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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF POSITIVE CONTINUOUS SOLUTIONS FOR A NONLINEAR ELLIPTIC SYSTEM IN THE HALF SPACE

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Abstract. This paper deals with the existence and the asymptotic behavior of positive continuous solutions of the nonlinear elliptic system $\Delta u = p(x)\, u^{\alpha}\, v^{r}$, $\Delta v = q(x)\, u^{s}\, v^{\beta}$ in the half space $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, $n \geq 2$, where $\alpha, \beta \geq 1$ and $r, s \geq 0$. The functions p and q are required to satisfy some appropriate conditions related to the Kato class $K^{\infty}(\mathbb{R}^n_+)$. Our approach is based on potential theory tools and the use of Schauder's fixed point theorem.

Keywords: asymptotic behavior, elliptic system, regular equation.

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1. INTRODUCTION

In this paper, we consider the upper half space $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ $(n \geq 2)$ and we are interested in the existence of positive continuous solutions (in the sense of distributions) to the following nonlinear elliptic system

$$\begin{cases} \Delta u = p(x)\, u^{\alpha}\, v^{r} \quad \text{in } \mathbb{R}^n_+, \\ \Delta v = q(x)\, u^{s}\, v^{\beta} \quad \text{in } \mathbb{R}^n_+, \\ \lim\limits_{x \to (\xi,0)} u(x) = a\varphi(\xi), \ \ \lim\limits_{x \to (\xi,0)} v(x) = c\psi(\xi), \ \ \forall \xi \in \mathbb{R}^{n-1}, \\ \lim\limits_{x_n \to \infty} \dfrac{u(x)}{x_n} = b, \ \ \lim\limits_{x_n \to \infty} \dfrac{v(x)}{x_n} = d, \end{cases} \tag{1.1}$$

where $\alpha, \beta \geq 1$ and $r, s \geq 0$. The constants a, b, c, d are nonnegative satisfying $(a + b)(c + d) > 0$, φ and ψ are non-trivial nonnegative bounded continuous functions on $\partial\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times \{0\}$ which we identify with $\mathbb{R}^{n-1}$. The functions p and q are nonnegative measurable in $\mathbb{R}^n_+$ satisfying some assumptions related to a certain Kato

class $K^{\infty}(\mathbb{R}^n_+)$ which was introduced by Bachar and Mâagli in [2] for $n \geq 3$ and by Bachar *et al.* in [3] for $n = 2$.
For reader convenience, we recall the definition of the class $K^{\infty}(\mathbb{R}^n_+)$.

Definition 1.1. A Borel measurable function q in $\mathbb{R}^n_+$ belongs to the Kato class $K^{\infty}(\mathbb{R}^n_+)$ if q satisfies

$$\lim_{\alpha \to 0} \left(\sup_{x \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+ \cap B(x,\alpha)} \frac{y_n}{x_n} G(x,y) \, |q(y)| \, dy \right) = 0$$

and

$$\lim_{M \to \infty} \left(\sup_{x \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+ \cap (|y| \geq M)} \frac{y_n}{x_n} G(x,y) \, |q(y)| \, dy \right) = 0.$$

Here and throughout this paper, $G(x,y)$ denotes the Green function of $(-\Delta)$ in $\mathbb{R}^n_+$. The elliptic Kato class $K^{\infty}(\mathbb{R}^n_+)$ is quite rich. In particular, it contains the classical Kato class $K^{\infty}_n(\mathbb{R}^n_+)$, for $n \geq 3$, used in the study of elliptic equations (see [18] for definition and properties).

In the following, we give some subclasses of functions belonging to $K^{\infty}(\mathbb{R}^n_+)$.

Proposition 1.2 ([2] and [3])**.** *(a) Let $p > \frac{n}{2}$ and $n \geq 3$. Then we have*

$$L^p(\mathbb{R}^n_+) \cap L^1(\mathbb{R}^n_+) \subset K^{\infty}(\mathbb{R}^n_+).$$

(b) Let $\lambda, \mu \in \mathbb{R}$ and $q(x) = \dfrac{1}{(1+|x|)^{\mu-\lambda} x_n^{\lambda}}$ for $x \in \mathbb{R}^n_+$. Then the function q is in $K^{\infty}(\mathbb{R}^n_+)$ if and only if $\lambda < 2 < \mu$.

