S, a contradiction. If $\mathrm{V}(]a', a[) \cap S = \emptyset$, then using similar arguments we get a contradiction and hence (4.3) is proved.

In order to prove (4.4) suppose that there is a vertex $b \in \mathrm{V}(]u, u_1])$ such that $\mathrm{N}_{P_1xP_2}(b) \neq \emptyset$. We have a vertex $z \in \mathrm{N}_{P_1xP_2}(b)$ and we assume that the vertices on the path P_1xP_2 are labeled as follows: $P_1xP_2 : \; p_1^1 \dots p_1^l x p_2^k \dots p_2^1$.

We shall consider three cases.

1. When $z = x$, then the following cycle:

$$C' : \quad uP_1xbb^{+} \dots u' \dots u_2 \dots u, \tag{4.9}$$

contains S, a contradiction.

2. If $z \in \mathrm{V}(P_1)$, then the following cycle:

$$C' : \quad up_1^1 \dots p_1^i zbb^{+} \dots u_1 \dots u' \dots u_2 \dots u, \tag{4.10}$$

contains $S \setminus \{x\}$ and has a C'-path

$$P' : \quad zp_1^{i+2} \dots xP_2u', \tag{4.11}$$

shorter then P, a contradiction with the choice of C and P.

3. If $z \in \mathrm{V}(P_2)$, then the following cycle:

$$C' : \quad uP_1xp_2^k \dots p_2^j zbb^+ \dots u' \dots u_2 \dots u, \tag{4.12}$$

contains S, a contradiction.

So we have $\mathrm{N}_{P_1xP_2}(b) = \emptyset$. Using similar arguments we can prove that for any $c \in \mathrm{V}(]u', u_2])$ $\mathrm{N}_{P_1xP_2}(c) = \emptyset$, and hence (4.4) is true.

We will prove now (4.5).

Suppose that $\mathrm{N}_R(z) \cap (\mathrm{N}(]u, u_1]) \cup \mathrm{N}(]u', u_2])) \neq \emptyset$.

So we have a vertex $w \in \mathrm{N}_R(z) \cap (\mathrm{N}(]u, u_1]) \cup \mathrm{N}(]u', u_2]))$. Without loss of generality we can assume that $w \in \mathrm{N}_R(z) \cap \mathrm{N}(]u, u_1])$. Let $a \in \mathrm{V}(]u, u_1])$, such that $aw \in \mathrm{E}$. From (4.4) we know that $w \notin \mathrm{V}(P_1xP_2)$.

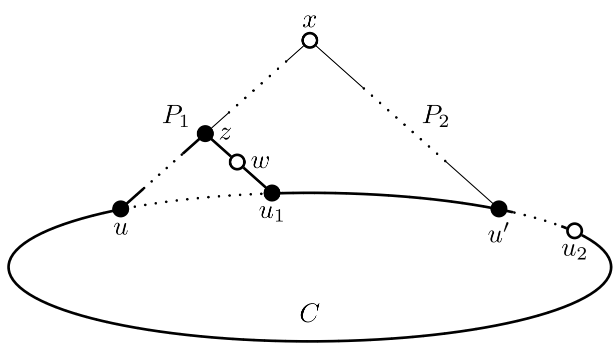

Fig. 3. u_2, w, $x \in X$ and u, u_1, u', $z \in Y$

As in the proof of (4.4), we shall consider three cases:

1. $z \in \mathrm{V}(P_1)$.
2. $z \in \mathrm{V}(P_2)$.
3. $z = x$.

Using similar arguments we get contradiction.

In any case we obtain a contradiction by replacing in (4.10), (4.12) and (4.9), the edge za by the path $zwua$. Hence (4.5) is true.

For $a = u_1$, you can find the illustrations of Cases 1 and 2 on Figures 3 and 4 respectively.

In order to prove (4.6), suppose that $b \in \mathrm{V}(]u, u_1])$, $c \in \mathrm{V}(]u', u_2])$ and $z \in \mathrm{N}_R(\{b, c\})$.

From (4.4) we know that $\mathrm{N}_{P_1xP_2}(b) = \mathrm{N}_c(P_1xP_2) = \emptyset$, and so $z \notin \mathrm{V}(P)$. In this case the cycle:

$$C' : \quad uP_1xP_2u'u'^- \dots u_1 \dots bzc \dots u_2u_2^+ \dots u$$

contains S, a contradiction.

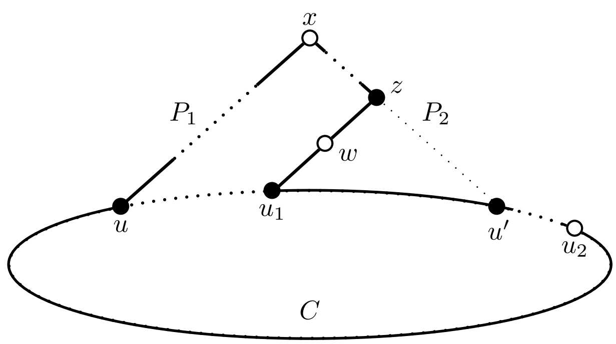

Fig. 4. u_2, w, $x \in X$ and u, u_1, u', $z \in Y$

Hence (4.6) is true.

