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A NOTE ON RADON-NIKODÝM DERIVATIVES AND SIMILARITY FOR COMPLETELY BOUNDED MAPS

Abstract. We point out a relation between the Arveson's Radon-Nikodým derivative and known similarity results for completely bounded maps. We also consider Jordan type decompositions coming out from Wittstock's Decomposition Theorem and illustrate, by an example, the nonuniqueness of these decompositions.

Keywords: Radon-Nikodym derivative, C^*-algebra, completely positive map, similarity.

Mathematics Subject Classification: 46L07.

1. INTRODUCTION

In this note we indicate a relation between the Arveson's Radon-Nikodým derivative and known similarity results for completely bounded maps as obtained by E. Christensen [3], U. Haagerup [5], and D. Hadwin [6]. This is done by reformulating the Paulsen's Decomposition Theorem, cf. [7]. To this end we first recall the construction of Radon-Nikodým derivatives for operator valued completely positive maps on C^*-algebras which is based on the Minimal Stinespring Representation.

Also, we consider Jordan type decompositions coming out from Wittsock's Decomposition Theorem [9] and illustrate, by an example, the nonuniqueness of these decompositions.

2. RADON-NIKODÝM DERIVATIVES OF COMPLETELY POSITIVE MAPS

2.1. COMPLETELY POSITIVE MAPS

Assume $\mathcal{A}$ is a unital C^*-algebra and let $\mathcal{H}$ be a Hilbert space. A linear mapping $\varphi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is positive if $\varphi(\mathcal{A}^+) \subseteq \mathcal{B}(\mathcal{H})^+$, that is, it maps positive elements into positive operators.

For $n \in \mathbb{N}$ let M_n denote the C^*-algebra of $n \times n$ complex matrices, identified with the C^*-algebra $\mathcal{B}(\mathbb{C}^n)$. The C^*-algebra $\mathcal{A} \otimes M_n$ identified with the C^*-algebra $M_n(\mathcal{A})$ of $n \times n$ matrices with entries in $\mathcal{A}$, has natural norm and order relation for selfadjoint elements, induced by the embedding $M_n(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n) = \mathcal{B}(\mathcal{H}^n)$, where $\mathcal{H}^n$ denotes the Hilbert space direct sum of n copies of $\mathcal{H}$. Using these considerations, a linear mapping $\varphi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is *completely positive* if for any $n \in \mathbb{N}$ the mapping $\varphi_n = \varphi \otimes I_n\colon \mathcal{A} \otimes M_n \to \mathcal{B}(\mathcal{H}^n)$ is positive. Note that, with respect to the identification $\mathcal{A} \otimes M_n = M_n(\mathcal{A})$, the mapping φ_n is given by

$$\varphi_n([a_{ij}]_{i,j=1}^n) = [\varphi(a_{ij})]_{i,j=1}^n, \quad [a_{ij}]_{i,j=1}^n \in M_n(\mathcal{A}). \tag{2.1}$$

A linear map $\varphi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is called *positive definite* if for all $n \in \mathbb{N}$, $(a_j)_{j=1}^n \in \mathcal{A}$, and $(h_j)_{j=1}^n \in \mathcal{H}$, we have

$$\sum_{i,j=1}^n \langle \varphi(a_j^* a_i) h_i, h_j \rangle \geq 0. \tag{2.2}$$

Since for any $(a_j)_{j=1}^n \in \mathcal{A}$ the matrix $[a_j^* a_i]_{i,j=1}^n$ is a nonnegative element in $M_n(\mathcal{A})$, if φ is positive definite then it is completely positive. Conversely, because any positive element in $M_n(\mathcal{A})$ can be written as a sum of elements of type $[a_j^* a_i]_{i,j=1}^n$, it follows that complete positivity is the same with positive definiteness.

$\mathrm{CP}(\mathcal{A}; \mathcal{H})$ denotes the set of all completely positive maps from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. If $\varphi, \psi \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$ one writes $\varphi \leq \psi$ if $\psi - \varphi \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$; this is the natural partial order (reflexive, antisymmetric, and transitive) on the cone $\mathrm{CP}(\mathcal{A}; \mathcal{H})$. With respect to the partial order relation $\leq$, $\mathrm{CP}(\mathcal{A}; \mathcal{H})$ is a strict convex cone.