Systems of type (1.1) have received considerable attention in the last few years. So several results have been obtained in both the bounded and unbounded domain $D \subset \mathbb{R}^n$ with different boundary conditions (see for example [6-9, 16, 17] and the references therein).

The motivation of our study of system (1.1) comes from the results proved in [4, 10, 11, 13-15]. In fact, in [4], Bachar *et al.* discussed the existence and the asymptotic behavior of solutions of the elliptic equation

$$\Delta u - u f(., u) = 0,$$

in $\mathbb{R}^n_+$ subject to some boundary conditions.

As is mentioned above, the main goal of this paper is to prove an existence result for system (1.1). For this aim, we shall study the existence of positive solutions for the following nonlinear elliptic problem

$$\begin{cases} \Delta u = p(x)\, u^{\alpha} \quad \text{in } \mathbb{R}^n_+, \\ \lim\limits_{x \to (\xi,0)} u(x) = a\varphi(\xi), \quad \forall \xi \in \mathbb{R}^{n-1}, \\ \lim\limits_{x_n \to \infty} \dfrac{u(x)}{x_n} = b, \end{cases} \tag{1.2}$$

where $\alpha \geq 1$. The constants a, b are nonnegative satisfying $a+b>0$, φ is a non-trivial nonnegative bounded continuous function in $\mathbb{R}^{n-1}$ and p is a nonnegative measurable function in $\mathbb{R}^n_+$.

Throughout this paper, we shall refer to the bounded continuous solution Hg of the following Dirichlet problem (see [1])

$$\begin{cases} \Delta u = 0 \quad \text{in } \mathbb{R}^n_+, \\ \lim\limits_{x \to (\xi,0)} u(x) = g(\xi), \quad \forall \xi \in \mathbb{R}^{n-1}, \end{cases}$$

where g is a non-trivial nonnegative bounded continuous function in $\mathbb{R}^{n-1}$.
Also, we refer to the potential of a nonnegative measurable function f, defined in $\mathbb{R}^n_+$ by

$$Vf(x) = \int\limits_{\mathbb{R}^n_+} G(x,y) f(y) dy.$$

We recall that the following assertions are equivalent for each nonnegative measurable function f in $\mathbb{R}^n_+$:

(i) $Vf \neq \infty$, and consequently $Vf \in L^1_{loc}(\mathbb{R}^n_+)$,
(ii) $\int_{\mathbb{R}^n_+} \frac{y_n}{(1+|y|)^n} f(y) dy < \infty$.

Hence for each nonnegative measurable function f in $\mathbb{R}^n_+$ such that $Vf \in L^1_{loc}(\mathbb{R}^n_+)$, we have

$$\Delta(Vf) = -f \quad \text{(in the sense of distributions)}.$$

The plan of this paper is organized as follows. In Section 2, we recapitulate some properties of functions belonging to $K^\infty(\mathbb{R}^n_+)$ developed in [2–4] and adopted to our interests. Section 3 is devoted to undertake a study of problem (1.2) by adopting similar techniques as in [4] based on potential theory tools. In fact, we consider two nonnegative real numbers a, b satisfying $a+b>0$ and φ a non-trivial nonnegative bounded continuous function in $\mathbb{R}^{n-1}$. Let ω and h be the harmonic functions defined in $\mathbb{R}^n_+$ by $\omega(x) = b\,x_n + a$ and $h(x) = b\,x_n + a\,H\varphi(x)$. The function p is required to satisfy the following hypothesis

(H_0) p is a nonnegative measurable function in $\mathbb{R}^n_+$ such that

$$x \to p(x)\,\omega^{\alpha-1}(x)$$

is in $K^\infty(\mathbb{R}^n_+)$.

Then we shall prove the following theorem.

Theorem 1.3. *Assume (H_0). Then problem (1.2) has a unique positive continuous solution u satisfying for each $x \in \mathbb{R}^n_+$,*

$$ch(x) \leq u(x) \leq h(x),$$

where $c \in (0,1)$.

In Section 4, we shall apply the result stated in Theorem 1.3 to investigate the existence and the behavior of positive solutions for system (1.1). By a positive solution of (1.1) we mean a pair of continuous functions (u, v) such that $u > 0$ and $v > 0$ in $\mathbb{R}^n_+$ and (u, v) satisfies (1.1).