In order to prove (4.7), suppose that there is a vertex $y \in \mathrm{N}_R(x)$ such that $\mathrm{N}_{]u,\,u_1]}(y) \neq \emptyset$. From (4.4) we know that $y \notin \mathrm{V}(P_1 x P_2)$. We have a vertex $b \in \mathrm{V}(]u, u_1])$ such that $yb \in \mathrm{E}$ and the following cycle:

$$C': \quad uP_1xybb^+ \dots u_1 \dots u' \dots u_2 \dots u$$

contains S, a contradiction and so $\mathrm{N}_{]u,\,u_1]}(y) = \emptyset$.

Using the same arguments we can prove $\mathrm{N}_{]u,\,u_2]}(y) = \emptyset$. Hence (4.7) is true and the proof of Lemma 4.3 is finished.

□

4.3. PROOF OF THEOREM 3.1

We may assume that $S_Y \neq \emptyset$ and $|S_Y| \geq |S_X|$. We will proceed by induction over the number of vertices in S_X.

If $|S_X| = 0$, then $S = S_Y$ and from Theorem 1.6 we know that S is cyclable in G. So the first step of the induction is finished.

Suppose now that S satisfies the assumptions of Theorem 3.1 and $|S_X| \geq 1$.

From the induction hypothesis, we assume that for any $x \in S_X$ the set $S \setminus \{x\}$ is cyclable in G, while S itself is not cyclable. Let us choose a vertex $x \in S_X$.

We have a cycle C containing $S \setminus \{x\}$ such that $x \notin \mathrm{V}(C)$. We recall that the cycle C and the C-path P are chosen such that P is shortest possible among all C-paths containing x for all cycles C containing $S \setminus \{x\}$. As in Section 4.1, u_1 is the first vertex from S on the cycle C after u and u_2 is the first vertex from S on the cycle C after u', R is the subgraph induced in G by $\mathrm{V}(G) \setminus \mathrm{V}(C)$.

It is clear that in this case R is a balanced bipartite graph.

Note that if $c = |\mathrm{V}(C)|$, $r = |\mathrm{V}(R)|$, then c and r are even and $n = \dfrac{c+r}{2}$.

From Remark 4.2 and Lemma 4.3 C and P satisfy (4.3) — (4.7).

We shall consider four cases:

1. $N_R(x) = \emptyset$.
2. $N_R(x) \neq \emptyset$ and $u_1, u_2 \in S_Y$.
3. $N_R(x) \neq \emptyset$ and u_1, $u_2 \in S_X$.
4. $N_R(x) \neq \emptyset$ and u_1 and u_2 are in different partite sets.

Case 1. $N_R(x) = \emptyset$

In this case P_1 and P_2 are empty and xu, $xu' \in E$.

Since R is balanced there is an $y \in Y \cap V(R)$. Since $xy \notin E$ then from (3.1) we have:

$$d(x) + d(y) \geq n + 1. \tag{4.13}$$

Since $N_R(x) = \emptyset$ we have:

$$d_R(x) = 0 \quad \text{and} \quad d_R(y) \leq \frac{r}{2} - 1\,. \tag{4.14}$$

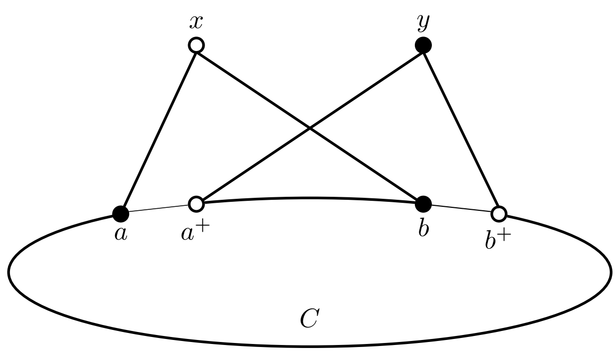

Fig. 5. a^+, b^+, $x \in X$ and a, b, $y \in Y$

Suppose that y has two neighbors a^+, b^+ in $N_C(x)^+$, then xa, $xb \in E$ and the cycle C' (see Fig. 5):

$$C': \quad xbb^- \ldots a^+yb^+b^{++} \ldots ax$$

contains S, a contradiction with noncyclability of S. So y has at most one neighbor in $N_C(x)^+$ and thus:

$$d_C(x) + d_C(y) \leq \frac{c}{2} + 1. \tag{4.15}$$

From (4.14) and (4.15) we have:

$$d(x) + d(y) \leq \frac{r}{2} - 1 + \frac{c}{2} + 1 \leq n,$$

a contradiction with (3.1).