Given $\theta \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$ we consider its *Minimal Stinespring Representation* $(\pi_\theta; \mathcal{K}_\theta; V_\theta)$ (cf. W.F. Stinespring [8]). Recall that $\mathcal{K}_\theta$ is the Hilbert space quotient-completion of the algebraic tensor product of the linear space $\mathcal{A} \otimes \mathcal{H}$ endowed with the inner product

$$\langle a \otimes h, b \otimes k \rangle_\theta = \langle \theta(b^* a) h, k \rangle, \text{ for all } a, b \in \mathcal{A},\ h, k \in \mathcal{H}. \tag{2.3}$$

π_θ is defined on elementary tensors by $\pi_\theta(a)(b \otimes h) = (ab) \otimes h$ for all $a, b \in \mathcal{A}$ and $h \in \mathcal{H}$, and then extended by linearity and continuity to a $*$-representation $\pi_\theta\colon \mathcal{A} \to \mathcal{K}_\theta$. Also, $V_\theta h = [1 \otimes h]_\theta \in \mathcal{K}_\theta$, for all $h \in \mathcal{H}$, where $[a \otimes h]_\theta$ denotes the equivalence class in the factor space $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}_\theta$, and $\mathcal{N}_\theta$ is the isotropic subspace corresponding to the inner product $\langle \cdot, \cdot \rangle_\theta$. The Minimal Stinespring Representation $(\pi_\theta; \mathcal{K}_\theta; V_\theta)$ of θ is uniquely defined, modulo unitary equivalence, subject to the following conditions:

(i) $\mathcal{K}_\theta$ is a Hilbert space and $V_\theta \in \mathcal{B}(\mathcal{H}, \mathcal{K}_\theta)$;
(ii) π_θ is a $*$-representation of $\mathcal{A}$ on $\mathcal{K}_\theta$ such that $\theta(a) = V_\theta^* \pi_\theta(a) V_\theta$ for all $a \in \mathcal{A}$;
(iii) $\pi_\theta(\mathcal{A}) V_\theta \mathcal{H}$ is total in $\mathcal{K}_\theta$.

In case θ is unital, the linear operator V_θ is an isometry and hence, due to the uniqueness, one can, and we always do, replace V with the canonical embedding $\mathcal{H} \hookrightarrow \mathcal{K}$.

2.2. RADON-NIKODÝM DERIVATIVES

Let $\varphi, \theta \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$ be such that $\varphi \leq \theta$ and consider the Minimal Stinespring Representation $(\pi_\varphi; \mathcal{K}_\varphi; V_\varphi)$ of φ, and similarly for θ. Then the identity operator $J_{\varphi,\theta} \colon \mathcal{A} \otimes \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$ has the property that $J_{\varphi,\theta} \mathcal{N}_\theta \subseteq \mathcal{N}_\varphi$, hence it can be factored to a linear operator $J_{\varphi,\theta} \colon (\mathcal{A} \otimes \mathcal{H})/\mathcal{N}_\theta \to (\mathcal{A} \otimes \mathcal{H})/\mathcal{N}_\varphi$ and then can be extended by continuity to a contractive linear operator $J_{\varphi,\theta} \in \mathcal{B}(\mathcal{K}_\theta, \mathcal{K}_\varphi)$. It is easy to see that

$$J_{\theta,\varphi} V_\theta = V_\varphi, \tag{2.4}$$

and that

$$J_{\theta,\varphi} \pi_\theta(a) = \pi_\varphi(a) J_{\theta,\varphi} \text{ for all } a \in \mathcal{A}. \tag{2.5}$$

Thus, letting

$$\mathrm{D}_\theta(\varphi) := J_{\theta,\varphi}^* J_{\theta,\varphi} \tag{2.6}$$

we get a contractive linear operator in $\mathcal{B}(\mathcal{K}_\theta)$. In addition, as a consequence of (2.5), $\mathrm{D}_\theta(\varphi)$ commutes with all operators $\pi_\theta(a)$ for $a \in \mathcal{A}$, briefly, $\mathrm{D}_\theta(\varphi) \in \pi_\theta(\mathcal{A})'$ (given a subset $\mathcal{T}$ of $B(\mathcal{H})$ we write $\mathcal{T}' = \{B \in \mathcal{B}(\mathcal{H}) \mid AB = BA \text{ for all } A \in \mathcal{T}\}$ for the commutant of $\mathcal{T}$) and

$$\varphi(a) = V_\theta^* \,\mathrm{D}_\theta(\varphi) \pi_\theta(a) V_\theta = V_\theta^* \,\mathrm{D}_\theta(\varphi)^{1/2} \pi_\theta(a) \,\mathrm{D}_\theta(\varphi)^{1/2} V_\theta \text{ for all } a \in \mathcal{A}. \tag{2.7}$$

