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Dedicated to Professor Semyon Yakubovich on the occasion of his 50th birthday

SYMBOLIC APPROACH TO THE GENERAL CUBIC DECOMPOSITION OF POLYNOMIAL SEQUENCES. RESULTS FOR SEVERAL ORTHOGONAL AND SYMMETRIC CASES

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Abstract. We deal with a symbolic approach to the cubic decomposition (CD) of polynomial sequences – presented in a previous article referenced herein – which allows us to compute explicitly the first elements of the nine component sequences of a CD. Properties are investigated and several experimental results are discussed, related to the CD of some widely known orthogonal sequences. Results concerning the symmetric character of the component sequences are established.

Keywords: symmetric polynomials, orthogonal polynomials, cubic decomposition, symbolic computations, Mathematica 8.0.1.0.

Mathematics Subject Classification: 33C45, 33D45, 42C05, 33F10, 68W30, 68-04.

INTRODUCTION

The most general cubic decomposition (CD) of a monic polynomial sequence (MPS) was presented recently in [9,12] and constitutes a complete polynomial CD of a given MPS. In this last reference, the reader can find extensive information and a bibliography about this subject. Indeed, this kind of CD is a natural feature of 2-orthogonal sequences and in wider contexts might lead us to the knowledge of new polynomial sequences or to the study of some properties of the polynomial sequences involved. Nevertheless, the extensive calculations involved can sometimes discourage an analytical treatment of certain conjectures. In the present work, we use symbolic computations in order to compute explicitly the first elements of each one of the nine component sequences of a CD of a given MPS $\{W_n\}_{n\geq 0}$, and to investigate some properties of these sequences, namely their linear independence and their orthogonal or symmetric character. The computer algebra manipulation software chosen to accomplish this purpose was *Mathematica 8.01.0* [13,14].

Once we have found the first polynomials of a component sequence, we can determine the correspondent first structure coefficients and, consequently, investigate its orthogonal character, or other properties. Indeed, we constructed commands whose aim is to calculate the structure coefficients of any monic polynomial sequence and to test its orthogonality. Other *Mathematica* commands were used in order to examine the six component sequences of a CD which are not necessarily linearly independent, neither monic.

In the last years, the use of computer algebra manipulation in the framework of orthogonal polynomials has been developed. We can refer to CAOP [5], a package for calculating formulas for orthogonal polynomials belonging to the Askey scheme in *Maple*, an approach based on special functions available on the internet elaborated by W. Koepf and R. Swarttouw. Also, using *Matlab*, W. Gautschi presents in [2–4] routines dealing with orthogonal polynomials and applications, in order to develop the constructive, computational and software aspects of the practice of this domain. For this matter, we remark that the instruments presented here aim to study polynomial sequences involved in a complete cubic decomposition of a given MPS not necessarily orthogonal.

Next, we summarize the contents of this work. In the first section, we present the basic notions and fundamental results needed in the sequel. Section two is devoted to the explanation of the implementation performed in *Mathematica*. We remark that the corresponding *Mathematica* notebook can be consulted in [11]. In section three, some concrete orthogonal sequences are taken as examples of study, with a special attention to symmetric sequences, for specific choices of the six parameters of the CD. Section four is reserved to the demonstration of some symmetry properties - fulfilled by any MPS - which caught our attention during the examination of the results obtained for the set of examples considered. We present, also, in that section, two tables whose aim is to summarize the conclusions obtained. Finally, we indicate some general comments to the symbolic work developed in this article.

1. BASIC NOTIONS AND FUNDAMENTAL RESULTS

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$. In the following, we will call polynomial sequence (PS) to any sequence $\{W_n\}_{n\geq 0}$ such that $\deg W_n = n$, for all $n \geq 0$. In this sense, a PS will always be a free sequence. We refer to PS so that, in each polynomial, the leading coefficient is equal to one, as a monic polynomial sequence (MPS).

Given a MPS $\{W_n\}_{n\geq 0}$, there are complex sequences $\{\beta_n\}_{n\geq 0}$ and $\{\chi_{n,\nu}\}_{0\leq\nu\leq n,\, n\geq 0}$ such that

$$W_0(x) = 1; \quad W_1(x) = x - \beta_0; \tag{1.1}$$

$$W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{n,\nu} W_\nu(x), \;\; n \geq 0. \tag{1.2}$$

This relation is called the structure relation of $\{W_n\}_{n\geq 0}$, and $\{\beta_n\}_{n\geq 0}$ and $\{\chi_{n,\nu}\}_{0\leq\nu\leq n,\ n\geq 0}$ are called the correspondent structure coefficients. They define each MPS and are known for a very wide range of MPSs.