To this end, let a, b, c, d be nonnegative constants satisfying $(a + b)(c + d) > 0$ and we fix φ and ψ two non-trivial nonnegative bounded continuous functions in $\mathbb{R}^{n-1}$. We set θ, ω, h and k the harmonic functions defined in $\mathbb{R}^n_+$ by $\omega(x) = b\,x_n + a$, $\theta(x) = d\,x_n + c$, $h(x) = b\,x_n + a\,H\varphi(x)$ and $k(x) = d\,x_n + c\,H\psi(x)$.
We need to assume the following hypothesis on functions p and q.

(H) p and q are nonnegative measurable functions in $\mathbb{R}^n_+$ such that

$$x \to p(x)\,\theta^r(x)\,\omega^{\alpha-1}(x) \ \text{ and } \ x \to q(x)\,\omega^s(x)\theta^{\beta-1}(x)$$

are in $K^\infty(\mathbb{R}^n_+)$.

Using the Schauder's fixed point theorem in a suitable closed convex subset in $(C(\overline{\mathbb{R}^n_+} \cup \{\infty\}))^2$, we obtain the following theorem.

Theorem 1.4. *Assume* (H). *Then system* (1.1) *has a positive continuous solution* (u, v) *satisfying for each* $x \in \mathbb{R}^n_+$,

$$c_1\,h(x) \le u(x) \le h(x) \ \text{ and } \ c_2\,k(x) \le v(x) \le k(x),$$

where $c_1, c_2 \in (0, 1)$.

2. NOTATIONS AND PRELIMINARIES

In this Section we discuss different notations and we recall some properties of functions belonging to the Kato class $K^\infty(\mathbb{R}^n_+)$.

As usual, we denote by $\mathcal{B}^+(\mathbb{R}^n_+)$ the set of nonnegative measurable functions in $\mathbb{R}^n_+$. We also denote by

$$C(\mathbb{R}^n_+) = \{w : \mathbb{R}^n_+ \to \mathbb{R} \colon w \text{ is continuous}\},$$

$$C_0(\mathbb{R}^n_+) = \{w \in C(\mathbb{R}^n_+) \colon \lim_{x \to (\xi, 0)} w(x) = 0 \text{ and } \lim_{|x| \to \infty} w(x) = 0\}$$

and

$$C_0(\overline{\mathbb{R}^n_+}) = \{w \in C(\overline{\mathbb{R}^n_+}) \colon \lim_{|x| \to \infty} w(x) = 0\}.$$

Next, we need to recall some potential theory tools. Fore more details, we refer the reader to [12] and [5]. Let $(X_t, t > 0)$ be the Brownian motion in $\mathbb{R}^n$ and P^x be the probability measure on the Brownian continuous paths starting at x. For a nonnegative measurable function q in $\mathbb{R}^n_+$, we denote by V_q the kernel defined by

$$V_q f(x) = E^x\left(\int_0^\tau e^{-\int_0^t q(X_s)ds}\, f(X_t)dt \right),$$

where E^x is the expectation on P^x and $\tau = \inf\{t > 0 : \ X_t \notin \mathbb{R}^n_+\}$ is the first exit time of $(X_t,\, t > 0)$ from $\mathbb{R}^n_+$.
Furthermore, if q satisfies $Vq < \infty$, then we have the following resolvent equation

$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{2.1}$$

So, for each measurable function u in $\mathbb{R}^n_+$ such that $V(q|u|) < \infty$, we have

$$(I - V_q(q.))(I + V(q.))u = (I + V(q.))(I - V_q(q.))u = u. \tag{2.2}$$

Now, we collect some preliminary results pertaining to the Kato class $K^\infty(\mathbb{R}^n_+)$ which will be used in the next Section.

Proposition 2.1 ([2] and [3]). *Let q be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$. Then we have:*

(i) $\alpha_q := \sup\limits_{x,y \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \dfrac{G(x,z)G(z,y)}{G(x,y)} q(z)dz < \infty$;

(ii) *the function* $x \to \dfrac{x_n}{(|x|+1)^n} q(x)$ *is in* $L^1(\mathbb{R}^n_+)$.