The property (2.7) uniquely characterizes the operator $\mathrm{D}_\theta(\varphi)$. The operator $\mathrm{D}_\theta(\varphi)$ is called the *Radon-Nikodým derivative* of φ with respect to θ.

It is immediate from (2.7) that, for any $n \in \mathbb{N}$, $(a_j)_{j=1}^n \in \mathcal{A}$, and $(h_j)_{j=1}^n \in \mathcal{H}$, the following formula holds

$$\sum_{i,j=1}^n \langle \varphi(a_j^* a_i) h_i, h_j \rangle = \| \,\mathrm{D}_\theta(\varphi)^{1/2} \sum_{j=1}^n \pi_\theta(a_j) V_\theta h_j \|^2. \tag{2.8}$$

This shows that for any $\varphi, \psi \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$ with $\varphi, \psi \leq \theta$, we have $\varphi \leq \psi$ if and only if $\mathrm{D}_\theta(\varphi) \leq \mathrm{D}_\theta(\psi)$.

In addition, if $\varphi, \psi \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$ are such that $\varphi, \psi \leq \theta$ then for any $t \in [0,1]$ the completely positive map $(1-t)\varphi + t\psi$ is $\leq \theta$ and

$$\mathrm{D}_\theta((1-t)\varphi + t\psi) = (1-t)\,\mathrm{D}_\theta(\varphi) + t\,\mathrm{D}_\theta(\psi). \tag{2.9}$$

The above considerations can be summarized in the following Theorem 2.1.

Theorem 2.1 (W.B. Arveson [1]). *Let $\theta \in \mathrm{CP}(\mathcal{A};\mathcal{H})$. The mapping $\varphi \mapsto \mathrm{D}_\theta(\varphi)$ defined in* (2.6)*, with its inverse given by* (2.7)*, is an affine and order-preserving isomorphism between the convex and partially ordered sets $\big(\{\varphi \in \mathrm{CP}(\mathcal{A};\mathcal{H}) \mid \varphi \leq \theta\};\leq\big)$ and $\big(\{A \in \pi_\theta(\mathcal{A})' \mid 0 \leq A \leq I\};\leq\big)$.*

One says that ψ *uniformly dominates* φ, and we write $\varphi \leq_{\mathrm{u}} \psi$, if for some $t > 0$ we have $\varphi \leq t\psi$. This is a partial preorder relation (only reflexive and transitive). It is immediate from Theorem 2.1 the following

Corollary 2.2. *For a given $\theta \in \mathrm{CP}(\mathcal{A};\mathcal{H})$, the mapping $\varphi \mapsto \mathrm{D}_\theta(\varphi)$ defined in* (2.6)*, with its inverse given by* (2.7)*, is an affine and order-preserving isomorphism between the convex cones $\big(\{\varphi \in \mathrm{CP}(\mathcal{A};\mathcal{H}) \mid \varphi \leq_{\mathrm{u}} \theta\};\leq\big)$ and $\big(\{A \in \pi_\theta(\mathcal{A})' \mid 0 \leq A\};\leq\big)$.*

3. SIMILARITY FOR OPERATOR VALUED COMPLETELY BOUNDED MAPS

In this section we show that the Radon-Nikodým derivatives can be naturally related with similarity problems in the operator spaces theory.

Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, and a bounded linear map $\rho\colon \mathcal{A} \to \mathcal{B}$, for arbitrary $n \in \mathbb{N}$ one considers the bounded linear map $\rho_n\colon M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by

$$\rho_n([a_{ij}]) = [\rho(a_{ij})], \quad [a_{ij}] \in M_n(\mathcal{A}),$$

and let

$$\|\rho\|_{\mathrm{cb}} := \sup_{n\in\mathbb{N}} \|\rho_n\|. \tag{3.1}$$

If $\|\rho\|_{\mathrm{cb}} < \infty$, ρ is called a *completely bounded map*. The set of all completely bounded maps $\mathrm{CB}(\mathcal{A},\mathcal{B})$ has a natural structure of vector space, $\|\cdot\|_{\mathrm{cb}}$ is a norm on it, and $(\mathrm{CB}(\mathcal{A},\mathcal{B});\|\cdot\|_{\mathrm{cb}})$ is a Banach space, e.g. see [4,7].