Case 2. $N_R(x) \neq \emptyset$ **and** $u_1, u_2 \in S_Y$

Subcase 2.1. There is a vertex $x_0 \in V(]u, u_1[) \cap X$ **or** $x_0 \in V(]u', u_2[) \cap X$

We can assume that $x_0 \in V(]u, u_1[) \cap X$.

Since $V(]u, u_1[) \cap S = \emptyset$, then from Lemma 4.3 (4.4) $xu_1 \notin E$.

Note that if $x_0u_2 \in E$, then the cycle:

$$C' : \quad uP_1xP_2u'u'^{-} \ldots u_1 \ldots x_0u_2u_2^{+} \ldots u$$

contains S, a contradiction.

Using the same arguments we can show that:

$$N_{]u,\, u_1[}(u_2) = \emptyset \text{ and } N_{]u',\, u_2[}(x_0) = \emptyset. \tag{4.16}$$

So we have u_1, $u_2 \in Y$, $xu_1 \notin E$, $x_0u_2 \notin E$ and from (3.1), (3.2) we have:

$$d(x_0) + d(x) + d(u_1) + d(u_2) \geq 2n + 2. \tag{4.17}$$

Consider the interval $]u_2, u_1[$.

If $a \in V(]u_2, u_1[)$ and $x_0a \in E$ then $a^+u_2 \notin E$.

Suppose that there is a vertex $a \in V(]u_2, u_1[)$ such that x_0a, $a^+u_2 \in E$. From (4.16) we have $a \in V(]u_2, u[)$ and in this case the cycle:

$$C' : \quad x_0aa^{-} \ldots u_2^{+}u_2a^{+} \ldots uP_1xP_2u'u'^{-} \ldots u_1 \ldots x_0 \tag{4.18}$$

contains S, a contradiction.

Using similar arguments we can show that if $a \in V(]u_1, u_2[)$ and $u_2a \in E$ then $a^+x_0 \notin E$.

Since also $u_1u_2 \notin E$ we have:

$$d_C(x_0) + d_C(u_2) \leq \frac{c}{2}. \tag{4.19}$$

If $a \in V(C) \setminus \{u\}$ and $xa \in E$ then $u_1a^+ \notin E$.

Suppose that there is a vertex $a \in V(C) \setminus \{u\}$ such that xa, $u_1a^+ \in E$. From (4.4) we know that $a \notin V(]u, u_1])$ and thus the cycle:

$$C' : \quad xaa^{-} \ldots u_1a^{+}a^{++} \ldots uP_1x$$

contains S, a contradiction.

So we have:

$$\mathrm{d}_C(x) + \mathrm{d}_C(u_1) \leq \frac{c}{2} + 1. \tag{4.20}$$

From Lemma 4.3 (4.5) we have:

$$\mathrm{d}_R(x) + \mathrm{d}_R(x_0) \leq \frac{r}{2}. \tag{4.21}$$

From Lemma 4.3 (4.6) we have:

$$\mathrm{d}_R(u_1) + \mathrm{d}_R(u_2) \leq \frac{r}{2}. \tag{4.22}$$

From (4.19) — (4.22) we have:

$$\mathrm{d}(x_0) + \mathrm{d}(x) + \mathrm{d}(u_1) + \mathrm{d}(u_2) \leq \frac{c}{2} + \frac{c}{2} + 1 + \frac{r}{2} + \frac{r}{2} = 2n + 1,$$

a contradiction with (4.17).

Subcase 2.2. $u^+ = u_1$ **and** $u'^+ = u_2$

If $u^+ = u_1$ and $u'^+ = u_2$ then u, $u' \in X$ and so $\mathrm{V}(P_1) \neq \emptyset$ and $\mathrm{V}(P_2) \neq \emptyset$ and from Remark 4.2 (4.2) we have $\mathrm{d}_C(x) = 0$.

Subcase 2.2.1. $\mathrm{N}_R(u_1) = \emptyset$ **or** $\mathrm{N}_R(u_2) = \emptyset$

We can assume that $\mathrm{N}_R(u_1) = \emptyset$. From Lemma 4.3 (4.4) $xu_1 \notin \mathrm{E}$ and so from (3.1) we have:

$$\mathrm{d}(x) + \mathrm{d}(u_1) \geq n + 1. \tag{4.23}$$

From Remark 4.2 (4.2) and the assumption that $\mathrm{N}_R(u_1) = \emptyset$ we have:

$$\mathrm{d}(x) + \mathrm{d}(u_1) = \mathrm{d}_C(x) + \mathrm{d}_R(x) + \mathrm{d}_C(u_1) + \mathrm{d}_R(u_1) \leq 0 + \frac{r}{2} + \frac{c}{2} + 0 = n,$$

a contradiction with (4.23).

