Opuscula Mathematica • Vol. 29 • No. 2 • 2009

Bogusław Bożek, Wiesław Solak, Zbigniew Szydełko

ON SOME QUADRATURE RULES WITH GREGORY END CORRECTIONS

Abstract. How can one compute the sum of an infinite series $s := a_1 + a_2 + \cdots$? If the series converges fast, i.e., if the term a_n tends to 0 fast, then we can use the known bounds on this convergence to estimate the desired sum by a finite sum $a_1 + a_2 + \cdots + a_n$. However, the series often converges slowly. This is the case, e.g., for the series $a_n = n^{-t}$ that defines the Riemann zeta-function. In such cases, to compute s with a reasonable accuracy, we need unrealistically large values n, and thus, a large amount of computation.

Usually, the n-th term of the series can be obtained by applying a smooth function $f(x)$ to the value n: $a_n = f(n)$. In such situations, we can get more accurate estimates if instead of using the upper bounds on the remainder infinite sum $R = f(n+1) + f(n+2) + \ldots$, we approximate this remainder by the corresponding integral I of $f(x)$ (from $x = n+1$ to infinity), and find good bounds on the difference $I - R$.

First, we derive sixth order quadrature formulas for functions whose 6th derivative is either always positive or always negative and then we use these quadrature formulas to get good bounds on $I - R$, and thus good approximations for the sum s of the infinite series. Several examples (including the Riemann zeta-function) show the efficiency of this new method. This paper continues the results from [3] and [2].

Keywords: numerical integration, quadrature formulas, summation of series.

Mathematics Subject Classification: 65D30, 65D32, 65G99, 65B10.

1. SOME QUADRATURE RULES WITH GREGORY END CORRECTIONS

We present one-parameter end corrections for elementary quadrature formula and we examine a property of this quadrature for the special values of the parameter. This paper continues the results from [3].

1.1. INTRODUCTION

One can compute the approximate value of the integral

$$I(f) = \int_a^b f(t)\,dt$$

by applying the quadrature formula in the form

$$Q(f) = \sum_{i=0}^{n} a_i f(t_i),$$

where quadrature nodes t_i belong to the interval $[a-c, b+c]$, $c \geq 0$. The quadrature coefficients $\{a_i\}$ satisfy the equation

$$\sum_{i=0}^{n} a_i = b - a.$$

If some nodes depend on β, i.e. $t_i = t_i(\beta)$ for $i \in A \subset \{0, 1, \ldots, n\}$, then we call this the quadrature formula with parameter. The value

$$EQ(f) = I(f) - Q(f)$$

is called the (global) quadrature error.

One of the methods to compute the error EQ is the method that comes from Peano. First we determine the quadrature range s and next we compute the Peano kernel defined as follows

$$K_s(x) = EQ\big(p(t)\big), \tag{1.1}$$

where

$$p(t) = \frac{(t-x)_+^{s-1}}{(s-1)!} \tag{1.2}$$
$$a_+ = \max\{a, 0\}, \qquad x - \text{parametr.}$$

Peano's theory (see [1]) says, that for the function $f \in C^{(s)}\big([a-c, b+c]\big)$ we have

$$EQ(f) = \int_{a-c}^{b+c} K_s(x) f^{(s)}(x)\,dx. \tag{1.3}$$

If $K_s(x)$ is of constant sign, then from (1.3) we obtain a useful formula

$$EQ(f) = f^{(s)}(\xi) \int_{a-c}^{b+c} K_s(x)dx, \qquad \xi \in [a-c, b+c]. \tag{1.4}$$

A quadrature formula obtained by adding some correction terms to the trapezoidal rule is called the Gregory type. One of the examples of such quadrature can be written as follows

$$Q^{\beta}_{n+5}(f) := T_{n+1}(f) + G_n(f,\beta), \tag{1.5}$$

where

$$G_n(f,\beta) = \frac{h}{24\beta}\big(-3(f_0+f_n) + 4(f_\beta + f_{n-\beta}) - (f_{2\beta}+f_{n-2\beta})\big),$$

$$f_t := f(a+th), \quad h = \frac{b-a}{n},$$

$$T_{n+1}(f) = \frac{h}{2}(f_0+f_n) + h\sum_{i=1}^{n-1} f_i$$

is the trapezoidal rule, and β is a parameter.
The polynomial

$$v_n(\beta) = \frac{EQ(t^4)}{\frac{1}{30}h^5} = 30\beta^3 - 20n\beta^2 + n \tag{1.6}$$

is called the characteristic polynomial of the quadrature Q^{β}_{n+5}. It is easy to verify that the quadrature (1.5) is of the fourth order if β is not a root of the characteristic polynomial v_n, and of the sixth order if β is a root of this polynomial.