We first reformulate the Paulsen's Decomposition Theorem, see [7] and the bibliography cited there.

Theorem 3.1. *Let $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be completely bounded. Then there exists a Hilbert space $\mathcal{G}$, a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{G})$, and $R \in \pi(\mathcal{A})'$ such that*

$$\varphi(a) = P_{\mathcal{H}} R\pi(a)|\mathcal{H}, \text{ for all } a \in \mathcal{A}. \tag{3.2}$$

Proof. By the Wittstock's Decomposition Theorem, $\varphi = \varphi_1 - \varphi_2 + \mathrm{i}(\varphi_3 - \varphi_4)$ for some $\varphi_i \in \mathcal{CP}(\mathcal{A},\mathcal{B}(\mathcal{H}))$. We may assume that $\varphi_1(1) + \varphi_2(1) + \varphi_3(1) + \varphi_4(1) = tI$, for some $t > 0$. Indeed by Arveson's Extension Theorem ([1,7]), for any $K \in B(\mathcal{H})^+$ there is a $\psi \in \mathcal{CP}(\mathcal{A},\mathcal{B}(\mathcal{H}))$ with $\psi(1) = K$. So, if necessary, by writing $\varphi = (\varphi_1+\psi) - (\varphi_2+\psi) + \mathrm{i}(\varphi_3 - \varphi_4)$ we may assume that the latter condition holds. Since $(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)/t$ is completely positive and unital it has a Stinespring representation $(\pi, V, \mathcal{K})$ where $V \in B(\mathcal{H},\mathcal{K})$ is an isometry. Let A_j be the Radon-Nikodým derivative of φ_j with respect to $(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)/t$ for $j = 1, 2, 3, 4$. Set $R = A_1 - A_2 + \mathrm{i}(A_3 - A_4)$. Since each $A_j \in \pi(\mathcal{A})'$, it follows that $R \in \pi(\mathcal{A})'$ and then (3.2) holds. □

We also remark that in the representation (3.2), the set $\pi(\mathcal{A})(\mathcal{H} \oplus 0)$ is total in $\mathcal{H} \oplus \mathcal{G}$. Since $R \in \pi(\mathcal{A})'$, it is uniquely determined.

Next we exemplify the use of the Radon-Nikodým derivative technique in proving the similarity result of E. Christensen [3], U. Haagerup [5], and D. Hadwin [6].

Theorem 3.2. *Let $\rho : \mathcal{A} \to B(\mathcal{H})$ be a unital homomorphism which is completely bounded. Then there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})^+$ such that $S^{-1}\rho S$ is a unital $*$-homomorphism.*

Proof. Since ρ is completely bounded it has a representation as in Theorem 3.1. Let V denote the embedding $\mathcal{H} \hookrightarrow \mathcal{H} \oplus \mathcal{G}$. We first observe that

$$\begin{aligned} \rho(ab) = \rho(a)\rho(b) \quad &\Rightarrow \quad V^*R\pi(ab)V = V^*R\pi(a)VV^*R\pi(b)V, \\ &\Rightarrow \quad V^*\pi(a)R\pi(b)V = V^*\pi(a)RVV^*R\pi(b)V, \\ &\Rightarrow \quad V^*\pi(a)\,(R - RVV^*R)\,\pi(b)V = 0 \text{ for all } a, b \in \mathcal{A}, \\ &\Rightarrow \quad R = RVV^*R. \end{aligned}$$