Subcase 2.2.2. $\mathrm{N}_R(u_1) \neq \emptyset$ **and** $\mathrm{N}_R(u_2) \neq \emptyset$

Take an $a \in \mathrm{N}_R(u_1)$. From Lemma 4.3 (4.4) $a \notin \mathrm{V}(P)$ and $u_1x \notin \mathrm{E}$. From (4.6) $u_2a \notin \mathrm{E}$ and so from (3.1), (3.2) we have:

$$\mathrm{d}(u_1) + \mathrm{d}(u_2) + \mathrm{d}(x) + \mathrm{d}(a) \geq 2n + 2. \tag{4.24}$$

Note that for any $b \in \mathrm{V}(]u_2, u[)$ if $ab \in \mathrm{E}$ then $u_2b^+ \notin \mathrm{E}$. Suppose that ab, $u_2b^+ \in \mathrm{E}$, then the cycle:

$$C: \quad abb^- \dots u_2b^+ \dots uP_1xP_2u'u'^- \dots u_1a$$

contains S, a contradiction.

Note that for any $b \in \mathrm{V}(]u_1, u'[)$ if $ab \in \mathrm{E}$ then $u_2b^- \notin \mathrm{E}$. Suppose that ab, $u_2b^- \in \mathrm{E}$, then the cycle:

$$C: \quad abb^+ \dots u'P_2xP_1uu^- \dots u_2b^- \dots u_1a$$

contains S, a contradiction.

Since S is not cyclable $N_{u_2}(]u, u_1]) = \emptyset$.
Note that in this case $u_2 = u'^+$ and so it is impossible that au', $u'^+u_2 \in E$.
From the above we have:

$$d_C(u_2) + d_C(a) \leq \frac{c}{2}. \tag{4.25}$$

Since from Remark 4.2 (4.2) we have $d_C(x) = 0$ then:

$$d_C(x) + d_C(u_1) \leq \frac{c}{2}$$

and

$$d_C(x) + d_C(a) + d_C(u_1) + d_C(u_2) \leq c. \tag{4.26}$$

Suppose now that we have a vertex $b \in V(R)$ such that ab, $xb \in E$. Then the cycle:

$$C: \quad u_1abxP_1uu^- \dots u_2u'u'^- \dots u_1$$

contains S, a contradiction and so we have:

$$d_R(x) + d_R(a) \leq \frac{r}{2}. \tag{4.27}$$

From Lemma 4.3 (3.1) for any vertex $b \in V(R)$ if $u_1b \in E$ then $u_2b \notin E$.
Since also $x \notin N(u_1) \cup N(u_2)$ we have:

$$d_R(u_1) + d_R(u_2) \leq \frac{r}{2} - 1. \tag{4.28}$$

From (4.26) — (4.28) we have

$$d(u_1) + d(u_2) + d(x) + d(a) \leq c + r - 1 \leq 2n - 1,$$

a contradiction with (4.24).

Case 3. $N_R(x) \neq \emptyset$ **and** u_1**,** $u_2 \in S_X$

Subcase 3.1. $V(]u, u_1[) \cap Y \neq \emptyset$ **or** $V(]u', u_2[) \cap Y \neq \emptyset$

Choose a vertex $y \in V(]u, u_1[) \cap Y$. From Lemma 4.3 (4.3) we have $xy \notin E$ and so from (3.1) we have:

$$d(x) + d(y) \geq n + 1. \tag{4.29}$$

Subcase 3.1.1. x **has a neighbor in** $R \setminus P_1$

Let y' be a neighbor of x from $R \setminus P_1$. Note that since S is not cyclable $u_1y' \notin E$ and from (3.1) we have:

$$d(x) + d(y) + d(y') + d(u_1) \geq 2n + 2. \tag{4.30}$$

Note that y and y' cannot have common neighbors in R, because if we have a vertex $b \in V(R) \cap X$ such that yb, $y'b \in E$ then the following cycle:

$$C: \quad uP_1xy'byy^+ \dots u_1 \dots u' \dots u_2 \dots u^-u$$

contains S, a contradiction.

Using the same arguments we can show that u_1 and x don't have common neighbors in R and thus:

$$\mathrm{d}_R(x) + \mathrm{d}_R(y) + \mathrm{d}_R(y') + \mathrm{d}_R(u_1) \leq \frac{r}{2} + \frac{r}{2} = r. \tag{4.31}$$

Subcase 3.1.1.1. $\mathrm{V}(P_1) \neq \emptyset$

Since $\mathrm{V}(P_1) \neq \emptyset$ and P is a shortest C-path containing x, we have $xu \notin \mathrm{E}$ and since from Lemma 4.3 (4.4) $\mathrm{d}_{]u,u_1]}(x) = 0$, we have $\mathrm{d}_{[u,u_1]}(x) = 0$.