A polynomial sequence $\{W_n(x)\}_{n\geq 0}$ is said to be symmetric if and only if $W_n(-x) = (-1)^n W_n(x)$, $n \geq 0$. For each MPS $\{W_n\}_{n\geq 0}$, the following statements are equivalent [6]:

a) $\{W_n\}_{n\geq 0}$ is symmetric;
b) $\beta_n = 0;\quad \chi_{2n+1,2\nu} = 0,\ \ 0 \leq \nu \leq n,\ \ n \geq 0;\ \chi_{2n,2\nu+1} = 0,\ \ 0 \leq \nu \leq n-1,\ \ n \geq 1.$

A PS $\{W_n\}_{n\geq 0}$ is regularly orthogonal with respect to the form u if and only if it fulfils:

$$\langle u, W_n W_m\rangle = 0,\ \ n \neq m,\ \ n,m \geq 0, \text{ and } \langle u, W_n^2\rangle \neq 0,\ n \geq 0 \ [6,8].$$

It is well known that the structure relation (1.1)–(1.2) of a regular monic orthogonal PS (MOPS) becomes the following second order recurrence relation [1, 7], since $\chi_{n,\nu} = 0$, $0 \leq \nu < n$, $n \geq 0$, and recalling that $\gamma_{n+1} = \chi_{n,n} \neq 0$, $n \geq 0$,

$$W_0(x) = 1;\quad W_1(x) = x - \beta_0;\quad W_{n+2}(x) = (x - \beta_{n+1})W_{n+1} - \gamma_{n+1}W_n(x),\ \ n \geq 0, \tag{1.3}$$

which characterizes the orthogonality of MPS $\{W_n\}_{n\geq 0}$. In this case, the structure coefficients are called recurrence coefficients.

1.1. CUBIC DECOMPOSITION OF A MONIC POLYNOMIAL SEQUENCE

In [9] the most general cubic decomposition of a given MPS was presented. Indeed, fixing a monic cubic polynomial

$$\varpi(x) = x^3 + p\,x^2 + q\,x + r; \tag{1.4}$$

by its three coefficients p, q and r, and three constants a, b and c, it was proved, using Euclidean division (of a polynomial $W(x)$ by $\varpi(x)$) and induction on n, the following result.

Proposition 1.1 ([9])**.** *Given any MPS $\{W_n\}_{n\geq 0}$, there are three MPSs $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, so that*

$$W_{3n}(x) = P_n(\varpi(x)) + (x-a)a^1_{n-1}(\varpi(x)) + (x-b)(x-c)a^2_{n-1}(\varpi(x)), \tag{1.5}$$
$$W_{3n+1}(x) = b^1_n(\varpi(x)) + (x-a)Q_n(\varpi(x)) + (x-b)(x-c)b^2_{n-1}(\varpi(x)), \tag{1.6}$$
$$W_{3n+2}(x) = c^1_n(\varpi(x)) + (x-a)c^2_n(\varpi(x)) + (x-b)(x-c)R_n(\varpi(x)), \tag{1.7}$$

with $\deg a^1_{n-1} \leq n-1$, $\deg a^2_{n-1} \leq n-1$, $\deg b^1_n \leq n$, $\deg b^2_{n-1} \leq n-1$, $\deg c^1_n \leq n$ *and* $\deg c^2_n \leq n$.

In the cubic decomposition (CD) (1.5)–(1.7) of $\{W_n\}_{n\geq 0}$, the sequences:

— $\{P_n\}_{n\geq 0}, \{Q_n\}_{n\geq 0}, \{R_n\}_{n\geq 0}$ are called the principal components;

— $\{a^1_{n-1}\}_{n\geq 0}$, $\{a^2_{n-1}\}_{n\geq 0}$, $\{b^1_n\}_{n\geq 0}$, $\{b^2_{n-1}\}_{n\geq 0}$, $\{c^1_n\}_{n\geq 0}$, $\{c^2_n\}_{n\geq 0}$ are called the secondary components.

In other words, the component sequences are divided in two sets: three principal components which are MPSs, in the sense mentioned before, more precisely, $\deg P_n = \deg Q_n = \deg R_n = n$, $n \geq 0$, and six secondary components which are not necessarily free subsets of $\mathcal{P}$, neither monic. Theorem 2.4 of reference [9] characterizes the component sequences of a CD of $\{W_n\}_{n\geq 0}$ in terms of its structure coefficients and it is enunciated as follows.

