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# ON THE SOLVABILITY OF DIRICHLET PROBLEM FOR THE WEIGHTED $p$-LAPLACIAN

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**Abstract.** The paper investigates the existence and uniqueness of weak solutions for a non-linear boundary value problem involving the weighted $p$-Laplacian. Our approach is based on variational principles and representation properties of the associated spaces.

**Keywords:** $p$-Laplacian, weak solutions, solvability.

**Mathematics Subject Classification:** 35A15, 35J20, 35J60.

## 1. INTRODUCTION

In this paper we are concerned with the existence and uniqueness of the weak solution to the boundary value problem

$$D(\Omega) : \begin{cases} -\Delta_{a,p} v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in which $\Delta_{a,p}$, with $1 < p < \infty$, denotes the $p$-Laplacian weighted by a vector-valued function $a = (a_1, \ldots, a_N)$, that can be (formally) given by

$$\Delta_{a,p} v = \operatorname{div}(a(x)|\nabla v|^{p-2}\overline{\nabla v}), \tag{1.2}$$

where $\nabla v$ denotes the weak gradient of a function $v$ and, respectively, div means the divergence operator (also understood in the weak sense). We treat the problem under general conditions on the weight function $a$, namely, we suppose that the components $a_j$ $(j = 1, \ldots, N)$ of $a$ are measurable functions on $\Omega$ such that

$$a_j(x) \geq 0 \text{ for } x \in \Omega \text{ a.e., } a_j \in L^1_{loc}(\Omega) \text{ and } 1/a_j \in L_\infty(\Omega) \ (j = 1, \ldots, N). \tag{1.3}$$

$\Omega$ is considered an arbitrary open domain in $\mathbb{R}^N$. We do not assume any smoothness conditions on its boundary $\partial\Omega$, it is not even assumed that the boundary has Lebesgue measure zero.

Boundary value problem involving the $p$-Laplacian and in general quasi-linear elliptic differential operators were extensively studied by many authors. We restrict ourselves to cite only the works [7, 9, 10] and references therein, and also the survey [4] (see also [3]) for more recent results. The methods used were mostly based on the technique of monotone operators developed by Leray-Lions [14] (see also [7], Section I.1.6, Leray-Lions Theorem, p.31). In this context, it should be noted the variational methods proposed in [2, 13, 15].

Our approach is based on a variational method related to that used in Hilbert spaces case. The essence of it is to interpret the problem as a generalized Dirichlet problem by involving a non-linear form defined on a suitable space which we denote by $W_a^{1,p}(\Omega)$. We prove that for any elements $f$ from the dual space $W_a^{-1,p}(\Omega)$ there exists a uniquely weak solution $v \in W_a^{1,p}(\Omega)$ of the boundary problem (1.1). Moreover the set of all weak solutions of the problem is covered by the entire space $W_a^{1,p}(\Omega)$ whence $f$ runs through on $W_a^{-1,p}(\Omega)$. The main results are presented by Theorem 2.2 in Section 2. The proof of the main results is given in Section 3.

## 2. DIRICHLET PROBLEM FOR THE WEIGHTED $P$-LAPLACIAN

The problem (1.1) will be considered as *the generelized Dirichlet problem* written in a variational form, namely, for a given locally sumable function $f$, we write

$$\int_\Omega a(x)\nabla u|\nabla v|^{p-2}\overline{\nabla v}\,dx = \int_\Omega u\overline{f}\,dx \quad \text{for all} \quad u \in C_0^\infty(\Omega). \tag{2.1}$$

Further on, we assume that the components $a_j$ $(j = 1, \ldots, N)$ of the vector-valued function $a$ are measurable functions satisfying conditions (1.3).

Under these conditions we consider the following (non-linear) form

$$a[u, v] = \int_\Omega a(x)\nabla u|\nabla v|^{p-2}\overline{\nabla v}\,dx \tag{2.2}$$

defined on functions $u, v \in C_0^\infty(\Omega)$.

It will need the following auxiliary results.

**Lemma 2.1.** *Under conditions* (1.3) *there holds the following inequality*

$$\int_\Omega |\nabla u|^p\,dx \le c\int_\Omega |a(x)\nabla u|^p\,dx \tag{2.3}$$

*for all* $u \in C_0^\infty(\Omega)$.