Note that by the choice of C and P, for any $a \in \mathrm{V}(]u_1, u[)$

$\mathrm{V}(P_1) \neq \emptyset$

$xa \in \mathrm{E}$ then $ya^+ \notin \mathrm{E}$. Suppose that xa, $ya^+ \in \mathrm{E}$ then the cycle:

$$C: \quad xaa^- \ldots ya^+a^{++} \ldots uP_1x$$

contains S, a contradiction.

From this:

$$\mathrm{d}_C(x) + \mathrm{d}_C(y) \leq \frac{c}{2}. \tag{4.32}$$

If $a \in \mathrm{V}(]u_1, u[)$ and $y'a \in \mathrm{E}$ then $u_1a^+ \notin \mathrm{E}$. Take a vertex $a \in \mathrm{V}(C) \setminus \{u\}$ and suppose that $y'a$, $u_1a^+ \in \mathrm{E}$, then the cycle:

$$C: \quad xy'aa^- \ldots u_1a^+a^{++} \ldots uP_1x$$

contains S, a contradiction.

From Lemma 4.3 (4.7) $\mathrm{d}_{y'}(]u, u_1]) = 0$.

Since it is possible that $y'u \in \mathrm{E}$, from the above we get:

$$\mathrm{d}_C(y') + \mathrm{d}_C(u_1) \leq \frac{c}{2} + 1. \tag{4.33}$$

From (4.32) and (4.33) we have:

$$\mathrm{d}_C(x) + \mathrm{d}_C(y) + \mathrm{d}_C(y') + \mathrm{d}_C(u_1) \leq c + 1. \tag{4.34}$$

Subcase 3.1.1.2. $\mathrm{V}(P_1) = \emptyset$

For any $a \in \mathrm{V}(C) \setminus \{u\}$ if $xa \in \mathrm{E}$ then $ya^+ \notin \mathrm{E}$, except xu and yu^+. Hence:

$$\mathrm{d}_C(x) + \mathrm{d}_C(y) \leq \frac{c}{2} + 1. \tag{4.35}$$

As in Subcase 3.1.1.1 for any $a \in \mathrm{V}(]u_1, u[)$ if $y'a \in \mathrm{E}$ then $u_1a^+ \notin \mathrm{E}$.

From Lemma 4.3 (4.7) $\mathrm{d}_{]u,u_1]}(y') = 0$.

Since in this case $xu \in \mathrm{E}$, we know that $u \in Y$ and $y'u \notin \mathrm{E}$, thus $\mathrm{d}_{y'}([u, u_1]) = 0$.

From the above:

$$\mathrm{d}_C(y') + \mathrm{d}_C(u_1) \leq \frac{c}{2}. \tag{4.36}$$

From (4.35) and (4.36) we have:

$$\mathrm{d}_C(x) + \mathrm{d}_C(y) + \mathrm{d}_C(y') + \mathrm{d}_C(u_1) \leq c + 1 \tag{4.37}$$

Conclusion from Subcases 3.1.1.1 and 3.1.1.2

Independently of the fact if $V(P_1)$ is empty or not, when x has a neighbor in $R \setminus P_1$ from (4.37) and (4.34)we have:

$$d_C(x) + d_C(y) + d_C(y') + d_C(u_1) \leq c + 1$$

and from (4.31):

$$d_R(x) + d_R(y) + d_R(y') + d_R(u_1) \leq r.$$

Hence from (4.31) and (4.37) we have:

$$d(x) + d(y) + d(y') + d(u_1) \leq r + c + 1 = 2n + 1,$$

a contradiction with (4.30). This ends the proof of Subcase 3.1.1.

Subcase 3.1.2. $N_R(x) \subset P_1$

We recall that $N_R(x) \neq \emptyset$. In this case $N_R(x) = \{y'\}$ and x has no other neighbors in R. We will get a contradiction by calculating the degree sum of the vertices x, y, y', and u_2.

We recall that $xy \notin E$ and we have the inequality (4.29):

$$d(x) + d(y) \geq n + 1.$$

Let y' be a neighbor of x in $R \cap P_1$. From (4.4) we have $u_2 y' \notin E$ and from (3.1):

$$d(y') + d(u_2) \geq n + 1. \tag{4.38}$$

Hence from (3.1) we have:

$$d(x) + d(y) + d(y') + d(u_2) \geq 2n + 2. \tag{4.39}$$

From Lemma 4.3 (4.4) and (4.7) we have:

$$d_R(y) + d_R(y') \leq \frac{r}{2}. \tag{4.40}$$

Since S is not cyclable x and u_2 cannot have common neighbors in R we have:

$$d_R(x) + d_R(u_2) \leq \frac{r}{2}. \tag{4.41}$$

From (4.40) and (4.41) we get:

$$d_R(x) + d_R(y) + d_R(y') + d_R(u_2) \leq r. \tag{4.42}$$

Using the same arguments as those used to show (4.32) in Subcase 3.1.1.1 we can show that:

$$d_C(x) + d_C(y) \leq \frac{c}{2}. \tag{4.43}$$

Since $N_R(x) \subset P_1$ we have $P_2 = \emptyset$ and $xu' \in ED$. Hence $u' \in Y$. Since y', $u' \in Y$ we know that $y'u' \notin E$ and since also S is not cyclable we have:

— $N_{[u', u_2[}(y') = \emptyset$.

