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Dedicated to Professor Semyon Yakubovich on the occasion of his 50th birthday

CONVOLUTIONS, INTEGRAL TRANSFORMS AND INTEGRAL EQUATIONS BY MEANS OF THE THEORY OF REPRODUCING KERNELS

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Abstract. This paper introduces a general concept of convolutions by means of the theory of reproducing kernels which turns out to be useful for several concrete examples and applications. Consequent properties are exposed (including, in particular, associated norm inequalities).

Keywords: Hilbert space, linear transform, reproducing kernel, linear mapping, convolution, norm inequality, integral equation, Tikhonov regularization.

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1. PRELIMINARIES

Following [25, 30] and [3], we shall introduce a general theory for linear mappings in the framework of Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space. Let E be an abstract set and $\mathbf{h}$ be a Hilbert $\mathcal{H}$-valued function on E. Then we shall consider the linear transform

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}, \tag{1.1}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued functions on E. In order to investigate the linear mapping (1.1), we form a positive definite quadratic form function $K(p, q)$ on E defined by

$$K(p, q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}} \text{ on } E \times E.$$

Then, we obtain the following:

(P1) *The range of the linear mapping* (1.1) *by* $\mathcal{H}$ *is characterized as the reproducing kernel Hilbert space* (*RKHS*) $H_K(E)$ *admitting the reproducing kernel* $K(p, q)$

whose characterization is given by the two following properties: $K(\cdot, q) \in H_K(E)$ *for any* $q \in E$ *and, for any* $f \in H_K(E)$ *and for any* $p \in E$, $(f(\cdot), K(\cdot . p))_{H_K(E)} = f(p)$.

(P2) *In general, we have the inequality* $\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}}$. *Here, for any member* f *of* $H_K(E)$ *there exists a uniquely determined* $\mathbf{f}^* \in \mathcal{H}$ *satisfying*

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}} \text{ on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}.$$

(P3) *In general, we have the inversion formula in* (1.1) *in the form*

$$f \mapsto \mathbf{f}^* \tag{1.2}$$

in (P2) *by using the reproducing kernel Hilbert space* $H_K(E)$.

However, this formula (1.2) is – in general – involved and consequently we need new arguments for each case. If we are in the case where the Hilbert space $\mathcal{H}$ itself is a reproducing kernel Hilbert space, then we can apply the Tikhonov regularization method in order to obtain the inversion numerically and, sometimes, analytically – as we shall see later. In this paper, we are assuming that the inversion formula (1.2) may be, in general, established.

Now we shall consider two systems

$$f_j(p) = (\mathbf{f}_j, \mathbf{h}_j(p))_{\mathcal{H}_j}, \quad \mathbf{f}_j \in \mathcal{H}_j,$$

in the above way by using $\{\mathcal{H}_j, E, \mathbf{h}_j\}_{j=1}^2$. Here, we assume that E is the same set for the two systems in order to have the output functions $f_1(p)$ and $f_2(p)$ on the same set E.

For example, we shall consider the operator $f_1(p)f_2(p)$ in $\mathcal{F}(E)$. Then, we can consider the following problem:

> How to represent the product $f_1(p)f_2(p)$ on E in terms of their inputs $\mathbf{f}_1$ and $\mathbf{f}_2$ through one system?

We shall show that by using the theory of reproducing kernels we can give a natural answer for this problem. Following similar ideas, we can consider various operators among Hilbert spaces. See [4] for consequent details. In particular, for the product of two Hilbert spaces, the idea gives generalizations of convolutions and the related natural convolution norm inequalities. These norm inequalities gave various generalizations and applications to forward and inverse problems for linear partial differential equations, see for example, [4,6–8,19–29]. Furthermore, surprisingly enough, for some very general nonlinear systems, we can consider similar problems (see [26] for details). Here, for its importance, we shall consider the product case which will give a general concept of convolutions and we shall refer to applications to integral equations.

