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EDGE CONDITION FOR HAMILTONICITY IN BALANCED TRIPARTITE GRAPHS

Abstract. A well-known theorem of Entringer and Schmeichel asserts that a balanced bipartite graph of order $2n$ obtained from the complete balanced bipartite $K_{n,n}$ by removing at most $n-2$ edges, is bipancyclic. We prove an analogous result for balanced tripartite graphs: If G is a balanced tripartite graph of order $3n$ and size at least $3n^2-2n+2$, then G contains cycles of all lengths.

Keywords: Hamilton cycle, pancyclicity, tripartite graph, edge condition.

Mathematics Subject Classification: 05C38, 05C35.

1. INTRODUCTION AND MAIN RESULT

A well-known theorem of Entringer and Schmeichel [4] asserts that a balanced bipartite graph of order $2n$ and size at least n^2-n+2 is bipancyclic. The bound is best possible: A graph obtained from $K_{n,n-1}$ by adding a single vertex adjacent to precisely one vertex in the colour class of n vertices, has size n^2-n+1 and contains no Hamilton cycle. One can consider an analogous problem for balanced tripartite graphs. It is readily seen that a balanced tripartite graph G obtained from the complete balanced tripartite $K_3(n)$ by removing $2n-1$ (that is, all but one) edges incident with a given vertex v (see Fig. 1), contains no Hamilton cycle. As the size of such G is $2n(n-1)+n^2+1$, at least $3n^2-2n+2$ edges are necessary to guarantee hamiltonicity of a balanced tripartite graph. The main result of this note asserts that this obvious necessary condition is, in fact, sufficient.

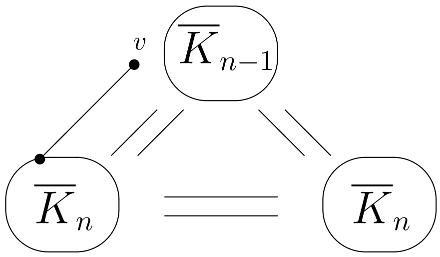

Fig. 1

Let $f_3(n) := 3n^2 - 2n + 2$ for $n \geq 2$. We prove the following sufficient condition for a balanced tripartite graph to contain a Hamilton cycle:

Theorem 1.1. *Let G be a balanced tripartite graph of order $3n$, $n \geq 2$, and size at least $f_3(n)$. Then G contains a Hamilton cycle.*

Remark 1.2. The result is best possible, as seen in Figure 1. Paired with a theorem of Bondy [1] (stating that a hamiltonian graph G satisfying $\|G\| \geq \frac{|G|^2}{4}$ is actually pancyclic), the condition $\|G\| \geq f_3(n)$ implies, in fact, that G contains cycles of all lengths (see Corollary 3.1).

Remark 1.3. The hamiltonicity criteria for balanced tripartite graphs analogous to the classical ones for bipartite graphs have been sought for and studied over the last decade or so (see, e.g., [2] and [5]). Notice however that the edge-type conditions have not yet been accounted for and our bound does not follow from neither Dirac-type minimal degree nor Ore-type degree sum conditions on tripartite graphs. (For the sake of completeness, recall that a balanced tripartite graph G with colour classes V_1, V_2, V_3 of cardinalities n and minimal degree $\delta(G)$ is known to be hamiltonian if $\delta(G) > 5n/4$ (by [2]), or $|N_G(x) \cap V_j| + |N_G(y) \cap V_i| \geq n + 1$ for every pair of nonadjacent vertices $x \in V_i, y \in V_j$ $(i \neq j)$ (by [5]).)

2. LEMMAS

Throughout the paper $\mathcal{G}_n$ will denote a family of balanced tripartite graphs G with the vertex set $V(G)$ a disjoint union of three colour classes V_1, V_2 and V_3 of cardinalities $|V_i| = n$, $n \geq 2$, and such that $\|G\| \geq f_3(n)$, where $f_3(n) = 3n^2 - 2n + 2$. As usual, $|G|$ denotes the order of a graph G and $\|G\|$ is the size of G. For a vertex v of G, we denote by $N(v)$ the set of vertices adjacent to v; note that $N(v) \subset V(G) \setminus V_i$ if $v \in V_i$, so in particular $|N(v)| \leq 2n$.