— If $a \in V([u_2, u'[)$ and $y'a \in E$ then $a^+ u_2 \notin E$.

From the above we have:

$$d_C(u_2) + d_C(y') \leq \frac{c}{2}. \tag{4.44}$$

From (4.42) — (4.44) we have:

$$d(x) + d(y) + d(y') + d(u_2) \le r + c \le 2n,$$

a contradiction with (3.1).

Subcase 3.2. $V(]u, u_1[) \cap Y = V(]u', u_2[) \cap Y = \emptyset$

From the main assumption in Case 1 we know that u_1, $u_2 \in X$. Since also $V(]u, u_1[) \cap Y = V(]u', u_2[) \cap Y = \emptyset$ we have:

$$]u, u_1[=]u', u_2[= \emptyset. \tag{4.45}$$

From (4.45): $u_1 = u^+$ and $u_2 = u'^+$ and thus u, $u' \in Y$.

Subcase 3.2.1. $N_R(u_1) = \emptyset$ **or** $N_R(u_2) = \emptyset$

We may assume that $N_R(u_1) = \emptyset$. In this case u, $u' \in Y$ and since $N_R(x) \ne \emptyset$, x has a neighbor in $R \setminus P_1$ or $R \setminus P_2$. We can assume that there is a vertex $y \in V(R \setminus P_1)$, such that $xy \in E$. From Lemma 4.3 (4.7) we know that $u_1 y \notin E$ and from (3.1) we have:

$$d(y) + d(u_1) \ge n + 1. \tag{4.46}$$

Note that for any $a \in V(]u_1, u^-[)$ if $ya \in E$ then $u_1 a^+ \notin E$, because if ya, $u_1 a^+ \in E$ then the cycle:

$$C: \quad xyaa^- \dots u_1 a^+ a^{++} \dots u^- u P_1 x$$

contains S, a contradiction.

Note that since S is not cyclable $yu^- \notin E$ and since u, $y \in Y$ we have $yu \notin E$ and so

$$d_C(y) + d_C(u_1) \le \frac{c}{2} \tag{4.47}$$

Since $u_1 y \notin E$ and $N_R(u_1) = \emptyset$ we have:

$$d_R(y) + d_R(u_1) \le \frac{r}{2} - 1 \tag{4.48}$$

and from (4.47), (4.48) we have:

$$d(y) + d(u_1) \le \frac{c}{2} + \frac{r}{2} - 1 \le n - 1,$$

a contradiction with (3.1).

Subcase 3.2.2. $N_R(u_1) \ne \emptyset$ **and** $N_R(u_2) \ne \emptyset$

From the noncyclability of S we know that u_1 and x cannot have common neighbors in R. We choose a vertex $y_1 \in N_R(u_1)$. Note that $y_1 \notin V(P)$. We chose also a $y \in N_R(x) \setminus \{y_1\}$.

Since $u_1y \notin \mathrm{E}$ and $xy_1 \notin \mathrm{E}$, from (3.1) we have:

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(u_1) + \mathrm{d}(y_1) \geq 2n + 2. \tag{4.49}$$

From the noncyclability of S we know that y and y_1 cannot have common neighbors in R and so:

$$\mathrm{d}_R(y) + \mathrm{d}_R(y_1) \leq \frac{r}{2}. \tag{4.50}$$

For the same reasons x and u_1 cannot have common neighbors in R and so:

$$\mathrm{d}_R(x) + \mathrm{d}_R(u_1) \leq \frac{r}{2}. \tag{4.51}$$

We recall that in this case $\mathrm{N}_R(x) \subset P_1$ and $\mathrm{N}_R(x) \neq \emptyset$, hence $xu \notin \mathrm{E}$. Note that for any $a \in \mathrm{V}(C) \setminus \{u\}$, if $ax \in \mathrm{E}$ then $a^+y_1 \notin \mathrm{E}$. Suppose that ax, $a^+y_1 \in \mathrm{E}$, then the cycle:

$$C: \quad xaa^- \ldots u_1y_1a^+a^{++} \ldots u^-uP_1x$$

contains S, a contradiction. From this:

$$\mathrm{d}_C(x) + \mathrm{d}_C(y_1) \leq \frac{c}{2}. \tag{4.52}$$

Using the same arguments we can show that for any $a \in \mathrm{V}(C)$, if $ay \in \mathrm{E}$ then $a^+u_1 \notin \mathrm{E}$ and thus:

$$\mathrm{d}_C(y) + \mathrm{d}_C(u_1) \leq \frac{c}{2}. \tag{4.53}$$

From (4.50) — (4.53) we have

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(y_1) + \mathrm{d}(u_1) \leq \frac{r}{2} + \frac{r}{2} + \frac{c}{2} + \frac{c}{2} = 2n,$$

a contradiction with (4.49).

