Opuscula Mathematica • Vol. 32 • No. 4 • 2012

http://dx.doi.org/10.7494/OpMath.2012.32.4.715

A NOTE ON THE INDEPENDENT ROMAN DOMINATION IN UNICYCLIC GRAPHS

Mustapha Chellali and Nader Jafari Rad

Abstract. A Roman dominating function (RDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. An RDF f in a graph G is independent if no two vertices assigned positive values are adjacent. The Roman domination number $\gamma_R(G)$ (respectively, the independent Roman domination number $i_R(G)$) is the minimum weight of an RDF (respectively, independent RDF) on G. We say that $\gamma_R(G)$ strongly equals $i_R(G)$, denoted by $\gamma_R(G) \equiv i_R(G)$, if every RDF on G of minimum weight is independent. In this note we characterize all unicyclic graphs G with $\gamma_R(G) \equiv i_R(G)$.

Keywords: Roman domination, independent Roman domination, strong equality.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *degree* of v, denoted by $d_G(v)$, is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. If v is a support vertex, then v is called *strong* if v is adjacent to at least two leaves.

For a graph G, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V = V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* **(RDF)** on G if every vertex u of G for which $f(u) = 0$ is adjacent to at least one vertex v of G for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. An RDF f in a graph G is independent if no two vertices assigned positive values are adjacent. The *Roman domination number* $\gamma_R(G)$ (respectively, the *independent Roman domination number* $i_R(G)$) is the minimum weight of an RDF (respectively, independent RDF) on G. A function $f = (V_0; V_1; V_2)$ is called a $\gamma_R(G)$-function or γ_R-function for G if it is a Roman dominating function on G and $f(V(G)) = \gamma_R(G)$. An $i_R(G)$-function or i_R-function for G is defined similarly. Let f be a $\gamma_R(G)$-function, and $f(x) = 0$ for some vertex x. Then we say that x is a *private neighbor* of a vertex y with $f(y) = 2$ if f is not an RDF for $G-xy$. Roman domination has been introduced by Cockayne et al. [3] and has been studied for example in [7]. The study of independent Roman domination has been initiated in [1].

We say that $\gamma_R(G)$ and $i_R(G)$ are *strongly equal* for G, denoted by $\gamma_R(G) \equiv i_R(G)$, if every $\gamma_R(G)$-function is an $i_R(G)$-function. In [2] a constructive characterization of all trees T with $\gamma_R(T) \equiv i_R(T)$ is provided. Note that strong equality between two parameters was considered first by Haynes and Slater [6]. Later Haynes, Henning and Slater gave in [4] and [5] constructive characterizations of trees with strong equality between some domination parameters.

In this note we characterize all unicyclic graphs G with $\gamma_R(G) \equiv i_R(G)$.

2. MAIN RESULT

We first describe the procedure given in [2] to built trees T with $\gamma_R(T) \equiv i_R(T)$. Let $\mathcal{T}$ be the family of trees T that can be obtained from k $(k \geq 1)$ disjoint stars of centers $x_1, x_2, ..., x_k$, where each star has order at least three, attached by edges from their center vertices either to a single vertex or to the same leaf of a path P_2. Such a vertex is called a special vertex of T. Let $\mathcal{F}$ be the collection of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees, where T_1 is a star $K_{1,t}$ with $t \geq 2$, $T = T_k$, and, if $k \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations:

— **Operation** $\mathcal{O}_1$: Assume y is a leaf of T_i with $f_i(y) = 0$ and whose support vertex z is either strong or satisfies $\gamma_R(T_i - z) > \gamma_R(T_i)$. Then T_{i+1} is obtained from T_i by adding a new vertex x and adding the edge xy.
— **Operation** $\mathcal{O}_2$: Assume y is a vertex of T_i. Then T_{i+1} is obtained from T_i by adding a tree $T \in \mathcal{T}$ of special vertex x and adding the edge xy with the condition that if x is a support vertex, then y satisfies $\gamma_R(T_i - y) \geq \gamma_R(T_i)$.
— **Operation** $\mathcal{O}_3$: Assume y is a vertex of T_i assigned 0 or 1 for every $\gamma_R(T_i)$-function. Then T_{i+1} is obtained from T_i by adding a path $P_3 = u\text{-}v\text{-}w$ and adding the edge wy.

Theorem 2.1 (Chellali and Jafari Rad [2]). *Let T be a tree. Then $\gamma_R(T) \equiv i_R(T)$ if and only if $T = K_1$ or $T \in \mathcal{F}$.*

Let $\mathcal{H}$ be the class of all graphs G such that G is obtained from a tree $T \in \mathcal{F}$ by joining two non-adjacent vertices v_1, v_2 such that:

(1) For every $\gamma_R(T)$-function f, $0 \in \{f(v_1), f(v_2)\}$,

(2) For $1 \leq i \neq j \leq 2$, there is no non-independent RDF f for $T - v_i$ with weight $\gamma_R(T)$ such that $f(v_j) = 2$.

Now we are ready to state our main result.

Theorem 2.2. *Let G be a unicyclic graph. Then $\gamma_R(G) \equiv i_R(G)$ if and only if $G \in \mathcal{H}$.*

Proof. Let G be a unicyclic graph, where C is its unique cycle. Assume that $\gamma_R(G) \equiv i_R(G)$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. By assumption f is independent. Let $x \in V(C) \cap V_0$, and let $N(x) \cap V(C) = \{y, z\}$. Clearly x cannot be a private neighbor for both y and z. Hence we assume that x is not a private neighbor of y and let $T = G - xy$. Then f is an IRDF for T, and so $\gamma_R(T) \leq i_R(T) \leq \gamma_R(G) = i_R(G)$. If $\gamma_R(T) < i_R(G)$, and f_1 is a $\gamma_R(T)$-function, then f_1 is an RDF for G with weight less than $\gamma_R(G)$, a contradiction. Thus $\gamma_R(T) = i_R(T) = i_R(G) = \gamma_R(G)$. Next we show that any $\gamma_R(T)$-function is independent. Assume to the contrary that f is a $\gamma_R(T)$-function and f is not independent. Since f is an RDF for G and $\gamma_R(G) = \gamma_R(T)$, we obtain that f is a $\gamma_R(G)$-function, contradicting the fact that $\gamma_R(G) \equiv i_R(G)$. Thus f is independent and consequently, $\gamma_R(T) \equiv i_R(T)$. We deduce that $T \in \mathcal{F}$.