In the paper [3] the properties of the quadrature Q^{β}_{n+5} for β from the interval $(0, \frac{1}{2}]$ are examined. The Peano kernel $K_4(x)$ is non-positive for $\beta \in [0.31, 0.5]$ and in this case the error of the quadrature formula for $f \in C^{(4)}[a,b]$ (c is equal zero) can be written in the form

$$EQ^{\beta}_{n+5}(f) = \frac{h^5}{720}v_n(\beta)f^{(4)}(\xi) \tag{1.7}$$

with some $\xi \in [a,b]$.

In this paper we investigate the properties of (1.5) for the roots of the characteristic polynomial v_n.

1.2. AN ANALYSIS OF GREGORY TYPE QUADRATURE FORMULAE

The roots of the characteristic polynomial $v_n(\beta)$ are

$$\alpha_n = \frac{2}{9}n\left(1 + 2\cos(\frac{\varphi_n + 2\pi}{3})\right),$$

$$\beta_n = \frac{2}{9}n\left(1 + 2\cos(\frac{\varphi_n + 4\pi}{3})\right),$$

$$\gamma_n = \frac{2}{9}n\left(1 + 2\cos(\frac{\varphi_n}{3})\right),$$

where

$$\varphi_n \in \big(0, \frac{\pi}{2}\big) \quad \text{and} \quad \varphi_n = \arccos\left(1 - \frac{243}{160n^2}\right).$$

It easy to verify, that

$$\lim_{n\to\infty}\varphi_n = 0,$$
$$\lim_{n\to\infty}\alpha_n = -\frac{\sqrt{5}}{10},$$
$$\lim_{n\to\infty}\beta_n = \frac{\sqrt{5}}{10},$$
$$\lim_{n\to\infty}\gamma_n = \infty,$$

moreover the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are decreasing, and $\alpha_n < 0$, $\beta_n > 0$ for $n = 1, 2, \ldots$.

Theorem 1.1. *The quadrature* (1.5) *with* $\beta = \alpha_n$ *is of the sixth order, and the error estimation for any function* $f \in C^{(6)}[a + 2\alpha_n h, b - 2\alpha_n h]$ *can be expressed by*

$$EQ^{\alpha_n}_{n+5}(f) = \frac{nh^7}{4320}\left(\frac{5\alpha_n^2 + 11}{15} n\alpha_n - \left(\frac{1}{7} + \alpha_n^2\right)\right) f^{(6)}(\eta) \tag{1.8}$$

for some $\eta \in [a + 2\alpha_n h, b - 2\alpha_n h]$.

Proof. It is clear that the support of the Peano kernel $K^{\alpha_n}_6(x)$ is the interval $[a + 2\alpha_n h, b - 2\alpha_n h]$. Taking advantage of the formula (1.4) it suffices to show, that the Peano kernel is negative in the interval $(a + 2\alpha_n h, b - 2\alpha_n h)$.
Directly from the definition, we can write the Peano kernel $K^{\alpha_n}_6(x)$ in the form:

$$K^{\alpha_n}_6(x) = \begin{cases} \phi_1\left(\frac{x-b}{h}\right) & \text{for} \quad x \in [b - \alpha_n h, b - 2\alpha_n h), \\ \phi_2\left(\frac{x-b}{h}\right) & \text{for} \quad x \in [b, b - \alpha_n h], \\ \phi_3^j\left(\frac{b-x}{h} - j\right) & \text{for} \quad x \in [b - (j+1)h, b - jh], \\ & \qquad j = 0, 1, \ldots, n-1, \\ \phi_2\left(\frac{a-x}{h}\right) & \text{for} \quad x \in [a + \alpha_n h, a], \\ \phi_1\left(\frac{a-x}{h}\right) & \text{for} \quad x \in (a + 2\alpha_n h, a + \alpha_n h], \end{cases} \tag{1.9}$$