Theorem 1.2 ([9])**.** *A MPS $\{W_n\}_{n\geq 0}$, with structure coefficients* (1.1)–(1.2)*, admits the CD* (1.5)–(1.7) *if and only if the following relations are fulfilled for $n \geq 0$,*

$$(Z_0) \qquad b^1_0(x) = a - \beta_0,$$

$$(Z_1) \qquad c^1_n(x) = -\sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}\, b^1_\nu(x) - (\beta_{3n+1} - a) b^1_n(x) + \Theta(x) b^2_{n-1}(x) - \\ - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}\, c^1_\nu(x) - \sum_{\nu=0}^{n} \chi_{3n,3\nu}\, P_\nu(x) - (a-b)(a-c) Q_n(x),$$

$$(Z_2) \qquad c^2_n(x) = -\sum_{\nu=0}^{n} \chi_{3n,3\nu}\, a^1_{\nu-1}(x) + b^1_n(x) + L b^2_{n-1}(x) - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}\, c^2_\nu(x) - \\ - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}\, Q_\nu(x) - (\beta_{3n+1} + a - b - c) Q_n(x),$$

$$(Z_3) \qquad R_n(x) = -\sum_{\nu=0}^{n} \chi_{3n,3\nu}\, a^2_{\nu-1}(x) - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+1}\, b^2_{\nu-1}(x) - \\ - (\beta_{3n+1} + b + c + p) b^2_{n-1}(x) + Q_n(x) - \sum_{\nu=0}^{n-1} \chi_{3n,3\nu+2}\, R_\nu(x),$$

$$(Z_4) \quad P_{n+1}(x) = -\sum_{\nu=0}^{n} \chi_{3n+1,3\nu}\, P_\nu(x) - (\beta_{3n+2} - a) c^1_n(x) - \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2}\, c^1_\nu(x) - \\ - \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1}\, b^1_\nu(x) - (a-b)(a-c) c^2_n(x) + \Theta(x) R_n(x),$$

$$(Z_5) \qquad a^1_n(x) = -\sum_{\nu=0}^{n} \chi_{3n+1,3\nu}\, a^1_{\nu-1}(x) + c^1_n(x) - \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2}\, c^2_\nu(x) - \\ - (\beta_{3n+2} + a - b - c) c^2_n(x) - \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1}\, Q_\nu(x) + L R_n(x),$$

$$(Z_6) \qquad a_n^2(x) = -\sum_{\nu=0}^{n} \chi_{3n+1,3\nu}\, a_{\nu-1}^2(x) - \sum_{\nu=0}^{n} \chi_{3n+1,3\nu+1}\, b_{\nu-1}^2(x) + c_n^2(x) - \\ - \sum_{\nu=0}^{n-1} \chi_{3n+1,3\nu+2}\, R_\nu(x) - (\beta_{3n+2} + b + c + p) R_n(x),$$

$$(Z_7) \qquad b_{n+1}^1(x) = -(a-b)(a-c)a_n^1(x) + \Theta(x) a_n^2(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1}\, b_\nu^1(x) - \\ - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2}\, c_\nu^1(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu}\, P_\nu(x) - (\beta_{3n+3} - a) P_{n+1}(x),$$

$$(Z_8) \quad Q_{n+1}(x) = -\sum_{\nu=0}^{n} \chi_{3n+2,3\nu}\, a_{\nu-1}^1(x) - (\beta_{3n+3} + a - b - c) a_n^1(x) + L a_n^2(x) - \\ - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2}\, c_\nu^2(x) + P_{n+1}(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1}\, Q_\nu(x),$$

$$(Z_9) \qquad b_n^2(x) = a_n^1(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu}\, a_{\nu-1}^2(x) - (\beta_{3n+3} + b + c + p) a_n^2(x) - \\ - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+1}\, b_{\nu-1}^2(x) - \sum_{\nu=0}^{n} \chi_{3n+2,3\nu+2}\, R_\nu(x),$$

where by convention $\sum_{\nu=0}^{-1} . = 0$, *and*

$$\Theta(x) = x - r + aL + bc(b + c + p), \tag{1.8}$$

$$L = bc - q - (b + c + p)(b + c). \tag{1.9}$$

1.2. CUBIC DECOMPOSITION OF A MONIC ORTHOGONAL POLYNOMIAL SEQUENCE

Let us suppose that $\{W_n\}_{n\geq 0}$ is a MOPS. Then, as a consequence of Theorem 1.2, the principal components fulfil the three relations that we reproduce in the following theorem, each one beginning as a recurrence relation of second order for each principal component and completed with elements of only two secondary component sequences.