Proposition 2.2 ([4]). *Let q be a function in $K^\infty(\mathbb{R}^n_+)$ and v be a nonnegative superharmonic function in $\mathbb{R}^n_+$. Then for each $x \in \mathbb{R}^n_+$, we have*

$$\int_{\mathbb{R}^n_+} G(x,y)v(y)|q(y)|dy \le \alpha_q\, v(x). \tag{2.3}$$

For a fixed nonnegative function q in $K^\infty(\mathbb{R}^n_+)$, we denote

$$\mathcal{M}_q := \{f \in K^\infty(\mathbb{R}^n_+) \colon |f| \le q\}.$$

Proposition 2.3 ([4]). *Let q be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$ and $\beta \in \{0,1\}$. Then the family of functions*

$$\left\{ \int_{\mathbb{R}^n_+} \left(\frac{y_n}{x_n}\right)^\beta G(x,y)f(y)dy : \ f \in \mathcal{M}_q \right\}$$

is relatively compact in $C_0(\mathbb{R}^n_+)$, for $\beta = 0$ and which is relatively compact in $C_0(\overline{\mathbb{R}^n_+})$, for $\beta = 1$.

The next proposition plays a key role in the proof of Theorem 1.3.

Proposition 2.4 ([4]). *Let q be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$ and v be a nonnegative superharmonic function in $\mathbb{R}^n_+$. Then for each $x \in \mathbb{R}^n_+$ such that $0 < v(x) < \infty$, we have*

$$\exp(-\alpha_q)v(x) \le (v - V_q(qv))(x) \le v(x).$$

3. PROOF OF THEOREM 1.3

In this Section, we aim at proving Theorem 1.3. So we need the following lemmas.

Lemma 3.1. *Let f and g be two nonnegative measurable functions in $\mathbb{R}^n_+$ such that $g \le f$ and Vf is continuous in $\mathbb{R}^n_+$. Then Vg is also continuous in $\mathbb{R}^n_+$.*

Proof. Let ϕ be a nonnegative measurable function in $\mathbb{R}^n_+$ such that $f = g + \phi$. It is obvious that $V\phi$ and Vg are lower semi-continuous in $\mathbb{R}^n_+$ and $V\phi$ is finite. So, we deduce that Vg is continuous in $\mathbb{R}^n_+$. □

Next, we recall that $\omega(x) = b\,x_n + a$, $x \in \mathbb{R}^n_+$, where a, b are nonnegative constants satisfying $a + b > 0$.

Lemma 3.2. *Let q be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$. Then we have:*

(i) *The family of functions*

$$\left\{ \int_{\mathbb{R}^n_+} \frac{\omega(y)}{\omega(x)} G(x,y)|f(y)|dy : \ f \in \mathcal{M}_q \right\}$$

is relatively compact in $C_0(\overline{\mathbb{R}^n_+})$.

(ii) $\lim\limits_{x \to (\xi,0)} V(\omega\, q)(x) = 0, \ \ \forall \xi \in \mathbb{R}^{n-1}$.

Proof. Since

$$\frac{\omega(y)}{\omega(x)} = \frac{b\,y_n + a}{b\,x_n + a} \le \max\left(1, \frac{y_n}{x_n}\right) \le 1 + \frac{y_n}{x_n},$$

then (i) follows from Proposition 2.3.

Next, we shall prove (ii). Using the fact that q is in $K^\infty(\mathbb{R}^n_+)$, we obtain from Proposition 2.3 that the function $v(x) = \int_{\mathbb{R}^n_+} \frac{y_n}{x_n} G(x,y)q(y)dy$ is in $C_0(\overline{\mathbb{R}^n_+})$. Hence v is bounded in $\mathbb{R}^n_+$.

Now, taking into account that

$$V(\omega\, q)(x) = b \int_{\mathbb{R}^n_+} y_n G(x,y)q(y)dy + a\, Vq(x) = bx_n\, v(x) + a\, Vq(x),$$

the assertion (ii) holds immediately from Proposition 2.3. □

In the sequel, consider φ a non-trivial nonnegative bounded continuous function in $\mathbb{R}^{n-1}$. Let h be the harmonic function defined in $\mathbb{R}^n_+$ by $h(x) = b\,x_n + a\,H\varphi(x)$ and p be a nonnegative measurable function in $\mathbb{R}^n_+$ satisfying (H_0). Put $\lambda = \max(1, \|\varphi\|_\infty)$ and $q(x) = \alpha\,\lambda^{\alpha-1} p(x) \omega^{\alpha-1}(x)$, $x \in \mathbb{R}^n_+$.