where

$$\phi_1(t) = \frac{-h^6}{720 \cdot 4\alpha_n}\big(t + 2\alpha_n\big)^5 \quad \text{for} \quad -\alpha_n \le t < -2\alpha_n,$$
$$\phi_2(t) = \frac{-h^6}{720 \cdot 4\alpha_n}\Big(\big(t + 2\alpha_n\big)^5 - 4\big(t + \alpha_n\big)^5\Big) \quad \text{for} \quad 0 \le t \le -\alpha_n,$$
$$\phi_3^j(t) = \frac{h^6}{720}\Big(\big(t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2\big) - 7\alpha_n^4 + \frac{1}{2}\big(1 - 20\alpha_n^2\big)\big(t + j\big)\big(t - (n - j)\big)\Big)$$
$$\text{for} \quad 0 \le t \le 1, \quad j = 0, 1, \ldots, n-1.$$

We will check that ϕ_1, ϕ_2, ϕ_3^j is negative in suitable intervals.

Because of $t < -2\alpha_n$, we have $\big(t + 2\alpha_n\big)^5 < 0$. Take into consideration the fact that $\alpha_n < 0$, we get $\phi_1(t) < 0$ in the interval $\big[-\alpha_n, -2\alpha_n\big)$, and moreover $\phi\big(-2\alpha_n\big) = 0$.

Next, we observe that $\phi_2(t) < 0$ if and only if $\big(\sqrt[5]{4}-1\big)t > \big(2-\sqrt[5]{4}\big)\alpha_n$. This inequality is evidently true as $\alpha_n < 0$ and $t \geq 0$.

Let us first define the auxiliary functions

$$f(t) = t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2,$$
$$g^j(t) = \frac{1}{2}\big(1-20\alpha_n^2\big)\big(t+j\big)\big(t-(n-j)\big) \qquad (j = 0,1,\dots,n-1).$$

A simple computation shows, that $f(t) < 0$ on $(0,1)$ and $f(0) = f(1) = 0$. For $j \in \{0,1,\dots,n-1\}$ we have $-j \leq 0$, $n-j \geq 1$, so $-j \leq 0 < 1 \leq n-j$ and these imply the inclusions $[0,1] \subset [-j, n-j]$. On the interval $[0,1]$ the parabola $(t+j)\big(t-(n-j)\big)$ is non-positive (it is negative in $(-j, n-j)$). Since $1-20\alpha_n^2 > 0$, we see that $g^j(t) \leq 0$ on $[0,1]$. From the above we have $\phi_3^j(t) < 0$ on $[0,1]$ because of

$$\phi_3^j(t) = \frac{h^6}{720}\big(f(t) + g^j(t) - 7\alpha_n^4\big).$$

This finishes the proof of the fact that the Peano kernel is negative. Integrating the Peano kernel over $[a+2\alpha_n h, b-2\alpha_n h]$ we have (1.8), which agrees with the formula (1.4). ☐

Theorem 1.2. *The quadrature* (1.5) *with* $\beta = \beta_n$ *is of the sixth order, and the error estimation for any function* $f \in C^{(6)}[a,b]$ *can be expressed by*

$$EQ_{n+5}^{\beta_n}(f) = \frac{nh^7}{4320}\left(\frac{5\beta_n^2+11}{15}n\beta_n - \big(\frac{1}{7}+\beta_n^2\big)\right) f^{(6)}(\xi) \tag{1.10}$$

with some $\xi \in [a,b]$.

Proof. Directly from the definition, we can write the Peano kernel $K_6^{\beta_n}(x)$ in the form:

$$K_6^{\beta_n}(x) = \begin{cases} \psi_1\big(\frac{x-a}{h}\big) & \text{for } x \in [a, a+\beta_n h], \\ \psi_2\big(\frac{x-a}{h}\big) & \text{for } x \in [a+\beta_n h, a+2\beta_n h], \\ \psi_3\big(\frac{x-a}{h}\big) & \text{for } x \in [a+2\beta_n h, a+h], \\ \psi_4^j\big(\frac{b-x}{h}-j\big) & \text{for } x \in [b-(j+1)h, b-jh], \\ & \quad j = 1,2,\dots,n-2, \\ \psi_3\big(\frac{b-x}{h}\big) & \text{for } x \in [b-h, b-2\beta_n h], \\ \psi_2\big(\frac{b-x}{h}\big) & \text{for } x \in [b-2\beta_n h, b-\beta_n h], \\ \psi_1\big(\frac{b-x}{h}\big) & \text{for } x \in [b-\beta_n h, b], \end{cases} \tag{1.11}$$