Theorem 1.3 ([9]). *For a MOPS with CD given by* (1.5)–(1.7)*, the correspondent principal components fulfill the following relations, for* $n \geq 0$.

$$P_{n+2}(x) = \big(\Theta(x) - A_{3n}\big) P_{n+1}(x) - B_{3n} P_n(x) - \\ - K_{3n} b_n^1(x) - H_{3n} b_{n+1}^1(x) - V_{3n} c_n^1(x) - S_{3n} c_{n+1}^1(x), \tag{1.10}$$

$$Q_{n+2}(x) = \big(\Theta(x) - A_{3n+1}\big) Q_{n+1}(x) - B_{3n+1} Q_n(x) - \\ - K_{3n+1} c_n^2(x) - H_{3n+1} c_{n+1}^2(x) - V_{3n+1} a_n^1(x) - S_{3n+1} a_{n+1}^1(x), \tag{1.11}$$

$$R_{n+2}(x) = \big(\Theta(x) - A_{3n+2}\big)R_{n+1}(x) - B_{3n+2}R_n(x) - \\ - K_{3n+2}a_n^2(x) - H_{3n+2}a_{n+1}^2(x) - V_{3n+2}b_n^2(x) - S_{3n+2}b_{n+1}^2(x), \tag{1.12}$$

where

$$A_n = \gamma_{n+3}(\beta_{n+2} + 2\beta_{n+3} + p) + \gamma_{n+4}(2\beta_{n+3} + \beta_{n+4} + p) + \\ +(\beta_{n+3}-a)\Big((\beta_{n+3}+a-b-c)(\beta_{n+3}+b+c+p)-L\Big)+(a-b)(a-c)(\beta_{n+3}+b+c+p);$$

$$B_n = \gamma_{n+1}\gamma_{n+2}\gamma_{n+3};$$

$$K_n = \gamma_{n+2}\gamma_{n+3}\,(\beta_{n+1} + \beta_{n+2} + \beta_{n+3} + p);$$

$$H_n = \gamma_{n+3} + \gamma_{n+4} + \gamma_{n+5} + (a-b)(a-c) - L + \\ +(\beta_{n+3} + a - b - c)(\beta_{n+3} + b + c + p) + (\beta_{n+4} - a)(\beta_{n+3} + \beta_{n+4} + a + p);$$

$$V_n = \gamma_{n+3}\Big(\gamma_{n+2} + \gamma_{n+3} + \gamma_{n+4} + (a-b)(a-c) - L + \\ +(\beta_{n+3} + a - b - c)(\beta_{n+3} + b + c + p) + (\beta_{n+2} - a)(\beta_{n+2} + \beta_{n+3} + a + p)\Big);$$

$$S_n = \beta_{n+3} + \beta_{n+4} + \beta_{n+5} + p.$$

Corollary 1.4. *In the context of Theorem 1.3, if*

$$K_n = H_n = V_n = S_n = 0,\ n \geq 0,$$

then all the principal components are orthogonal.

Proof. Under the hypotheses taken, and considering Theorem 1.3, the three principal components fulfil recurrence relations of order two of type (1.3), assuring the orthogonality of these sequences. □

2. SYMBOLIC IMPLEMENTATION OF THE CUBIC DECOMPOSITION

2.1. RECURSIVE COMPUTATION OF ALL COMPONENT SEQUENCES

The symbolic implementation of relations $(Z_0) - (Z_9)$, (1.8)–(1.9) permits us to compute the first elements of the nine component sequences for any given MPS. The required initial data are:

— the polynomial $\varpi(x)$, by its coefficients p, q and r;
— the zeros a, b and c of the auxiliary polynomials;
— the structure coefficients definitions of $\{\beta_n\}_{n\geq 0}$ and $\{\chi_{n,\nu}\}_{0\leq\nu\leq n,\, n\geq 0}$ for every n, or their first elements $\{\beta_n\}_{n=0,\ldots,3nmax+1}$ and $\{\chi_{n,\nu}\}_{0\leq\nu\leq n,\, n=0,\ldots,3nmax}$.

However, the components of a CD of any polynomial $W(x)$ can, also, be computed exactly as they appear in (1.5)–(1.7), that is, as linear combinations of elements of the set $\{(\varpi(x))^n, (x-a)(\varpi(x))^m, (x-b)(x-c)(\varpi(x))^k,\ n, m, k$ positive integers$\}$. Such a procedure, with arguments $W(x)$, p, q, r, a, b and c, requires a previous definition of the MPS $\{W_n\}_{n\geq 0}$ that we intend to decompose, given its structure coefficients, and posterior definitions of the component sequences for final retrieval. Nevertheless, comparing the two approaches, we find that the first approach given by the relations $(Z_0) - (Z_9)$ is more efficient.

For any kind of procedure, it is useful to assemble the nine component sequences in the following matrix (which was used in the demonstration of Theorem 1.2)

$$M_n(x) = \begin{pmatrix} P_n(x) & a^1_{n-1}(x) & a^2_{n-1}(x) \\ b^1_n(x) & Q_n(x) & b^2_{n-1}(x) \\ c^1_n(x) & c^2_n(x) & R_n(x) \end{pmatrix},$$

and to present the first $nmax$ matrices $M_0, M_1, \ldots, M_{nmax}$ of any CD of $\{W_n\}_{n\geq 0}$, for a fixed non-negative integer $nmax$.