We begin by showing the following three simple lemmas.

Lemma 2.1. *Let $G \in \mathcal{G}_n$ $(n \geq 2)$ and assume that the minimal degree of G satisfies $\delta(G) \leq 2n - 2$. Then there exist $i \neq j$ and a pair of non-adjacent vertices $x \in V_i$, $y \in V_j$ such that both x and y have neighbours in the third colour class V_k.*

Proof. Pick $y \in V(G)$ with $d(y) \le 2n-2$, say $y \in V_j$. There exists at least one pair x_1, x_2 of distinct non-neighbours of y, with $x_1, x_2 \in V(G) \setminus V_j$. For every such pair, we have $d(x_1) + d(x_2) \ge 2n$. Indeed, as G is obtained from the complete tripartite graph $K_3(n)$ by removing at least $(2n-1) + (2n-1) + 1 - d(x_1) - d(x_2)$ edges, then $d(x_1) + d(x_2) \le 2n - 1$ implies $\|G\| \le 3n^2 - 2n < f_3(n)$; a contradiction.

Hence at least one of the x_1, x_2 has degree greater than $n-1$. Consequently, we may choose $x \in V_i$ $(i \ne j)$ such that $xy \notin E(G)$, $yz \in E(G)$ for some z from the third colour class V_k, and $d(x) \ge n$. This last inequality together with $xy \notin E(G)$ implies that x also has a neighbour in V_k. □

Lemma 2.2. *Let $G \in \mathcal{G}_n$ $(n \ge 2)$ and assume $\delta(G) \le 2n-2$. Then there exist $i \ne j$ and a pair of non-adjacent vertices $x \in V_i$, $y \in V_j$ such that $N(x) \cap N(y) \ne \emptyset$ (i.e., x and y have a common neighbour in the third class).*

Proof. By Lemma 2.1, we may choose a pair of non-adjacent vertices $x \in V_i$, $y \in V_j$ such that both x and y have neighbours in the third colour class V_k. Suppose that, for every z a neighbour of x in V_k, z is not a neighbour of y. Pick such $z \in N(x) \cap V_k$. We may assume that z and y share no neighbour in V_i; otherwise, if, say, $x' \in N(z) \cap N(y)$, replace $(x, y) \in V_i \times V_j$ with $(z, y) \in V_k \times V_j$ and get $zy \notin E(G)$, $zx' \in E(G)$ and $yx' \in E(G)$, as required.

Now, no vertex of V_k is a common neighbour of x and y, no vertex of V_i is a common neighbour of z and y, and both x and z have at most $n-1$ neighbours in V_j. Counting the total number of neighbours of x, y and z, we thus get

$$d(x) + d(y) + d(z) \le |V_i| + |V_k| + 2(|V_j| - 1) = 4n - 2,$$

so that

$$\|G\| \le \|G - \{x, y, z\}\| + d(x) + d(y) + d(z) \le 3(n-1)^2 + 4n - 2 < f_3(n);$$

a contradiction. This shows that at least one neighbour of x in V_k is simultanously adjacent to y. □

Let G_n^* denote a graph obtained from the complete tripartite $K_3(n)$, with colour classes V_1, V_2, V_3, by removing a complete $V_1 - V_2$ matching; i.e., if $V_1 = \{x_1, \ldots, x_n\}$, $V_2 = \{y_1, \ldots, y_n\}$, then

$$G_n^* = K_3(n) - \{x_1y_1, x_2y_2, \ldots, x_ny_n\}.$$

Lemma 2.3. *Let $G \in \mathcal{G}_n$ be as in Lemma 2.2. Then either G contains (a copy of) G_n^* or else there is a triple of vertices $x \in V_1$, $y \in V_2$, $z \in V_3$ such that $xy \notin E(G)$, $xz \in E(G)$, $yz \in E(G)$ and $\|G - \{x, y, z\}\| \ge f_3(n-1)$.*