Lemma 3.3. *Let u be a continuous function satisfying $0 \le u \le h$ in $\mathbb{R}^n_+$. Then u is a solution of problem (1.2) if and only if u satisfies the integral equation*

$$u + V(p\, u^\alpha) = h. \tag{3.1}$$

Proof. Suppose that u is a solution of problem (1.2) satisfying $0 \le u \le h$. Since

$$h(x) \le b\lambda x_n + a\,\|\varphi\|_\infty \le \lambda\,\omega(x), \tag{3.2}$$

we deduce that

$$pu^\alpha \le \alpha p h^\alpha \le \lambda\omega q. \tag{3.3}$$

According to Lemma 3.2 (i) and Lemma 3.1, we can see that $V(pu^\alpha) \in C(\mathbb{R}^n_+)$. So u satisfies

$$\begin{cases} \Delta(u + V(p\,u^\alpha)) = 0, \\ \lim\limits_{x\to(\xi,0)} u(x) = a\varphi(\xi), \quad \forall \xi \in \mathbb{R}^{n-1}, \\ \lim\limits_{x_n\to\infty} \dfrac{u(x)}{x_n} = b. \end{cases}$$

Thus u satisfies the integral equation (3.1).

Conversely, from (3.3) and by using Lemma 3.2 (i), we have $\frac{1}{\omega}V(pu^\alpha) \in C_0(\overline{\mathbb{R}^n_+})$. Hence by (3.1), we conclude that u satisfies (in the sense of distributions) the equation

$$\Delta u = pu^\alpha \text{ in } \mathbb{R}^n_+.$$

On the other hand, since $\frac{1}{\omega}V(pu^\alpha) \in C_0(\overline{\mathbb{R}^n_+})$ and $H\varphi$ is continuous and bounded on $\mathbb{R}^n_+$ satisfying $\lim\limits_{x\to(\xi,0)} H\varphi(x) = \varphi(\xi)$, we deduce from (3.1), (3.3) and Lemma 3.2 (ii) that $\lim\limits_{x\to(\xi,0)} u(x) = a\varphi(\xi)$ and $\lim\limits_{x_n\to\infty} \dfrac{u(x)}{x_n} = b$. □

Now, we are ready to state the proof of Theorem 1.3.

Proof of Theorem 1.3. We consider the non-empty closed convex set Λ given by

$$\Lambda := \{u \in \mathcal{B}^+(\mathbb{R}^n_+) :\ \exp(-\alpha_q)h \le u \le h\},$$

where α_q is the constant given in Proposition 2.1.

We define the operator T on Λ by

$$Tu(x) := h(x) - V_q(qh)(x) + V_q(qu - pu^\alpha)(x).$$

We need to check that the operator T has a fixed point u in Λ. To this end, we first prove that $T\Lambda \subset \Lambda$. Indeed, for $u \in \Lambda$, we have

$$Tu(x) \le h(x) + V_q(qu)(x) - V_q(qh)(x) \le h(x).$$

This implies by (3.2) that for $u \in \Lambda$,

$$q - pu^{\alpha-1} = \alpha\,\lambda^{\alpha-1} p\,\omega^{\alpha-1} - pu^{\alpha-1} \ge p(h^{\alpha-1} - u^{\alpha-1}) \ge 0. \tag{3.4}$$

Hence $Tu \ge h - V_q(qh)$ and by Proposition 2.4, we obtain

$$Tu \ge \exp(-\alpha_q)h.$$

Next, we claim that the operator T is nondecreasing on Λ. Let $u, v \in \Lambda$ such that $u \leq v$ and consider the function $L : t \to t(q(x) - t^{\alpha-1}p(x)\omega^{\alpha-1}(x))$. By differentiation, it is clear that L is nondecreasing on $[0, \lambda]$. Then we deduce that

$$Tv - Tu = V_q((q - pv^{\alpha-1})v - (q - pu^{\alpha-1})u) = V_q\left(\omega\Big(L\Big(\frac{v}{\omega}\Big) - L\Big(\frac{u}{\omega}\Big)\Big)\right) \geq 0.$$