For each example (for each set of data, as indicated before), we may observe, for the first elements of component sequences, the following aspects: existence of zero secondary components, the degrees of secondary components, the symmetric character, among other aspects. Next, we cite one example of a non-symmetric orthogonal sequence, where $a = b = c = 0$ and $p = q = r = 0$. Introducing the structure coefficients definitions of the Laguerre sequence [1], with parameter 0, which are $\beta_n = 2n+1$; $\chi_{n,n} = (n+1)^2$; $\chi_{n,\nu} = 0$, we obtain for $n = 0, 1, 2$, respectively,

$$M_0(x) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}, \quad M_1(x) = \begin{pmatrix} x-6 & 18 & -9 \\ -16x+24 & x-96 & 72 \\ 200x-120 & -25x+600 & x-600 \end{pmatrix},$$

$$M_2(x) = \begin{pmatrix} x^2-2400x+720 & 450x-4320 & -36x+5400 \\ -49x^2+29400x-5040 & x^2-7350x+35280 & 882x-52920 \\ 1568x^2-376320x+40320 & -64x^2+117600x-322560 & x^2-18816x+564480 \end{pmatrix}.$$

For this example, observing the first $M_n(x)$, for $n = 0, \ldots, nmax$, we can remark that there are no secondary components vanishing and that $\deg a^1_{n-1}(x) = \deg a^2_{n-1}(x) = \deg b^2_{n-1}(x) = n-1$ and $\deg b^1_n(x) = \deg c^1_n(x) = \deg c^2_n(x) = n$. These empirical observations may lead one to conjecture that they hold for general n.

2.2. STUDY OF THE PRINCIPAL COMPONENTS

Regarding the principal components, the commands implemented are the following.

–A command, called $OrthoPCdirectTest_{\zeta,nmax}$, with arguments ζ and $nmax$, that investigates if a principal component sequence fulfils a recurrence relation of second order of type (1.3), giving as output the message "$\{\zeta_n\}_{n\geq 0}$ is not orthogonal" or " $\{\zeta_n\}_{n\geq 0}$ satisfies the orthogonal recurrence relation of second order up to $nmax$". Obviously, we cannot conclude, by this way, that a principal component is orthogonal, because this iterative process is finite; nevertheless we can conclude that a principal component is not orthogonal, for the chosen set of parameters a, b, c, p, q and r.

–If the given MPS $\{W_n\}_{n\geq 0}$ is orthogonal, we can also search for conditions that assure the principal components orthogonality by computing the coefficients A_n, B_n, K_n, H_n, V_n and S_n, mentioned before. If we are able to give the definitions of the structure coefficients, for every n – which is possible in a large set of interesting cases – then we can compute these coefficients for all n obtaining their closed formulas.

–Denoting by β^ζ_n and $\chi^\zeta_{n,\nu}$ the structure coefficients of a given MPS $\{\zeta_n\}_{n\geq 0}$, we defined the commands $\beta SC_{\zeta,n}$, with arguments ζ and n, which calculates the coefficient β_n^ζ, and $\chi SC_{\zeta,n,\nu}$ which computes the coefficient $\chi_{n,\nu}^\zeta$. A command, called $PrintSC_{\zeta,nmax}$, prints the set of structure coefficients β_n^ζ and $\{\chi_{n,\nu}^\zeta,\ 0 \le \nu \le n\}$ of the MPS $\{\zeta_n\}_{n\ge 0}$, from $n = 0$ to $n = nmax$, where $nmax$ is a given non-negative integer.

2.3. STUDY OF THE SECONDARY COMPONENTS

The secondary components are not necessarily free sequences, that is, the degree of a_{n-1}^1, a_{n-1}^2 and b_{n-1}^2 can be less than $n-1$, and the degree of b_n^1, c_n^1 and c_n^2 can be less than n. In order to investigate their linear independence, we consider a command $deg_{\zeta,n}$ which gives as output the degree of the polynomial ζ_n, by the use of the *Mathematica* function Exponent [13]. Notice that, if $\zeta_n(x) = 0$ then $deg_{\zeta,n}$ returns $-\infty$, which is helpful to distinguish a nonzero constant polynomial from the zero polynomial. In fact, for some secondary components we have: $deg_{\zeta,0} = -\infty$, $deg_{\zeta,1} = 0$, $deg_{\zeta,2} = 1$, $deg_{\zeta,3} = 2$, giving us the idea that the sequence $\{l\zeta_{n,x}\}_{n\ge 0}$ with

$$l\zeta_{n,x} = \zeta_{n+1,x},\ n \ge 0,$$

might be a free sequence. Hence, we define these $l\zeta$ sequences, and the degree of each element of these new sequences is again reported by $deg_{\zeta,n}$, considering $\zeta = la^1, la^2, lb^1, lb^2, lc^1$ and lc^2.