Proof. Let $x \in V_1$, $y \in V_2$, $z \in V_3$ be a triple guaranteed by Lemma 2.2. We have $\|G - \{x, y, z\}\| \ge f_3(n) - d(x) - d(y) - d(z) + 2$, with the last summand arising from counting xz and yz twice in $d(x) + d(y) + d(z)$. As $xy \notin E(G)$, then $d(x) \le 2n-1$ and $d(y) \le 2n - 1$, and the above inequality yields

$$\|G - \{x, y, z\}\| \ge f_3(n) - 6n + 4 = 3n^2 - 8n + 6,$$

whilst $f_3(n-1) = 3n^2 - 8n + 7$. It follows that $\|G-\{x,y,z\}\| \geq f_3(n-1)$ unless $d(x) = d(y) = 2n-1$ and $d(z) = 2n$.

Suppose the latter holds. Then we may replace z by another $z' \in V_3$ and repeat the above argument with a triple $\{x,y,z'\}$. We get again either $\|G-\{x,y,z'\}\| \geq f_3(n-1)$ or else $d(x) = d(y) = 2n-1$ and $d(z') = 2n$.

Suppose then that $d(z') = 2n$ for all $z' \in V_k$. If there is no other pair of vertices $x' \in V_1$ and $y' \in V_2$ with $x'y' \notin E(G)$, then $G = K_3(n)-\{xy\}$ contains G_n^*. Otherwise, pick $x' \in V_1$ and $y' \in V_2$ with $x'y' \notin E(G)$ and repeat the argument with $\{x',y',z\}$. If $\|G-\{x',y',z\}\| < f_3(n-1)$, repeat the argument with a triple $\{x',y',z'\}$ for some $z' \in V_3 \setminus \{z\}$, and so on.

It is readily seen that in this way we find a triple $\tilde{x} \in V_1$, $\tilde{y} \in V_2$, $\tilde{z} \in V_3$ with $\|G-\{\tilde{x},\tilde{y},\tilde{z}\}\| \geq f_3(n-1)$ unless there exist subsets $\{x_1,\ldots,x_s\} \subset V_1$ and $\{y_1,\ldots,y_s\} \subset V_2$, $s \leq n$, such that $G = K_3(n)-\{x_1y_1, x_2y_2, \ldots, x_sy_s\}$ contains G_n^*. $\square$

3. PROOF OF THE MAIN RESULT

We are now ready to prove Theorem 1.1. Let G be a balanced tripartite graph of order $3n$, $n \geq 2$, and size at least $f_3(n) = 3n^2-2n+2$. We proceed by induction on n.

As $f_3(2) = 10$, a balanced tripartite graph G on 6 vertices with $\|G\| \geq f_3(2)$ is obtained from $K_3(2)$ by removing at most two edges. One easily verifies that every such a graph is hamiltonian.

Suppose then that $n \geq 3$ and the assertion of the theorem holds for $n-1$. If $\delta(G) \geq 2n-1$, then G is hamiltonian by Dirac's theorem [3], as $2n-1 \geq \frac{|G|}{2}$ for $n \geq 2$. We may thus assume that $\delta(G) \leq 2n-2$, and hence Lemma 2.3 applies to G.

Denote, as before, the colour classes of G by V_1, V_2 and V_3. Recall that by G_n^* we denote a graph obtained from $K_3(n)$ by removing a complete V_1-V_2 matching. If G contains a subgraph isomorphic to G_n^*, then we can define explicitly a Hamilton cycle as follows: Write $V_1 = \{x_1,\ldots,x_n\}$, $V_2 = \{y_1,\ldots,y_n\}$ and $V_3 = \{z_1,\ldots,z_n\}$, where G contains all the x_iy_j, x_iz_k, y_jz_k edges except at most $x_1y_1,\ldots,x_ny_n$. Then $x_1y_2z_2x_2y_3z_3\ldots x_{n-1}y_nz_nx_ny_1z_1$ is a required cycle in G.