2. PRODUCT AND CONVOLUTION

2.1. GENERAL THEORY

We start by observing that for any two positive definite quadratic form functions $K_1(p,q)$ and $K_2(p,q)$ on E, the usual product $K(p,q) = K_1(p,q)K_2(p,q)$ is again a positive definite quadratic form function on E by Schur's theorem. Then the reproducing kernel Hilbert space H_K admitting the kernel $K(p,q)$ is the restriction of the tensor product $H_{K_1}(E) \otimes H_{K_2}(E)$ to the diagonal set; that is given by the following proposition.

Proposition 2.1. *Let $\{f_j^{(1)}\}_j$ and $\{f_j^{(2)}\}_j$ be some complete orthonormal systems in $H_{K_1}(E)$ and $H_{K_2}(E)$, respectively. Then, the reproducing kernel Hilbert space H_K is comprised of all functions on E which are represented as*

$$f(p) = \sum_{i,j} \alpha_{i,j} f_i^{(1)}(p) f_j^{(2)}(p) \quad \text{on} \quad E, \quad \sum_{i,j} |\alpha_{i,j}|^2 < \infty, \tag{2.1}$$

in the sense of absolutely convergence on E, and its norm in H_K is given by

$$\|f\|_{H_K}^2 = \min \sum_{i,j} |\alpha_{i,j}|^2.$$

By (P1), for $K_j(p,q) = (\mathbf{h}_j(q), \mathbf{h}_j(p))_{\mathcal{H}_j}$ on $E \times E$, and for $f_1 \in H_{K_1}(E)$ and $f_2 \in H_{K_2}(E)$, we note that for the reproducing kernel Hilbert space $H_{K_1K_2}(E)$ admitting the reproducing kernel $K_1(p,q)K_2(p,q)$ on E, in general, we have the inequality $\|f_1 f_2\|_{H_{K_1K_2}(E)} \le \|f_1\|_{H_{K_1}(E)} \|f_2\|_{H_{K_2}(E)}$.

For the positive definite quadratic form function K_1K_2 on E, we assume an expression of the form

$$K_1(p,q)K_2(p,q) = (\mathbf{h}_P(q), \mathbf{h}_P(p))_{\mathcal{H}_P} \quad \text{on } E \times E \tag{2.2}$$

with a Hilbert space $\mathcal{H}_P$-valued function $\mathbf{h}_P(p)$ on E and further we assume that

$$\{\mathbf{h}_P(p) \colon p \in E\} \text{ is complete in } \mathcal{H}_P. \tag{2.3}$$

Such a representation is, in general, possible by the fundamental result of Kolmogorov. Then we can consider conversely the linear mapping from $\mathcal{H}_P$ onto $H_{K_1K_2}(E)$, $f_P(p) = (\mathbf{f}_P, \mathbf{h}_P(p))_{\mathcal{H}_P}$, $\mathbf{f}_P \in \mathcal{H}_P$, and we obtain the isometric identity

$$\|f_P\|_{H_{K_1K_2}(E)} = \|\mathbf{f}_P\|_{\mathcal{H}_P}.$$

Hence, for such representations (2.2) with (2.3), we obtain the isometric mapping between the Hilbert spaces $\mathcal{H}_P$ and $H_{K_1K_2}(E)$.

Now, for the product $f_1(p)f_2(p)$ there exists a uniquely determined $\mathbf{f}_P \in \mathcal{H}_P$ satisfying

$$f_1(p)f_2(p) = (\mathbf{f}_P, \mathbf{h}_P(p))_{\mathcal{H}_P} \text{ on } E.$$

Then, $\mathbf{f}_P$ may be considered as a product of $\mathbf{f}_1$ and $\mathbf{f}_2$ through these transforms and so, we shall introduce the notation

$$\mathbf{f}_S = \mathbf{f}_1[\times]\mathbf{f}_2.$$

For the members $\mathbf{f}_1 \in \mathcal{H}_1$ and $\mathbf{f}_2 \in \mathcal{H}_2$, this product is introduced through the three transforms induced by $\{\mathcal{H}_j,\ E,\ \mathbf{h}_j\}$ $(j = 1, 2)$ and $\{\mathcal{H}_P,\ E,\ \mathbf{h}_P\}$.