In order to list the first values of $deg_{\zeta,n}$, we define a command called $SCDegreeTest_{\zeta,nmax}$ that, given a non-negative integer $nmax$, prints, for i from 0 to $nmax$, the constant $deg_{\zeta,i}$. When the application of this command indicates that the sequence $\{\zeta_{n,x}\}_{n\ge 0}$ – or $\{l\zeta_{n,x}\}_{n\ge 0}$ – might be free, that is, the elements $\zeta_{0,x}, \zeta_{1,x}, \ldots, \zeta_{nmax,x}$ constitute a basis of $\mathcal{P}_{nmax}$ (vectorial space of polynomial functions of maximum degree $nmax$), we can normalize the sequence, calculate its structure coefficients, like we did before for the principal components, and investigate its orthogonality. The normalized (monic) sequences are called $m\zeta$, where $\zeta = a^1, a^2, b^1, b^2, c^1, c^2,\ la^1, la^2, lb^1, lb^2, lc^1$ and lc^2.

2.4. SYMBOLIC RESULTS FOR SOME ORTHOGONAL EXAMPLES

In this subsection, we begin to present some results obtained from the commands described in the preceding two subsections for a specific orthogonal sequence. Let us consider the MOPS $\{W_n\}_{n\ge 0}$ such that $\beta_n = \beta$, $\gamma_n = \alpha$, with $\alpha \ne 0$. Notice that this sequence is a shift of the Chebychev polynomials of the second kind. More precisely, $W_n(x) = A^{-n} U_n(Ax + B),\ n \ge 0$, where $A = \frac{1}{2\sqrt{\alpha}}$, $B = -\frac{\beta}{2\sqrt{\alpha}}$ and $\{U_n\}_{n\ge 0}$ denotes the monic Chebyshev polynomials of the second kind [1]. Then, the coefficients of Theorem 1.3 of [9], for all parameters a, b, c, p, q and r are the following.

$$A_n = -ab^2 - abc + b^2c - ac^2 + bc^2 - abp - acp + bcp - aq + 2p\alpha + q\beta + 6\alpha\beta + p\beta^2 + \beta^3,$$

$$\Theta(x) - A_n = x - r - 2p\alpha - q\beta - 6\alpha\beta - p\beta^2 - \beta^3,$$

$$B_n = \alpha^3,\ \ K_n = \alpha^2(p + 3\beta),\ \ H_n = q + 3\alpha + 2p\beta + 3\beta^2,$$

$$V_n = \alpha(q + 3\alpha + 2p\beta + 3\beta^2),\ \ S_n = p + 3\beta.$$

Consequently, we conclude that if $p = -3\beta$ and $q = -3\alpha+3\beta^2$, then $K_n = H_n = V_n = S_n = 0$ and the principal components are orthogonal. Also, we can write precisely the recurrence coefficients of the principal components, using the expressions of $\Theta(x)-A_n$ and B_n, as follows (Table 1):

$$\beta_0^P = r - a\alpha - 2\alpha\beta + \beta^3, \quad \beta_n^P = r - 3\alpha\beta + \beta^3, \;\; n \geq 1,$$

$$\beta_n^Q = \beta_n^R = r - 3\alpha\beta + \beta^3, \quad \gamma_{n+1}^P = \gamma_{n+1}^Q = \gamma_{n+1}^R = \alpha^3, \;\; n \geq 0.$$

The computation of the first elements of the component sequences of a CD of a given sequence might yield extensive polynomials, if the recurrence coefficients are just a bit more complicated. Therefore, besides the last MOPS indicated (a shift of the Chebyshev polynomials of the second kind), we took as objects of experimentation some symmetric orthogonal sequences, making the coefficients $\beta_n, \; n \geq 0$ disappear. The sequences taken are the Hermite sequence, the Chebyshev polynomials of the second kind, modified Lommel with parameter 1 and Tricomi-Carlitz with parameter equal to 1 or 2 [1]. For these sequences, when $a = b = c = p = q = r = 0$, by $OrthoPCdirectTest_{P,3}$, we know that all principal components are not orthogonal. As a further matter, each principal component seems to be symmetric. Also, the sequences $\{ma_n^1\}_{n\geq 0}, \{mla_n^2\}_{n\geq 0}, \{mlb_n^1\}_{n\geq 0}, \quad \{mb_n^2\}_{n\geq 0}, \{mc_n^1\}_{n\geq 0}$ and $\{mlc_n^2\}_{n\geq 0}$ seem to be symmetric MPSs.