*Proof.* By using the Hölder inequality we have

$$\int_\Omega |\nabla u|^p dx \le \|a^{-1}\|_{L_\infty(\Omega)} \int_\Omega |a(x)\nabla u|\nabla u|^{p-2}\overline{\nabla u}|dx \le$$

$$\le \|a^{-1}\|_{L_\infty(\Omega)} \Big(\int_\Omega |a(x)\nabla u|^p dx\Big)^{\frac{1}{p}} \Big(\int_\Omega |\nabla u|^{(p-1)q} dx\Big)^{\frac{1}{q}},$$

where $q$ is the conjugate number of $p$. Hence

$$\Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{1}{p}} \le \|a^{-1}\|_{L_\infty(\Omega)} \Big(\int_\Omega |a(x)\nabla u|^p dx\Big)^{\frac{1}{p}},$$

that is the desired inequality. □

Next, we consider on $C_0^\infty(\Omega)$ the functional

$$\|u\|_a = \Big(\int_\Omega |a(x)\nabla u|^p dx\Big)^{\frac{1}{p}} \quad \text{for} \quad u \in C_0^\infty(\Omega)$$

which obviously is a norm on $C_0^\infty(\Omega)$. We will denote the completion of $C_0^\infty(\Omega)$ with respect to the metric of this norm $\|\cdot\|_a$ by $W_a^{1,p}(\Omega)$. We will need some properties of the obtained space $W_a^{1,p}(\Omega)$. In this context, note that it is a uniformly convex space (for the concept of the uniformly convex spaces see, for instance, [12]) and therefore there holds a representation theorem for linear continuous functionals defined on it (see [12, Theorem 8.2, p. 288]). Besides, the space $W_a^{1,p}(\Omega)$ can be realized by elements of the Sobolev space $W_0^{1,p}(\Omega)$, more exactly $W_a^{1,p}(\Omega)$ can be embedded continuously in $W_0^{1,p}(\Omega)$. In fact, for any $u \in W_a^{1,p}(\Omega)$ there exists a sequence of elements $u_n \in C_0^\infty(\Omega)$ such that

$$\|u_n - u\|_a \to 0, \quad n \to \infty.$$

By Lemma 2.1 one also has

$$\int_\Omega |\nabla(u_n - u_m)|^p dx \to 0, \quad n, m \to \infty,$$

or, what is the same,

$$\|u_n - u_m\|_{W_0^{1,p}(\Omega)} \to 0, \quad n, m \to \infty.$$

Due to the fact that $W_0^{1,p}(\Omega)$ is a complete space there exists an element $v \in W_0^{1,p}(\Omega)$ such that

$$\|u_n - v\|_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as} \quad n \to \infty.$$

The element $v$ depends only on $u$ and it does not depend on the chosen sequence $(u_n)$. So, the elements $u$, $v$ can be identified provided that the norm $\|\cdot\|_a$ is compatible with the Sobolev norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$. The compatibility means that if

$$\|u_n\|_{W_0^{1,p}(\Omega)} \to 0 \quad \text{and} \quad \|u_n - u\|_a \to 0,$$

then $u = 0$. To prove this fact, without loss generality, we can assume that $\nabla u_n(x) \to 0$ almost everywhere (otherwise we may pass to a suitable subsequence of $(u_n)$). For any $\varepsilon > 0$ one has

$$\|u_m - u_n\|_a < \varepsilon$$

for sufficiently large $n$ and $m$. By applying Fatou's Lemma,

$$\|u_n\|_a^p = \int\limits_\Omega |a(x)\nabla u_n|^p dx = \int\limits_\Omega \lim_{m\to\infty} |a(x)\nabla(u_n - u_m)|^p dx \leq$$

$$\leq \liminf_{m\to\infty} \int\limits_\Omega |a(x)\nabla(u_n - u_m)|^p dx \leq \varepsilon^p,$$

we see that $u_n \to 0$ with respect to the topology norm of $W_a^{1,p}(\Omega)$, and thus $u = 0$.

Next, we denote by $W_a^{-1,p}(\Omega)$ the dual space of $W_a^{1,p}(\Omega)$. Descriptions of dual Sobolev type spaces are found in [1] (see also [11] for related results concerning weighted Sobolev spaces). Besides, we also note the works [5, 6] for some abstract more general results, but for Hilbert spaces case.

Our main result is the following.