Also $\rho(1) = V^*RV = I$. So it is easy to see that

$$R = \begin{bmatrix} I & Y \\ Z & ZY \end{bmatrix} \tag{3.3}$$

for some $Y : \mathcal{G} \to \mathcal{H}$ and $Z : \mathcal{H} \to \mathcal{G}$. Clearly, $I + Z^*Z$ is positive and invertible in $\mathcal{B}(\mathcal{H})$, and it satisfies

$$[(I + Z^*Z)^{-1} \quad 0]R^*R = V^*R.$$

Hence, for any $a \in \mathcal{A}$ we have

$$\rho(a) = [(I + Z^*Z)^{-1} \quad 0]R^*R\pi(a)|\mathcal{H}.$$

Here $R^* \in \pi(\mathcal{A})'$. Therefore, letting $S = (I + Z^*Z)^{-1/2}$ we get the result. □

We now consider Jordan decompositions. A linear map $\rho\colon \mathcal{A} \to \mathcal{B}$ is *selfadjoint* if $\rho(a^*) = \rho(a)^*$ for all $a \in \mathcal{A}$. According to the Wittstock's Decomposition Theorem [9], *if $\rho \in \mathrm{CB}(\mathcal{A};\mathcal{H})$ is selfadjoint then there exists $\rho_\pm \in \mathrm{CP}(\mathcal{A};\mathcal{H})$ such that $\rho = \rho_+ - \rho_-$.* Note that, since any $\rho \in \mathrm{CB}(\mathcal{A};\mathcal{H})$ can be (uniquely) decomposed $\rho = \rho_{\mathrm{re}} + \mathrm{i}\rho_{\mathrm{im}}$, where $\rho_{\mathrm{re}}, \rho_{\mathrm{im}} \in \mathrm{CB}(\mathcal{A};\mathcal{H})$ are selfadjoint, it follows that $\mathrm{CB}(\mathcal{A};\mathcal{H})$ is linearly generated by its cone $\mathrm{CP}(\mathcal{A};\mathcal{H})$.

Let φ and ψ be two completely positive maps from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. φ is called *ψ-singular* if the only map $\rho \in \mathrm{CP}(\mathcal{A};\mathcal{H})$ such that $\rho \le \varphi, \psi$ is 0. Note that φ is ψ-singular if and only ψ is φ-singular and, in this case, we call φ and ψ *mutually singular*.

Proposition 3.3. *In the Wittstock Decomposition, one can always choose $\rho_\pm$ such that they are mutually singular.*

Proof. To see this, by Wittstock's Decomposition Theorem, let $\varphi, \psi \in \mathcal{CP}(\mathcal{A},\mathcal{H})$ be such that $\rho = \varphi - \psi$. Let $(\pi, V, \mathcal{K})$ be the Minimal Stinespring Representation for

$\varphi+\psi$, and let F and $I-F$ be the Radon-Nikodým derivatives of φ and ψ, respectively, with respect to $\varphi+\psi$. Then, clearly,

$$\rho = \varphi - \psi = V^*(2F - I)\pi(\cdot)V.$$

Let $2F - I = X - Y$ be the Jordan decomposition of the positive operator $2F - I$, that is, $X, Y \geq 0$ and $XY = 0$, equivalently, they have orthogonal supports. By continuous functional calculus both X and Y are in $\mathcal{C}^*(I, F)$ and consequently, they commute with $\pi(a)$ for all $a \in \mathcal{A}$. Therefore, $\rho_+ := V^* X\pi(\cdot)V$ and $\rho_- := V^* Y\pi(\cdot)V$ are completely positive and clearly $\rho = \rho_+ - \rho_-$. Then $\rho_\pm$ are mutually singular, e.g. by Theorem 2.1. □

A different approach to get this remark, within the Krein space theory, can be found in [2]).

Jordan decompositions in this non-commutative setting, unlike the Jordan decomposition for signed measures, are not unique.

Example 3.4. Consider the projections $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $Q = \frac{1}{2}\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ in M_2. Let $\mathcal{A}$ be the commutant of the C^*-algebra generated by I, P and Q. For $X \in \mathcal{B}(\mathbb{C}^2)$ we define $\mu_X : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by $\mu_X(Y) = XY$. Then $\mu_{I-P}, \mu_Q, \mu_{I-Q}$ and μ_P are all completely positive. Now it is easy to show that μ_{I-P} is μ_Q-singular and μ_{I-Q} is μ_P-singular. This means that the completely bounded selfadjoint map μ_{I-P-Q} has two distinct Jordan decomposition $\mu_{I-P-Q} = \mu_{I-P} - \mu_Q = \mu_{I-Q} - \mu_P$.

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