Now, we consider the sequence (u_k) defined by

$$u_0 = h - V_q(qh) \text{ and } u_{k+1} = Tu_k \text{ for } k \in \mathbb{N}.$$

Since $T\Lambda \subset \Lambda$, then by (3.4) and the monotonicity of T, we obtain

$$u_0 \leq u_1 \leq ... \leq u_k \leq u_{k+1} \leq h.$$

Hence by (2.3) and the dominated convergence theorem, the sequence (u_k) converges to a function $u \in \Lambda$, which satisfies

$$u = h - V_q(qh) + V_q(qu - pu^\alpha),$$

which means that

$$(I - V_q(q.))u = (I - V_q(q.))h - V_q(pu^\alpha).$$

So applying the operator $(I + V(q.))$ on both sides of the last equality, we deduce by (2.1) and (2.2) that u satisfies the integral equation

$$u = h - V(pu^\alpha).$$

Now, using (3.3) and Lemma 3.2 (i), we have $\frac{1}{\omega}V(pu^\alpha) \in C_0(\overline{\mathbb{R}^n_+})$. So we deduce from Lemma 3.3 that u is a positive continuous solution of problem (1.2).

For the remainder of the proof, we aim to show that u is the unique solution of problem (1.2) satisfying $0 \leq u \leq h$.

To this end, suppose that problem (1.2) has two positive continuous solutions u and v such that $0 \leq u \leq h$ and $0 \leq v \leq h$.

It follows from Lemma 3.3 that

$$(u - v) + V(p(u^\alpha - v^\alpha)) = 0.$$

This yields

$$(u - v) + V((u - v)k) = (I + V(k.))(u - v) = 0,$$

where

$$k(x) = \begin{cases} \dfrac{u^\alpha(x) - v^\alpha(x)}{u(x) - v(x)}p(x) & \text{if } u(x) \neq v(x), \\ 0 & \text{if } u(x) = v(x). \end{cases}$$

Now, since u and v satisfy (3.1), we have from (3.3) and (2.3) that

$$V(k|u - v|) \leq V(p(u^\alpha + v^\alpha)) \leq 2\lambda\, V(\omega\, q) \leq 2\lambda\, \alpha_q\, \omega.$$

Then we deduce from (2.2) that $u = v$. This completes the proof. $\square$

4. PROOF OF THEOREM 1.4

Proof of Theorem 1.4. Let us consider two non-trivial nonnegative bounded continuous functions φ and ψ in $\mathbb{R}^{n-1}$ and assume that hypothesis (H) is satisfied. We recall that $\omega(x) = b\,x_n + a$, $\theta(x) = d\,x_n + c$, $h(x) = b\,x_n + a\,H\varphi(x)$ and $k(x) = d\,x_n + c\,H\psi(x)$, where a, b, c, d are nonnegative constants satisfying $(a+b)(c+d) > 0$. Put $m = \max(1, \|\varphi\|_\infty, \|\psi\|_\infty)$,

$$\begin{aligned}\widetilde{p}(x) &= \alpha\, m^{r+\alpha-1} p(x)\omega^{\alpha-1}(x)\theta^r(x) \quad \text{and}\\ \widetilde{q}(x) &= \beta\, m^{s+\beta-1} q(x)\omega^{s}(x)\theta^{\beta-1}(x) \quad \text{for} \quad x \in \mathbb{R}^n_+.\end{aligned}$$

Let Λ be the non-empty closed convex set given by

$$\Lambda = \Big\{(u,v) \in (C(\overline{\mathbb{R}^n_+} \cup \{\infty\}))^2 : 0 \le u \le (1-\exp(-\alpha_{\widetilde{p}}))\frac{h}{\omega}, 0 \le v \le (1-\exp(-\alpha_{\widetilde{q}}))\frac{k}{\theta}\Big\}.$$

By applying the result stated in Theorem 1.3, we shall define the operator T on Λ by

$$T(u,v) = \Big(\frac{h-y}{\omega}, \frac{k-z}{\theta}\Big),$$

where (y,z) is the unique solution of the following system

$$\begin{cases} \Delta y = p(x)\, y^\alpha\, (k - \theta\, v)^r, \\ \Delta z = q(x)\, (h - \omega\, u)^s\, z^\beta, \\ \lim\limits_{x \to (\xi,0)} y(x) = a\varphi(\xi), \quad \lim\limits_{x \to (\xi,0)} z(x) = c\psi(\xi), \\ \lim\limits_{x_n \to \infty} \dfrac{y(x)}{x_n} = b, \quad \lim\limits_{x_n \to \infty} \dfrac{z(x)}{x_n} = d, \end{cases}$$

satisfying

$$\exp(-\alpha_{\widetilde{p}})\, h \le y \le h \ \text{ and } \ \exp(-\alpha_{\widetilde{q}})\, k \le z \le k. \tag{4.1}$$