Next we prove (1). Suppose that there is a $\gamma_R(T)$-function f such that $0 \notin \{f(x), f(y)\}$. If $\{f(x), f(y)\} = \{2, 1\}$ and $f(x) = 1$, then g defined on G by $g(x) = 0$ and $g(u) = f(u)$ if $u \neq x$ is an RDF for G with weight less than $\gamma_R(G)$, a contradiction. Thus $\{f(x), f(y)\} \neq \{2, 1\}$ but then f would be a non-independent $\gamma_R(G)$-function, a contradiction since $\gamma_R(G) \equiv i_R(G)$.

Finally, let us prove (2). Assume that there is a non-independent RDF f for $T - x$ with weight $\gamma_R(T)$ such that $f(y) = 2$. Then f is a $\gamma_R(G)$-function which is not independent, a contradiction.

Conversely, assume that $G \in \mathcal{H}$. Let G be obtained from a tree $T \in \mathcal{F}$ by joining two vertices x and y such that (1) and (2) hold. First notice that $\gamma_R(G) \leq \gamma_R(T)$. Assume to the contrary that $\gamma_R(G) < \gamma_R(T)$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. If $\{f(x), f(y)\} \neq \{0, 2\}$, then f is an RDF for T with weight less than $\gamma_R(T)$, a contradiction. Thus $\{f(x), f(y)\} = \{0, 2\}$. Suppose that $f(y) = 0$. Then $N(y) \cap V_2 = \{x\}$. Now g defined on T by $g(y) = 1$ and $g(u) = f(u)$ if $u \neq y$, is an RDF for T. Then $w(g) = \gamma_R(T)$ for otherwise g is an RDF for T with weight less than $\gamma_R(T)$ which is impossible. Hence g is a $\gamma_R(T)$-function and $0 \notin \{g(x), g(y)\}$, contradicting (1). Therefore $\gamma_R(G) = \gamma_R(T)$. Now let h be an $i_R(T)$-function. Note that h is a $\gamma_R(T)$-function since $\gamma_R(T) \equiv i_R(T)$. If h is not an IRDF for G, then $0 \notin \{h(x), h(y)\}$, and h is a $\gamma_R(T)$-function that does not satisfy (1), a contradiction. Thus h is an IRDF for G, and so $i_R(G) \leq \gamma_R(T) = \gamma_R(G) \leq i_R(G)$, implying that $i_R(G) = \gamma_R(G) = \gamma_R(T) = i_R(T)$. So h is an $i_R(G)$-function. We next show that each $\gamma_R(G)$-function is independent. Assume to the contrary that $f = (V_0, V_1, V_2)$ is a $\gamma_R(G)$-function and f is not independent. If $0 \notin \{f(x), f(y)\}$, then f is a $\gamma_R(T)$-function which is not independent, contradicting the fact that $T \in \mathcal{F}$. Thus $0 \in \{f(x), f(y)\}$, and we may assume that $f(y) = 0$. Furthermore, $N(y) \cap V_2 = \{x\}$. Then $f|_{T-y}$ is an IRDF for $T - y$ with weight $\gamma_R(T)$ and $f(x) = 2$, a contradiction with (2). We deduce that $\gamma_R(G) \equiv i_R(G)$. □

Acknowledgements

This research was supported by "Programmes Nationaux de Recherche: Code 8/u09/510".

This research of the second author was supported by Shahrood University of Technology.

REFERENCES

[1] M. Adabi, E. Ebrahimi Targhi, N. Jafari Rad, M. Saied Moradi, *Properties of independent Roman domination in graphs*, Australasian Journal of Combinatorics **52** (2012), 11–18.

[2] M. Chellali, N. Jafari Rad, *Strong equality between the Roman domination and independent Roman domination numbers in trees*, Discussiones Mathematicae Graph Theory, to appear.

[3] E.J. Cockayne, P.M. Dreyer Jr., S.M. Hedetniemi, S.T. Hedetniemi, *On Roman domination in graphs*, Discrete Mathematics **278** (2004), 11–22.

[4] T.W. Haynes, M.A. Henning, P.J. Slater, *Strong equality of domination parameters in trees*, Discrete Mathematics **260** (2003), 77–87.

[5] T.W. Haynes, M.A. Henning P.J. Slater, *Strong equality of upper domination and independence in trees*, Utilitas Math. **59** (2001), 111–124.

[6] T.W. Haynes, P.J. Slater, *Paired-domination in graphs*, Networks **32** (1998), 199–206.

[7] M.A. Henning, *A characterization of Roman trees*, Discussiones Mathematicae Graph Theory **22** (2002), 325–334.

Mustapha Chellali
m_chellali@yahoo.com

University of Blida
LAMDA-RO Laboratory, Department of Mathematics
B.P. 270, Blida, Algeria

Nader Jafari Rad
n.jafarirad@shahroodut.ac.ir

Shahrood University of Technology
Department of Mathematics
Shahrood, Iran

Received: January 10, 2011.
Revised: May 2, 2012.
Accepted: May 7, 2012.