Case 4. $\mathrm{N}_R(x) \neq \emptyset$ **and** u_1 **and** u_2 **are in different partite sets**

We can assume that $u_1 \in X$ and $u_2 \in Y$.

Subcase 4.1. $\mathrm{N}_R(x) \cap \big(\mathrm{V}(R \setminus P_1)\big) \neq \emptyset$

We choose a vertex $y \in \mathrm{N}_R(x) \cap \big(\mathrm{V}(R \setminus P_1)\big)$. From Lemma 4.3 (4.4) we know that $xu_2 \notin \mathrm{E}$ and so from (3.1) we have:

$$\mathrm{d}(x) + \mathrm{d}(u_2) \geq n + 1. \tag{4.54}$$

From Lemma 4.3 (4.7) we know that $yu_1 \notin \mathrm{E}$ and so from (3.2) we have:

$$\mathrm{d}(u_1) + \mathrm{d}(y) \geq n + 1. \tag{4.55}$$

Thus from (4.54) and (4.55) we have:

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(u_1) + \mathrm{d}(u_2) \geq 2n + 2. \tag{4.56}$$

For any $a \in \mathrm{V}(]u_2, u'[)$ if $xa \in \mathrm{E}$ then $u_2a^+ \notin \mathrm{E}$. Suppose that xa, $u_2a^+ \in \mathrm{E}$ then the cycle:

$$C: \quad xaa^- \ldots u_2^+u_2a^+a^{++} \ldots uu^+ \ldots u_1 \ldots u'P_2x$$

contains S, a contradiction.

Note that however from Lemma 4.3 (4.4) we know that $\mathrm{N}_{]u', u_2]}(x) = \emptyset$, but if $a = u'$ it may happen that xa, $u_2a^+ \in \mathrm{E}$ and so:

$$\text{if } xu' \notin \mathrm{E} \quad \text{then} \quad \mathrm{d}_C(x) + \mathrm{d}_C(u_2) \leq \frac{c}{2}, \tag{4.57}$$

$$\text{if } xu' \in \mathrm{E} \quad \text{then} \quad \mathrm{d}_C(x) + \mathrm{d}_C(u_2) \leq \frac{c}{2} + 1. \tag{4.58}$$

Using the same arguments we can show that for any $a \in \mathrm{V}(]u_1, u[)$ if $ya \in \mathrm{E}$ then $u_1a^+ \notin \mathrm{E}$. Note that however from Lemma 4.3 (4.7) we know that $\mathrm{N}_{]u, u_1]}(y) = \emptyset$, but if $a = u$ it may happen that ya, $u_1a^+ \in \mathrm{E}$ and so:

$$\text{if } yu \notin \mathrm{E} \quad \text{then} \quad \mathrm{d}_C(y) + \mathrm{d}_C(u_1) \leq \frac{c}{2}, \tag{4.59}$$

$$\text{if } yu \in \mathrm{E} \quad \text{then} \quad \mathrm{d}_C(y) + \mathrm{d}_C(u_1) \leq \frac{c}{2} + 1. \tag{4.60}$$

Suppose now that $\mathrm{d}_C(x) + \mathrm{d}_C(u_2) = \frac{c}{2} + 1$ and $\mathrm{d}_C(y) + \mathrm{d}_C(u_1) = \frac{c}{2} + 1$. In this case we have xu', $yu \in \mathrm{E}$. Since $xu' \in \mathrm{E}$ we have $y \notin \mathrm{V}(P_2)$ and since $yu \in \mathrm{E}$ we have $u \in X$. Since $u \in X$ we have $\mathrm{V}(P_1) \neq \emptyset$. Hence from Remark 4.2 (4.1) $\mathrm{d}_C(x) = 1$.

From the noncyclability of S we have $u_1u_2 \notin \mathrm{E}$ and so $\mathrm{d}_C(u_1) \leq \frac{c}{2} - 1$ and $\mathrm{d}_C(u_2) \leq \frac{c}{2} - 1$.

From the above if xu', $yu \in \mathrm{E}$ then:

$$\mathrm{d}_C(x) + \mathrm{d}_C(u_2) \leq \frac{c}{2} \tag{4.61}$$

and this improves upon the inequality (4.58). In fact we cannot have $\mathrm{d}_C(y) + \mathrm{d}_C(u_1) = \mathrm{d}_C(x) + \mathrm{d}_C(u_2) = \frac{c}{2} + 1$, and so from (4.57) — (4.61) we know that in any case:

$$\mathrm{d}_C(x) + \mathrm{d}_C(u_2) + \mathrm{d}_C(y) + \mathrm{d}_C(u_1) \leq c + 1. \tag{4.62}$$

From Lemma 4.3 (4.7) vertices u_2 and y cannot have common neighbors in R and so:

$$\mathrm{d}_R(u_2) + \mathrm{d}_R(y) \leq \frac{r}{2}. \tag{4.63}$$

For the same reasons we have:

$$\mathrm{d}_R(u_1) + \mathrm{d}_R(x) \leq \frac{r}{2}. \tag{4.64}$$

From (4.62) — (4.64) we have:

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(u_1) + \mathrm{d}(u_2) \leq c + 1 + r = 2n + 1,$$

a contradiction with (4.56).