where

$$\psi_1(t) = \frac{h^6}{720} t^5 \Big(t + 3\big(\frac{1}{4\beta_n} - 1\big)\Big) \quad \text{for} \quad 0 \le t \le \beta_n,$$

$$\psi_2(t) = \frac{h^6}{720}\Big(t^6 - \big(3 + \frac{1}{4\beta_n}\big) t^5 + 5t^4 - 10\beta_n t^3 + 10\beta_n^2 t^2 - 5\beta_n^3 t + \beta_n^4 \Big)$$
$$\text{for} \quad \beta_n \le t \le 2\beta_n,$$

$$\psi_3(t) = \frac{h^6}{720}\Big(t^6 - 3t^5 + \frac{5}{2}t^4 - 10\beta_n^2 t^2 + 15\beta_n^3 t - 7\beta_n^4 \Big) \quad \text{for} \quad 2\beta_n \le t \le 1,$$

$$\psi_4(t) = \frac{h^6}{720}\Big(\big(t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2\big) - 7\beta_n^4 + \frac{1}{2}\big(1 - 20\beta_n^2\big)(t+j)\big(t - (n-j)\big)\Big)$$
$$\text{for} \quad 0 \le t \le 1, \quad j = 1, 2, \ldots, n-2.$$

We can now proceed analogously to the proof of the previous theorem. We prove that the kernel $K_6^{\beta_n}(x)$ is non-negative and from (1.4) after the integration of the Peano kernel we have (1.10). □

Figure 1 illustrates the graphs of Peano kernels $K_6^{\alpha_n}$, $K_6^{\beta_n}$ for $[a, b] = [0, 1]$ and $n = 16$.

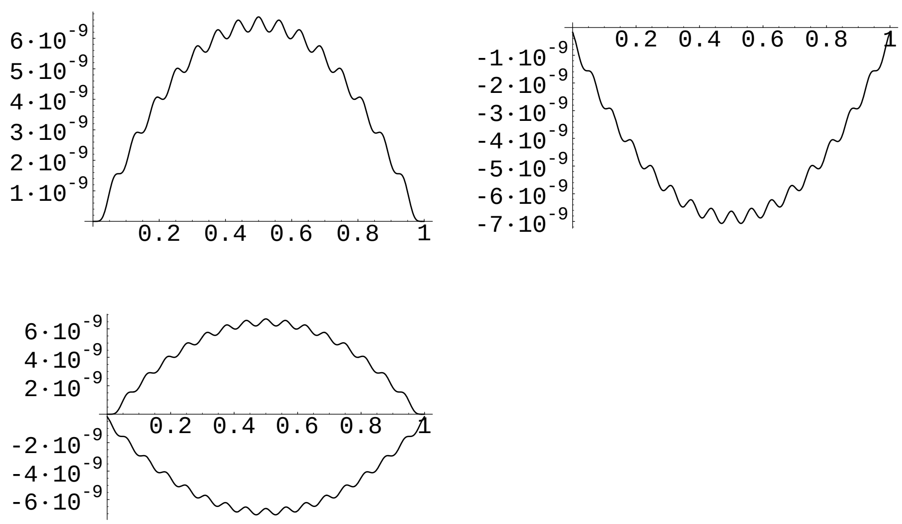

Fig. 1. The kernels $K_6^{\alpha_{16}}$, $K_6^{\beta_{16}}$, and the both kernels in one figure

Theorem 1.3. *If the function* f *is of the class* $C^6[a + 2\alpha_n h, b - 2\alpha_n)h]$ *and* $f^{(6)}$ *is positive in this interval, then*

$$Q_{n+5}^{\beta_n}(f) < I(f) < Q_{n+5}^{\alpha_n}(f). \tag{1.12}$$

If $f^{(6)}$ is negative, then

$$Q^{\alpha_n}_{n+5}(f) < I(f) < Q^{\beta_n}_{n+5}(f). \tag{1.13}$$

Proof. It is easy to see, that

$$\frac{5\alpha_n^2+11}{15} n\alpha_n - \Big(\frac{1}{7}+\alpha_n^2\Big) < 0$$

and

$$\frac{5\beta_n^2+11}{15} n\beta_n - \Big(\frac{1}{7}+\beta_n^2\Big) > 0$$

for all $n \geq 2$. These inequalities, the estimations (1.8), (1.10), and the sign of the derivative $f^{(6)}$ imply the inequalities (1.12), (1.13) of the definition of the error of the quadrature. □