Assume then that G contains no G_n^*, and hence by Lemma 2.3, there is a triple of vertices $x \in V_1$, $y \in V_2$ and $z \in V_3$ such that $xy \notin E(G)$, $xz \in E(G)$, $yz \in E(G)$ and $\|G-\{x,y,z\}\| \geq f_3(n-1)$. Put $H := G-\{x,y,z\}$. By the inductive hypothesis, H contains a Hamilton cycle C.

Observe that $\delta(G) \geq 2$, for otherwise G would have at least $2n-1$ edges less than $K_3(n)$ and hence $\|G\| \leq 3n^2-2n+1 < f_3(n)$; a contradiction. Therefore, as $xy \notin E(G)$, both x and y have a neighbour on C, say w_x and w_y respectively.

Observe next that $d(x)+d(y) \geq 2n+1$, for otherwise, as $d(z) \leq 2n$, would have $\|G\| = \|H\|+d(x)+d(y)+d(z)-2 \leq 3(n-1)^2+2n+2n-2 < f_3(n)$; a contradiction. Hence at least one of the vertices x, y has more than one neighbour on C and we may assume that $w_x \neq w_y$ (see Fig. 2). Now, taking $C+xz+zy+yw_y$ and splitting C at w_y, we obtain a Hamilton path $xzyw_y\ldots v_x$ in G, and by reversing the orientation of C, another Hamilton path $xzyw_y\ldots v_x'$. Similarly, G contains two Hamilton paths starting at y: $yzxw_x \dots v_y$ and $yzxw_x \dots v'_y$ (see Fig. 2). As $n \geq 3$, $|C| \geq 6$ and at least one of the pairs (v_x, v_y), (v'_x, v'_y) is a pair of distinct vertices; say $v_x \neq v_y$.

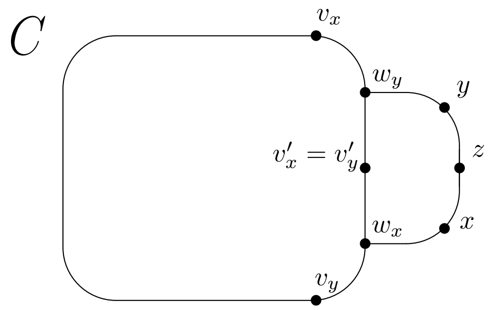

Fig. 2

Suppose first that $d_G(x) + d_G(y) + d_H(v_x) + d_H(v_y) > 6n - 4$. Then at least one of $d_G(x) + d_H(v_x)$ and $d_G(y) + d_H(v_y)$ is greater than $3n - 2$, say

$$d_G(x) + d_H(v_x) \geq 3n - 1.$$

Consider the Hamilton $x-v_x$ path P in G; write $P = xzyv_1v_2 \dots v_{3n-4}v_x$. We may assume that $xv_x \notin E(G)$, for otherwise $P + v_x x$ is a Hamilton cycle in G. Define

$$\tilde{N}_P(x) = \{v_i : xv_{i+1} \in E(G)\} \qquad \text{and} \qquad N_P(v_x) = \{v_i : v_i v_x \in E(G)\}.$$

We have $|\tilde{N}_P(x)| \geq d_G(x) - 2$ and $|N_P(v_x)| = d_H(v_x)$, hence $|\tilde{N}_P(x)| + |N_P(v_x)| \geq 3n-3$. By the pigeonhole principle, there exists $1 \leq i \leq 3n-5$ such that $v_i v_x \in E(G)$ and $xv_{i+1} \in E(G)$, hence a Hamilton cycle $xzyv_1 \dots v_i v_x v_{3n-4} \dots v_{i+1}$ in G.