We intend to prove that T has a fixed point in Λ. Let $(u,v) \in \Lambda$. Then, we have

$$p\, y^\alpha\, (k - \theta\, v)^r \le p h^\alpha\, k^r \le \frac{m}{\alpha}\omega\, \widetilde{p} \tag{4.2}$$

and

$$q\, z^\beta\, (h - \omega\, u)^s \le q k^\beta\, h^s \le \frac{m}{\beta}\theta\, \widetilde{q}. \tag{4.3}$$

Now, since y and z satisfy the integral equations

$$y = h - V(p\, y^\alpha\, (k - \theta\, v)^r) \tag{4.4}$$

and

$$z = k - V(q\, z^\beta (h - \omega\, u)^s),$$

we deduce by using (4.2), (4.3), (H) and Lemma 3.2 (i) that the family of functions

$$T\Lambda := \Big\{x \to \frac{1}{\omega(x)}\, V(p\, y^\alpha\, (k - \theta\, v)^r)(x), x \to \frac{1}{\theta(x)}\, V(q\, z^\beta (h - \omega\, u)^s)(x) \,:\, (u,v) \in \Lambda\Big\}$$

is relatively compact in $(C_0(\overline{\mathbb{R}^n_+}))^2$. Next, for $(u,v)\in\Lambda$, we have clearly from (4.1)

$$T(u,v)=\Big(\frac{h-y}{\omega},\frac{k-z}{\theta}\Big)\in\Lambda.$$

To achieve the proof, we need to prove the continuity of the operator T with respect to the norm $\|\cdot\|$ defined by $\|(u,v)\|=\|u\|_\infty+\|v\|_\infty$. Let $((u_j,v_j))$ be a sequence in Λ which converges to $(u,v)\in\Lambda$ with respect to the norm $\|.\|$. Let

$$T(u_j,v_j)=\Big(\frac{h-y_j}{\omega},\ \frac{k-z_j}{\theta}\Big)$$

and

$$T(u,v)=\Big(\frac{h-y}{\omega},\ \frac{k-z}{\theta}\Big).$$

Then, we have

$$\Big\|T(u_j,v_j)-T(u,v)\Big\|=\Big\|\frac{y_j-y}{\omega}\|_\infty+\|\frac{z_j-z}{\theta}\Big\|_\infty.$$

According to (4.4), we obtain

$$\begin{aligned}y_j-y&=V(p\,y^\alpha\,(k-\theta\,v)^r)-V(p\,y_j^\alpha\,(k-\theta\,v_j)^r)=\\&=V(p[y^\alpha((k-\theta\,v)^r-(k-\theta\,v_j)^r)+(k-\theta\,v_j)^r(y^\alpha-y_j^\alpha)]).\end{aligned}$$

Put

$$K_j(x)=\begin{cases}\dfrac{y_j^\alpha(x)-y^\alpha(x)}{y_j(x)-y(x)}\,(k(x)-\theta(x)\,v_j(x))^r\,p(x) & \text{if } \ y_j(x)\neq y(x),\\ 0 & \text{if } \ y_j(x)=y(x).\end{cases}$$

Then, we have

$$(I+V(K_j.))(y_j-y)=V(p\,y^\alpha((k-\theta\,v)^r-(k-\theta\,v_j)^r)). \tag{4.5}$$

By using the fact that $\widetilde{p}\in K^\infty(\mathbb{R}^n_+)$, (4.2) and (2.3), we get

$$V(K_j|y_j-y|)\le V(p\,y_j^\alpha(k-\theta\,v_j)^r)+V(py^\alpha(k-\theta\,v_j)^r)\le\frac{2m}{\alpha}\alpha_{\widetilde{p}}\,\omega.$$