Example 1. Consider the integral

$$I(f) = \int_0^{\frac{\pi}{4}} f(x)\,dx,$$

where $f(x) := \sqrt{\cos x}$. We can see that

$$I(f) = \sqrt{\frac{2}{\pi}}\Big(\Gamma\big(\frac{3}{4}\big)\Big)^2 \approx 0.74\,430\,307.$$

The derivative $f^{(6)}$ is given by

$$f^{(6)}(x) = -\frac{19}{8}\sqrt{\cos x} - \frac{289\sin^2 x}{16\cos^{3/2} x} - \frac{975\sin^4 x}{32\cos^{7/2} x} - \frac{945\sin^6 x}{64\cos^{11/2} x}$$

therefore $f^{(6)}(x) < 0$ for all $x \in \big[0, \frac{\pi}{4}\big]$. For example we compute:

$$Q^{\alpha_{20}}_{25}(f) = 0.743\,721\,22 < I(f) < 0.744\,660\,93 = Q^{\beta_{20}}_{25}(f),$$
$$Q^{\alpha_{30}}_{35}(f) = 0.744\,043\,07 < I(f) < 0.744\,464\,67 = Q^{\beta_{30}}_{35}(f).$$

Example 2. Consider the integral

$$I(f) = \int_1^{2} f(x)\,dx,$$

where $f(x) := \frac{e^x}{x}$.The derivative $f^{(6)}$ is given by

$$f^{(6)}(x) = \Big(\frac{720}{x^7} - \frac{720}{x^6} + \frac{360}{x^5} - \frac{120}{x^4} + \frac{30}{x^3} - \frac{6}{x^2} + \frac{1}{x}\Big)e^x;$$

therefore, $f^{(6)}(x) > 0$, for all $x \in [1,2]$. For example we compute:

$$Q^{\beta_{20}}_{25}(f) = 3.056\,553\,592 < I(f) < 3.063\,275\,128 = Q^{\alpha_{20}}_{25}(f),$$
$$Q^{\beta_{30}}_{35}(f) = 3.057\,961\,330 < I(f) < 3.060\,972\,732 = Q^{\alpha_{30}}_{35}(f).$$

Remark 1.4. Comparing the quadrature formulas $Q^{\alpha_n}_{n+5}(f)$, $Q^{\beta_n}_{n+5}(f)$ with Gauss sixth order quadrature formula

$$Q^G_{3n}(f) := \frac{h}{18}\sum_{j=1}^{n}\Big(5f_{j-\frac{1}{2}-\frac{\sqrt{5}}{10}} + 8f_{j-\frac{1}{2}} + 5f_{j-\frac{1}{2}+\frac{\sqrt{5}}{10}}\Big)$$

we can see that the quadrature $Q^G_{3n}(f)$ has $3n$ function calls whereas the quadratures $Q^{\alpha_n}_{n+5}(f)$, $Q^{\beta_n}_{n+5}(f)$ have $n+5$ function calls. Evidently, $n+5 < 3n$ for $n > 2$.

2. SERIES ESTIMATION VIA BOUNDARY CORRECTIONS WITH PARAMETERS

The sum of a series

$$s := \sum_{n=1}^{\infty} a_n \tag{2.1}$$

can be approximated by a finite sum $\sum_{n=1}^{N} a_n$. The error of this estimation can be represented as the sum of the series $\sum_{n=N+1}^{\infty} a_n$.

Therefore, if we have a method of estimating the sum of an infinite series, then this method will enable us to estimate the error of the N–term approximation. One way to estimate the sum of the series is to take into consideration the fact that a series can be viewed as an integral over an infinite domain

$$I(f) = \int_1^{\infty} f(x)dx \tag{2.2}$$

for some function $f : [1,\infty) \to \mathbb{R}$ for which $f(n) = a_n$ for all n. Therefore, if for a given series, we know an explicitly integrable function $f(x)$ with this property, then we can take the value $I(f)$ of the integral as an estimate for s.