Subcase 4.2. $\mathrm{N}_R(x) \subset P_1$

Since $\mathrm{N}_R(x) \subset P_1$ we have: $xu' \in \mathrm{E}$ and there is a vertex $y \in \mathrm{V}(P_1)$ such that $xy \in \mathrm{E}$ and so from Lemma 4.3 we have $\mathrm{d}_C(x) \leq 1$.

Note that from noncyclability of S $xu_2 \notin \mathrm{E}$. From Lemma 4.3 (4.4) we know that $yu_1 \notin \mathrm{E}$ and from (3.1) we have:

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(u_1) + \mathrm{d}(u_2) \geq 2n + 2. \tag{4.65}$$

Using the same arguments as in Subcase 4.1 to show (4.57) and (4.58) we have:

$$\mathrm{d}_C(x) + \mathrm{d}_C(u_2) \leq \frac{c}{2} + 1. \tag{4.66}$$

From Lemma 4.3 (4.4) we have $\mathrm{V}(]u, u_1]) \cap \mathrm{N}(y) = \emptyset$. For any $a \in \mathrm{V}(]u_1, u[)$ if $ya \in \mathrm{E}$ then $u_1a^+ \notin \mathrm{E}$. In this case $xy \in \mathrm{E}$, $y \in \mathrm{V}(P_1)$ and the vertices of the path $P = P_1xP_2$ are labelled in the following way: $p_1^1 \dots p_l^1 x p_2^k \dots p_2^1$. Since P is the shortest C-path $y = p_l^1$.

Suppose that ya, $u_1a^+ \in \mathrm{E}$, then the cycle:

$$C' : \quad yaa^- \dots u_1a^+ \dots up_1^1 \dots p_{l-1}^1 y$$

contains $S \setminus \{x\}$ and has a C-path $P' :\ yxu'$, shorter than P, a contradiction with the choice of C and P. However it is possible that if $a = u$ then ya, $u_1a^+ \in \mathrm{E}$. So we have shown that:

$$\text{if } yu \notin \mathrm{E} \text{ then } \mathrm{d}_C(y) + \mathrm{d}_C(u_1) \leq \frac{c}{2}. \tag{4.67}$$

Suppose now that $yu \in \mathrm{E}$. Since $xu' \in \mathrm{E}$ we have $u' \in Y$ and $u_2 \in Y$. From this $\mathrm{V}(]u', u_2[) \cap X \neq \emptyset$. We choose a vertex $v \in \mathrm{V}(]u', u_2[) \cap X$. From the noncyclability of S $u_1v^+ \notin \mathrm{E}$ and $yv \notin \mathrm{E}$.

Since if $yu \in \mathrm{E}$, then we have a vertex $v \in \mathrm{V}(C) \cap X$ such that $u_1v^+ \notin \mathrm{E}$ and $yv \notin \mathrm{E}$, so we have:

$$\mathrm{d}_C(y) + \mathrm{d}_C(u_1) \leq \frac{c}{2}. \tag{4.68}$$

From (4.67) and (4.68) in any case:

$$\mathrm{d}_C(y) + \mathrm{d}_C(u_1) \leq \frac{c}{2}. \tag{4.69}$$

From Lemma 4.3 (4.7) vertices u_2 and y cannot have common neighbors in R and so:

$$\mathrm{d}_R(u_2) + \mathrm{d}_R(y) \leq \frac{r}{2}. \tag{4.70}$$

For the same reasons we have:

$$\mathrm{d}_R(u_1) + \mathrm{d}_R(x) \leq \frac{r}{2}. \tag{4.71}$$

From (4.66) and (4.69) — (4.71) we have:

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(u_1) + \mathrm{d}(u_2) \leq \frac{c}{2} + 1 + \frac{c}{2} + \frac{r}{2} + \frac{r}{2} = 2n + 1,$$

a contradiction with (4.65).

We have shown that in any case we get a contradiction with the hypothesis that S is not cyclable, so the proof of Theorem 3.1 is finished. □

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Denise Amar,
amar@labri.u-bordeaux.fr

LaBRI Université de Bordeaux 1,
351 Coursde la Liberation, 33405 Talence, France

Evelyne Flandrin
fe@lri.fr

LRI,UMR8623, Bâtiment 490,
Université Paris-Sud
91405 Orsay Cedex France

Grzegorz Gancarzewicz (corresponding author)
gancarz@agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Cracow, Poland

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