Other particular choices of the CD parameters considered were somewhat general, like, for example, choose $a = b = c = 0$ and leave the parameters p, q and r free, or choose $p = q = r = 0$ and leave the parameters a, b and c free. The application of the commands $OrthoPCdirectTest_{\zeta,nmax}$, $PrintSC_{\zeta,nmax}$ and $SCDegreeTest_{\zeta,nmax}$, with these choices of parameters, yielded similar conclusions. For these choices, the properties fulfilled by the first elements of each polynomial sequence are indicated in Table 2.

3. THE SYMMETRIC CASE

In the next result we aim to prove, for the case when $p = q = r = 0$, that is $\varpi(x) = x^3$, the symmetric character observed in the experimental essays. In order to simplify the presentation of the following theorem, we will say that a sequence $\{F_n\}_{n\geq 0}$, in $\mathcal{P}$, is symmetric if it fulfils $F_n(-x) = (-1)^n F_n(x), \; n \geq 0$.

Theorem 3.1. *Let $\{W_n\}_{n\geq 0}$ be a symmetric MPS defined by* (1.5)–(1.7)*, where $p = q = r = 0$. Then, we have:*

— *$\{R_n\}_{n\geq 0}$, $\{la_n^2\}_{n\geq 0}$ and $\{b_n^2\}_{n\geq 0}$ are symmetric;*
— *if $a = 0$, then $\{P_n\}_{n\geq 0}$, $\{lb_n^1\}_{n\geq 0}$ and $\{c_n^1\}_{n\geq 0}$ are symmetric;*
— *if $b + c = 0$, then $\{Q_n\}_{n\geq 0}$, $\{a_n^1\}_{n\geq 0}$ and $\{lc_n^2\}_{n\geq 0}$ are symmetric.*

Proof. Writing every component sequence in terms of the canonical sequence, we have: $W_n(x) = \sum_{k=0}^{n} w_{n,k}x^k$, or $a_{n-1}^1(x) = \sum_{k=0}^{n-1} a_{n-1,k}^1 x^k$, and similarly for all the other component sequences, where, by convention, $\sum_{k=0}^{-1} \cdot = 0$.

The MPS $\{W_n\}_{n\geq 0}$ fulfils $W_n(-x) = (-1)^n W_n(x),\quad n \geq 0$, therefore, identity (1.5) can be written as follows, considering $p = q = r = 0$ and depending of the parity of n.

$$\Lambda(x) = \sum_{k=0}^{(3n)/2} w_{3n,2k} x^{2k}, \text{ if } n \text{ is even,} \tag{3.1}$$

$$\Lambda(x) = \sum_{k=0}^{(3n-1)/2} w_{3n,2k+1} x^{2k+1} \text{ if } n \text{ is odd, where} \tag{3.2}$$

$$\Lambda(x) = \sum_{k=0}^{n} p_{n,k} x^{3k} - a\sum_{k=0}^{n-1} a^1_{n-1,k} x^{3k} + bc\sum_{k=0}^{n-1} a^2_{n-1,k} x^{3k} + \sum_{k=0}^{n-1} a^1_{n-1,k} x^{3k+1} - \\ - (b+c)\sum_{k=0}^{n-1} a^2_{n-1,k} x^{3k+1} + \sum_{k=0}^{n-1} a^2_{n-1,k} x^{3k+2}.$$

Let us remark that the terms in x^{3k}, x^{3n+1} and x^{3m+2} are all different for every set of positive integers k, n and m, and, also, that: $3k$ is even if and only if k is even; $3k+1$ is even if and only if k is odd; $3k+2$ is even if and only if k is even. These two properties are subjacent to the following arguments.

Looking carefully at identities (3.1) and (3.2) and the correspondent terms of type x^{3k+2}, we conclude that $a^2_{n-1}(x)$ is even, when n is even, and $a^2_{n-1}(x)$ is odd, when n is odd, and therefore, $la^2_n(-x) = (-1)^n la^2_n(x),\ \ n \geq 0$. Also, analysing the part written in terms of x^{3k+1}, we get that if $b + c = 0$, then $a^1_{n-1}(x)$ is odd, when n is even, and $a^1_{n-1}(x)$ is even, when n is odd, thus, $a^1_n(-x) = (-1)^n a^1_n(x),\ \ n \geq 0$. Let us suppose that $a = 0$ and analyse the part written in terms of x^{3k}. Taking into account the already obtained property for $a^2_{n-1}(x)$, we can conclude that $\{P_n\}_{n\geq 0}$ is symmetric.