Theorem 2.1. *We assume that the function f is such that:*

(1) *f is either positive and decreasing, or negative and increasing,*
(2) *$\int_1^{\infty} f(x)\,dx$ is convergent,*
(3) *$f \in C^6\big([1-\frac{2\sqrt{5}}{5},\infty)\big)$,*
(4) *$f^{(6)}$ is either positive or negative on $\big[1-\frac{2\sqrt{5}}{10},\infty\big)$. Under this assumptions, if $f^{(6)} > 0$ then*

$$\begin{aligned}\sum_{i=1}^{n-1} f(i) + \frac{f(n)}{2} + \int_n^{\infty} f(x)dx + P_n(-\sqrt{5}, f) < s <\\ < \sum_{i=1}^{n-1} f(i) + \frac{f(n)}{2} + \int_n^{\infty} f(x)dx + P_n(\sqrt{5}, f),\end{aligned} \tag{2.3}$$

where

$$P_n(t,f) := -\frac{t}{12}\left(-3f(n)+4f\Big(n+\frac{t}{10}\Big)-f\Big(n+\frac{t}{5}\Big)\right).$$

If $f^{(6)} < 0$, then we get a similar inequality, but with the right-hand side instead of the left-hand side, and vice versa.

Proof. Let us rewrite the inequality (1.12) in an equivalent form

$$\int\limits_a^{a+nh} f(x)\,dx - G_n(f,\alpha_n) < T_{n+1}(f) < \int\limits_a^{a+nh} f(x)\,dx - G_n(f,\beta_n). \tag{2.4}$$

Bearing in mind the assumptions we can apply the Theorem 1.3 for the function f with $a=m$, $h=1$, $n\geq 4$. In our situation we have

$$T_{n+1}(f) = \sum_{i=m}^{m+n-1} a_i - \frac{1}{2}a_m + \frac{1}{2}a_{m+n},$$

$$\int\limits_a^{a+nh} f(x)\,dx = \int\limits_m^{m+n} f(x)\,dx,$$

$$G_n(f,\zeta) = \frac{1}{24\zeta}\Big(-3\big(f(m)+f(m+n)\big)+$$
$$+4\big(f(m+\zeta)+f(m+n-\zeta)\big)-\big(f(m+2\zeta)+f(m+n-2\zeta)\big)\Big).$$

Passing with n to ∞ in the inequality (2.4) we obtain

$$\int\limits_m^{\infty} f(x)\,dx + P_m(-\sqrt{5},f) \leq \sum_{i=m}^{\infty} a_i - \frac{1}{2}a_m \leq \int\limits_m^{\infty} f(x)\,dx + P_m(\sqrt{5},f). \tag{2.5}$$

We complete the proof by adding the term

$$\frac{1}{2}a_m + \sum_{i=1}^{m-1} a_i$$

to the both sides of the inequality (2.5). □

Theorem 2.1 generalizes results from [2].

Example 3. Let us estimate the value of the Riemann function

$$\zeta(t) = \sum_{i=1}^{\infty}\left(\frac{1}{i}\right)^t$$

for $t\in(1,+\infty)$. In this case

$$f(x) = \frac{1}{x^t}, \qquad f^{(6)}(x) = \frac{t(t+1)(t+2)(t+3)(t+4)(t+5)}{x^{t+6}} > 0$$

and

$$\int_{n}^{+\infty} f(x)\,dx = \frac{1}{t-1} n^{1-t}.$$

In Figure 2 we show the algebraic difference between the upper and lower estimates by the formula (2.3) for $n = 10$ and $n = 15$.

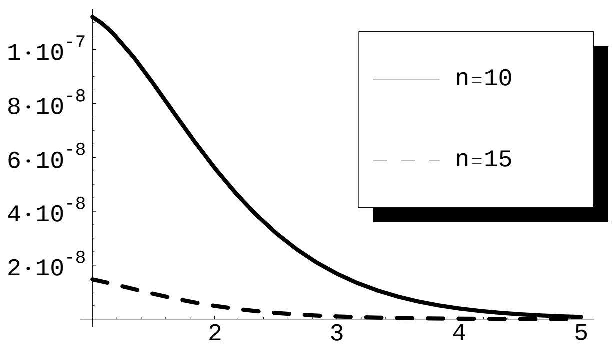

Fig. 2. The difference between the upper and lower estimates by the formula (2.3)