**Theorem 2.2.** *Suppose that the conditions* (1.3) *are fulfilled. For $f \in W_a^{-1,p}(\Omega)$, the Dirichlet problem* (2.1) *has a unique weak solution $v \in W_a^{1,p}(\Omega)$, i.e.*

$$\int\limits_\Omega a(x)\nabla u|\nabla v|^{p-2}\overline{\nabla v}\, dx = \langle u, f\rangle$$

*for all $u \in C_0^\infty(\Omega)$ (or, equivalently, for any $u \in W_a^{1,p}(\Omega)$). Moreover, the set of all weak solutions, where $f$ runs through $f \in W_a^{-1,p}(\Omega)$ is the entire space $W_a^{1,p}(\Omega)$.*

Next, for $p < N$ we let $p^* = Np/(N-p)$ for the critical Sobolev exponent, and denote $s'$ for the conjugate number of any $s \in [p, p^*]$. It follows from the Sobolev embedding theorems that any element $f \in L_{s'}(\Omega)$ can be viewed as an element in $W_0^{-1,p}(\Omega)$ (see, for instance, [8, Theorem 3.7, p. 230]. Thus, from the fact that $W_a^{1,p}(\Omega)$ is embedded continuously in $W_0^{1,p}(\Omega)$, $f$ can be treated as an element of the space $W_a^{-1,p}(\Omega)$. Taking into account this fact, we can formulate.

**Corollary 2.3.** *Under conditions* (1.3) *for every $f \in L_{s'}(\Omega)$ there exists a unique weak solution $v \in W_a^{1,p}(\Omega)$ solving problem* (2.1).

## 3. PROOF OF THEOREM 2.2

Due to the estimate (2.3) in Lemma 2.1 the form $a[u,v]$ defined by (2.2) can be extended on elements of the space $W_a^{1,p}(\Omega)$. To this form we can associate an operator $A$ from $W_a^{1,p}(\Omega)$ into $W_a^{-1,p}(\Omega)$ as follows. For any $v \in W_a^{1,p}(\Omega)$, we consider the functional $f$ defined on $W_a^{1,p}(\Omega)$ by

$$\langle u, f\rangle = \int_\Omega a(x)\nabla u|\nabla v|^{p-2}\overline{\nabla v}\,dx, \quad u \in W_a^{1,p}(\Omega).$$

This functional is linear and bounded, hence $f \in W_a^{-1,p}(\Omega)$. Moreover the operator $A$ satisfies the following variational equation

$$\langle u, Av\rangle = a[u,v] \quad \text{for all} \quad u, v \in W_a^{1,p}(\Omega), \tag{3.1}$$

where $\langle u, v\rangle = \int_\Omega u\overline{v}\,dx$.

Next, we change the topology of the space $W_a^{1,p}(\Omega)$ by defining the following metric on this space

$$d_a(u,v) = \sup_{\|w\|_a=1} |a[w,u] - a[w,v]|, \tag{3.2}$$

i.e.,

$$d_a(u,v) = \sup_{\|w\|_a=1} \Big| \int_\Omega a(x)\nabla w\Big(|\nabla u|^{p-2}\overline{\nabla u} - |\nabla v|^{p-2}\overline{\nabla v}\Big)\,dx\Big|, \quad u, v \in W_a^{1,p}(\Omega).$$

$W_a^{1,p}(\Omega)$ equipped with the metric $d_a$ becomes a complete metric space. Moreover, for any $u, v \in W_a^{1,p}(\Omega)$ we have

$$\|Au - Av\|_{W_a^{-1,p}(\Omega)} = \sup_{\|w\|_a=1} |\,\langle w, Au - Av\rangle\,| = \sup_{\|w\|_a=1} |a[w,u] - a[w,v]| = d_a(u,v),$$

hence $A$ is an isometry viewed as an operator from the metric space $(W_a^{1,p}(\Omega), d_a)$ into $W_a^{-1,p}(\Omega)$. Due to the fact that the space $W_a^{1,p}(\Omega)$ is a uniformly convex space (that implies it is a strictly convex space) the operator $A$ is injective. Besides, $A$ is surjective from the representation theorem, i.e. to every continuous linear functional $f \in W_a^{-1,p}(\Omega)$ there exists a unique element $v \in W_a^{1,p}(\Omega)$ such that

$$\langle u, f\rangle = a[u,v] \quad \text{for all} \quad u \in W_a^{1,p}(\Omega).$$

Hence, $A$ is a bijection between the space $W_a^{1,p}(\Omega)$ and $W_a^{-1,p}(\Omega)$ and there exists the inverse operator $A^{-1}$. Therefore, for $f \in W_a^{-1,p}(\Omega)$ the element $v = A^{-1}f$ is a weak solution of the generalized Dirichlet problem and the set of those solutions covers the entire space $W_a^{1,p}(\Omega)$ whence $f$ runs through the dual space $W_a^{-1,p}(\Omega)$. The proof of Theorem 2.2 is complete. □

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