So applying the operator $(I-V_{K_j}(K_j.))$ on both sides of (4.5), we deduce by (2.1) and (2.2) that

$$y_j-y=V_{K_j}(p\,y^\alpha((k-\theta\,v)^r-(k-\theta\,v_j)^r)).$$

On the other hand, from (4.2), we have

$$p\,y^\alpha|(k-\theta\,v)^r-(k-\theta\,v_j)^r|\le p\,y^\alpha((k-\theta\,v)^r+(k-\theta\,v_j)^r)\le\frac{2m}{\alpha}\,\omega\widetilde{p}.$$

So, from hypothesis (H), (2.3) and by the dominated convergence theorem we deduce that
$$\lim_{j\to\infty} V(p\,y^{\alpha}|(k-\theta\,v)^r-(k-\theta\,v_j)^r|)=0.$$
Now, since
$$V_{K_j}(p\,y^{\alpha}|(k-\theta\,v)^r-(k-\theta\,v_j)^r|)\le V(p\,y^{\alpha}|(k-\theta\,v)^r-(k-\theta\,v_j)^r|),$$
it follows that for each $x\in\mathbb{R}^n_+$, $(y_j(x))$ converges to $y(x)$ as $j\to\infty$.

A similar argument as above, shows that for each $x\in\mathbb{R}^n_+$, $(z_j(x))$ converges to $z(x)$ as $j\to\infty$.

Consequently, as $T(\Lambda)$ is relatively compact in $(C_0(\overline{\mathbb{R}^n_+}))^2$, we deduce that the pointwise convergence implies uniform convergence, namely, $\|\frac{y_j-y}{\omega}\|_\infty+\|\frac{z_j-z}{\theta}\|_\infty$ converges to 0 as $j\to\infty$.

This shows that T is a continuous mapping from Λ into itself. Then by Schauder's fixed point theorem, there exists $(u,v)\in\Lambda$ such that $T(u,v)=(u,v)$.

Now, put $\widetilde{u}=h-\omega\,u$ and $\widetilde{v}=k-\theta\,v$, we obtain that $(\widetilde{u},\widetilde{v})$ is a positive continuous solution for system (1.1) satisfying for each $x\in\mathbb{R}^n_+$
$$\exp(-\alpha_{\widetilde{p}})\,h(x)\le\widetilde{u}(x)\le h(x)\quad\text{and}\quad\exp(-\alpha_{\widetilde{q}})\,k(x)\le\widetilde{v}(x)\le k(x).\qquad\square$$

Example 4.1. Let $\alpha\ge1$, $\beta\ge1$, $r\ge0$, $s\ge0$ and a,b,c,d be nonnegative real numbers with $(a+b)(c+d)>0$. Let p and q be two nonnegative measurable functions in $\mathbb{R}^n_+$ satisfying
$$p(x)\le\frac{1}{(|x|+1)^{\mu-\gamma+r+\alpha-1}\,x_n^{\gamma}}$$
and
$$q(x)\le\frac{1}{(|x|+1)^{\nu-\delta+s+\beta-1}\,x_n^{\delta}},$$
where $\gamma<2<\mu$ and $\delta<2<\nu$. Put $\omega(x)=b\,x_n+a$ and $\theta(x)=d\,x_n+c$ for each $x\in\mathbb{R}^n_+$. Then, there exists $\widetilde{c}>0$ such that for each $x\in\mathbb{R}^n_+$, we have
$$p(x)\,\theta^r(x)\,\omega^{\alpha-1}(x)\le\widetilde{c}\frac{(x_n+1)^{r+\alpha-1}}{(|x|+1)^{\mu-\gamma+r+\alpha-1}\,x_n^{\gamma}}\le\frac{\widetilde{c}}{(|x|+1)^{\mu-\gamma}\,x_n^{\gamma}}$$
and
$$q(x)\,\omega^s(x)\,\theta^{\beta-1}(x)\le\frac{\widetilde{c}}{(|x|+1)^{\nu-\delta}\,x_n^{\delta}}.$$
Hence, hypothesis (H) is well satisfied. So for φ and ψ two non-trivial nonnegative bounded continuous functions in $\mathbb{R}^{n-1}$, system (1.1) has a positive continuous solution (u,v) satisfying for each $x\in\mathbb{R}^n_+$,
$$c_1(b\,x_n+a\,H\varphi(x))\le u(x)\le b\,x_n+a\,H\varphi(x)$$
and
$$c_2(d\,x_n+c\,H\psi(x))\le v(x)\le d\,x_n+c\,H\psi(x),$$
where $c_1,c_2\in(0,1)$.

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