For instance for $n = 15$ we have

$$1.644\,934\,064\,14 \le \zeta(2) = \frac{\pi^2}{6} \approx 1.644\,934\,066\,84 \le 1.644\,934\,069\,06$$

and

$$1.082\,323\,233\,62 \le \zeta(4) = \frac{\pi^4}{90} \approx 1.082\,323\,233\,71 \le 1.082\,323\,233\,77.$$

Example 4. Let us estimate the sum of the series

$$q = \sum_{i=1}^{\infty} \frac{(-1)^i \ln i}{i}.$$

To apply our method we must represent the sum of this series in the desired form. This can be done by combining together the neighboring terms of opposite sign, for instance

$$q = \sum_{i=1}^{\infty} \frac{(-1)^i \ln i}{i} = s_c + s$$

where

$$s_c = \frac{\ln 2}{2} + \sum_{i=1}^{5} \left(\frac{\ln(2i+2)}{2i+2} - \frac{\ln(2i+1)}{2i+1} \right) \approx 0.260\,832\,628\,568\,947\,6,$$

$$s = \sum_{i=1}^{\infty} \left(\frac{\ln(2i+12)}{2i+12} - \frac{\ln(2i+11)}{2i+11} \right).$$

In this case

$$f(x)=\frac{\ln(2x+12)}{2x+12}-\frac{\ln(2x+11)}{2x+11},$$

f is decreasing on interval $[1,+\infty)$,

$$f^{(6)}(x)=112\,896\left(\frac{1}{(2x+11)^7}-\frac{1}{(2x+12)^7}\right)+\\ +46\,080\left(\frac{\ln(2x+12)}{(2x+12)^7}-\frac{\ln(2x+11)}{(2x+11)^7}\right)<0$$

and

$$\int_n^{+\infty} f(x)\,dx=\frac{1}{4}\left(\ln^2(2n+11)-\ln^2(2n+12)\right).$$

Applying the estimate (2.3) with $n=20$ we get

$$-0.100\,963\,724\,864\,2\le s\le -0.100\,963\,724\,784\,6$$

and in consequence

$$0.159\,868\,903\,704\,6\le q\le 0.159\,868\,903\,784\,2.$$

Both estimations in Theorem 2.1 differ from each other by the term $P_n(\cdot,f)$. The arithmetic mean of the upper and the lower estimates is a good approximation of the sum of the series $\sum_{i=1}^{\infty} f(i)$ so we have

$$s:=\sum_{i=1}^{\infty} f(i)\approx s_n:=\sum_{i=1}^{n-1} f(i)+\frac{f(n)}{2}+\int_n^{\infty} f(x)\,dx+S_n(f), \tag{2.6}$$

where

$$S_n(f)=\frac{\sqrt{5}}{24}\left(4\Big(f\big(n-\frac{\sqrt{5}}{10}\big)-f\big(n+\frac{\sqrt{5}}{10}\big)\Big)-\Big(f\big(n-\frac{\sqrt{5}}{5}\big)-f\big(n+\frac{\sqrt{5}}{5}\big)\Big)\right).$$

Example 5. We consider the series

$$s=\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{2i-1}.$$

We can write this series in the form

$$s=\sum_{j=1}^{\infty}\Big(\frac{1}{4j-3}-\frac{1}{4j-1}\Big)=\frac{\pi}{4}.$$

In this case

$$f(x) = \frac{1}{4x-3} - \frac{1}{4x-1},$$
$$f^{(6)}(x) = 2\,949\,120\Big(\frac{1}{(4x-3)^7} - \frac{1}{(4x-1)^7}\Big) > 0$$

and

$$\int_n^\infty f(x)\,dx = \frac{\log(4n-1) - \log(4n-3)}{4}.$$

In the table (tab. 1) below we give some exemplary values of s_n by using the formula (2.6).

Table 1. Exemplary values of s_n

	s_n	$s - s_n$
$n = 10$	0.785 398 162 658 706 361 34	$7.387\,42 \cdot 10^{-10}$
$n = 40$	0.785 398 163 397 413 894 17	$3.441\,54 \cdot 10^{-14}$

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Bogusław Bożek
bozek@uci.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Krakow, Poland

Wiesław Solak
solak@uci.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Krakow, Poland

Zbigniew Szydełko

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Krakow, Poland

Received: July 3, 2008.
Revised: April 8, 2009.
Accepted: April 8, 2009.