In the same manner, considering identities (1.6) and (1.7), the remaining conclusions are easily obtained. □

4. SUMMARY OF THE RESULTS

Finally, we present two tables that organize the results advanced for each choice of parameters, obtained with the software PolySeqCubicDecomposition2012.nb version 1.0 [11]. Let us remark that in the left column we indicate properties which are established by Theorem 1.3, Corollary 1.4 and Theorem 3.1. In the right column, we present properties fulfilled by the first elements of each polynomial sequence studied, that is, for $n = 0, \ldots, nmax$, for fixed values of $nmax$. These properties, not yet proven for all n, can be the object of posterior theoretical studies that are out of the scope of this work. For instance, the CD of an orthogonal MPS where $\{a^2_n\}_{n\geq 0}$ and $\{b^2_n\}_{n\geq 0}$ vanish can be studied analogously to the case where $a^1_n = a^2_n = 0,\ n \geq 0$, which is treated in [9], although with an increase in technical difficulties.

Table 1. A shift of the Chebyshev polynomials of the second kind: $\beta_n = \beta$, $\gamma_{n+1} = \alpha$, $n \geq 0$, $\alpha \neq 0$ (non-symmetric orthogonal case)

Results due to Theorem 1.3 and Corollary 1.4	Results fulfilled for $n = 0, \ldots, nmax$
$p = -3\beta$ and $q = -3\alpha + 3\beta^2 \quad (\beta \neq a)$	
• The principal components are orthogonal, with recurrence coefficients: $\beta_0^P = r - a\alpha - 2\alpha\beta + \beta^3$, $\beta_n^P = r - 3\alpha\beta + \beta^3, \ n \geq 1$, $\beta_n^Q = \beta_n^R = r - 3\alpha\beta + \beta^3, \ n \geq 0$, $\gamma_{n+1}^P = \gamma_{n+1}^Q = \gamma_{n+1}^R = \alpha^3, \ n \geq 0$.	• a_n^2 and b_n^2 vanish; • ma_n^1, mb_n^1, mc_n^1 and mc_n^2 are MOPSs, with recurrence coefficients: $\beta_n^\zeta = r - 3\alpha\beta + \beta^3, \ n \geq 0, \ \zeta = ma^1, mc^1, mc^2$, $\beta_0^{mb^1} = \frac{-ar+\alpha^2+r\beta+3a\alpha\beta-3\alpha\beta^2-a\beta^3+\beta^4}{\beta-a}$, $\beta_n^{mb^1} = r - 3\alpha\beta + \beta^3, \ n \geq 1$, $\gamma_{n+1}^\zeta = \alpha^3, \ \zeta = ma^1, mb^1, mc^1, mc^2$.

Table 2. Hermite, Chebyshev of the second kind, modified Lommel and Tricomi-Carlitz sequences (symmetric orthogonal cases)

Results due to Theorem 3.1	Results fulfilled for $n = 0, \ldots, nmax$
$p = q = r = 0$	
• $\{R_n\}_{n\geq 0}$, $\{mla_n^2\}_{n\geq 0}$ and $\{mb_n^2\}_{n\geq 0}$ are symmetric.	• mla_n^2 and mb_n^2 are MPSs.
$p = q = r = 0$ and $a = 0$	
• $\{P_n\}_{n\geq 0}$, $\{mlb_n^1\}_{n\geq 0}$ and $\{mc_n^1\}_{n\geq 0}$ are symmetric.	• ma_n^1, mlb_n^1, mc_n^1 and mc_n^2 are MPSs.
$p = q = r = 0$ and $b + c = 0$	
• $\{Q_n\}_{n\geq 0}$, $\{ma_n^1\}_{n\geq 0}$ and $\{mlc_n^2\}_{n\geq 0}$ are symmetric.	• ma_n^1, mb_n^1, mc_n^1 and mlc_n^2 are MPSs.
$a = b = c = 0$ and $p = 0$	
	• ma_n^1, mla_n^2, mlb_n^1, mb_n^2, mc_n^1 and mlc_n^2 are MPSs. • For $\zeta = P, Q, R$, ma^1, mla^2, mlb^1, mb^2, mc^1 and mlc^2, we have: $\beta_n^\zeta = r, \quad \chi_{2n+1,2\nu}^\zeta = 0, \ 0 \leq \nu \leq n$, $\chi_{2n,2\nu+1}^\zeta = 0, \ 0 \leq \nu \leq n-1$.

5. CONCLUSIONS

The symbolic implementation described along with this work allowed us to study some characteristics of all polynomial sequences connected to a CD of a given MPS, but in fact, some of the commands established can be applied to any polynomial sequence, even if we are working outside the framework of the CD. On the other hand, the list of examples considered was restricted to some very famous orthogonal sequences, having only the initial purpose of illustrating the use and interest of the commands. Nonetheless, some symmetry behavior was remarked and conducted to a theoretical result. Therefore, we consider that this implementation constitutes a useful tool for CD analysis, and might be an efficient method of testing some future ideas, avoiding, in a few cases, the extensive analytical calculations that are involved in this kind of decomposition.

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