Suppose now that

$$d_G(x) + d_G(y) + d_H(v_x) + d_H(v_y) \leq 6n - 4. \tag{3.1}$$

As H is obtained from $K_3(n-1)$ by removing at least $4n-5-d_H(v_x)-d_H(v_y)$ edges, we have

$$\|H\| \leq 3(n-1)^2 - 4n + 5 + d_H(v_x) + d_H(v_y). \tag{3.2}$$

Then $\|G\| = \|H\| + d_G(x) + d_G(y) + d_G(z) - 2$, together with (3.1), (3.2) and $d_G(z) \leq 2n$, yield

$$\begin{aligned}3n^2 - 2n + 2 = f_3(n) \leq \|G\| \leq \\ &\leq 3(n-1)^2 - 4n + 5 + d_H(v_x) + d_H(v_y) + d_G(x) + d_G(y) + 2n - 2 \leq \\ &\leq 3n^2 - 2n + 2.\end{aligned}$$

This is only possible if $\|H\|$ actually equals $3(n-1)^2 - 4n + 5 + d_H(v_x) + d_H(v_y)$; i.e., for every pair of distinct vertices $v_1, v_2 \in V(H) \setminus \{v_x, v_y\}$, either v_1, v_2 belong to the same colour class of G or else they are adjacent. Note that for any such pair, H is obtained from $K_3(n-1)$ by removing at least $4n-5-d_H(v_1)-d_H(v_2)+1$ edges, so that

$$\|H\| \le 3(n-1)^2 - 4n + 4 + d_H(v_1) + d_H(v_2). \tag{3.3}$$

Now, if $v'_x \neq v'_y$, then we can repeat the above calculations with (3.3) in place of (3.2), to get

$$\|G\| \le 3(n-1)^2 - 4n + 4 + d_H(v'_x) + d_H(v'_y) + d_G(x) + d_G(y) + 2n - 2 \le 3n^2 - 2n + 1,$$

provided $d_G(x) + d_G(y) + d_H(v'_x) + d_H(v'_y) \le 6n-4$. This however contradicts $\|G\| \ge f_3(n)$, hence without loss of generality $d_G(x) + d_H(v'_x) \ge 3n-1$, and we produce a Hamilton cycle from the path $xzyw_y \dots v'_x$, as above.

It does remain to consider the case $v'_x = v'_y$. Then the Hamilton x–v'_x path P' in G is as in Figure 2; i.e., of the form $P' = xzyw_yv_x \dots w_xv'_x$. Since $d(x) + d(y) \ge 2n+1$, then without loss of generality $d(y) \ge n+1 \ge 4$, and hence y has a neighbour in G, say w'_y, different from z, w_y and w_x. It follows that w'_y on P' has a neighbour v''_x different from v_y, v'_y and v_x. In particular, v'_y and v''_x are adjacent, else from the same colour class. We now repeat our calculations with the endvertices of the Hamilton paths y–v'_y and $xzyw'_y$–v''_x, with v'_y and v''_x in place of v_1 and v_2 in (3.3), to get that $d_G(y) + d_H(v'_y) \ge 3n-1$ or $d_G(x) + d_H(v''_x) \ge 3n-1$. This again implies a Hamilton cycle, which completes the proof. □

Corollary 3.1. *Let G be a balanced tripartite graph of order $3n$ and size at least $3n^2 - 2n + 2$. Then G is pancyclic.*

Proof. By a theorem of Bondy [1], pancyclicity of G follows from its hamiltonicity, provided $\|G\| \ge \frac{|G|^2}{4}$. But $f_3(n) = 3n^2 - 2n + 2 \ge \frac{(3n)^2}{4}$ for all $n \in \mathbb{N}$. □

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REFERENCES

[1] J.A. Bondy, *Pancyclic graphs I*, J. Combinatorial Theory B **11** (1971), 80–84.

[2] G. Chen, R.J. Faudree, R.J. Gould, M.S. Jacobson, L. Lesniak, *Hamiltonicity in balanced k-partite graphs*, Graphs and Combinatorics **11** (1995), 221–231.

[3] G.A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. **3** (1952) 2, 69–81.

[4] R.C. Entringer, E. Schmeichel, *Edge conditions and cycle structure in bipartite graphs*, Ars Combinatoria **26** (1988), 229–232.

[5] K. Yokomura, *A degree sum condition on hamiltonian cycles in balanced* 3*-partite graphs*, Discrete Math. **178** (1998), 293–297.

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