The operator $\mathbf{f}_1[\times]\mathbf{f}_2$ is expressible in terms of $\mathbf{f}_1$ and $\mathbf{f}_2$ by the inversion formula

$$(\mathbf{f}_1, \mathbf{h}_1(p))_{\mathcal{H}_1}(\mathbf{f}_2, \mathbf{h}_2(p))_{\mathcal{H}_2} \mapsto \mathbf{f}_1[\times]\mathbf{f}_2$$

in the sense (P2) from $H_{K_1K_2}(E)$ onto $\mathcal{H}_P$. Then, from (P2) and (2.3) we have the following *Schwarz type inequality.*

Proposition 2.2. *We have the inequality*

$$\|\mathbf{f}_1[\times]\mathbf{f}_2\|_{\mathcal{H}_P} \leq \|\mathbf{f}_1\|_{\mathcal{H}_1}\|\mathbf{f}_2\|_{\mathcal{H}_2}.$$

If $\{\mathbf{h}_j(p)\colon p \in E\}$ are complete in $\mathcal{H}_j$ $(j = 1, 2)$, then $\mathcal{H}_j$ and H_{K_j} are isometrical. By using the isometric mappings induced by the Hilbert space valued functions $\mathbf{h}_j$ $(j = 1, 2)$ and $\mathbf{h}_P$, we can introduce the product space of $\mathcal{H}_1$ and $\mathcal{H}_2$ in the form

$$\mathcal{H}_1[\times]\mathcal{H}_2$$

through these transforms.

We can also obtain the corresponding sum version.

Proposition 2.3. *For two positive definite quadratic form functions $K_1(p, q)$ and $K_2(p, q)$ on E, the sum $K_S(p, q) = K_1(p, q) + K_2(p, q)$ is a positive definite quadratic form function on E. The RKHS H_{K_S} admitting the reproducing kernel $K_S(p, q)$ on E is composed of all functions*

$$f = f_1 + f_2 \quad (f_j \in H_{K_j}(E)) \tag{2.4}$$

and the norm in H_{K_S} is given by

$$\|f\|^2_{H_{K_S}} = \min\{\|f_1\|^2_{H_{K_1}(E)} + \|f_2\|^2_{H_{K_2}(E)}\},$$

where the minimum is taken over all the expressions (2.4) *for f. In particular, we obtain the triangle inequality*

$$\|f_1 + f_2\|^2_{H_S} \leq \|f_1\|^2_{H_{K_1}(E)} + \|f_2\|^2_{H_{K_2}(E)} \tag{2.5}$$

for $f_j \in K_{K_j}(E)$ for $j = 1, 2$.

For the positive definite quadratic form function $K_1 + K_2$ on E, we assume the expression in the form

$$K_1(p, q) + K_2(p, q) = (\mathbf{h}_S(q), \mathbf{h}_S(p))_{\mathcal{H}_S} \quad \text{on } E \times E$$

with a Hilbert space $\mathcal{H}_S$-valued function $\mathbf{h}_S(p)$ on E and further we assume that

$$\{\mathbf{h}_S(p)\colon p \in E\} \text{ is complete in } \mathcal{H}_S.$$

Such a representation is, in general, possible by the fundamental result of Kolmogorov. Then, we can consider conversely the linear mapping from $\mathcal{H}_S$ onto $H_{K_1+K_2}(E)$

$$f_S(p) = (\mathbf{f}_S, \mathbf{h}_S(p))_{\mathcal{H}_S}, \ \mathbf{f}_S \in \mathcal{H}_S \tag{2.6}$$

and we obtain the isometric identity

$$\|f_S\|_{H_{K_1+K_2}(E)} = \|\mathbf{f}_S\|_{\mathcal{H}_S}. \tag{2.7}$$

Hence, for such representations (2.6) with (2.7), we obtain the isometric mapping between the Hilbert spaces $\mathcal{H}_S$ and $H_{K_1+K_2}(E)$.

Now, for the sum $f_1(p) + f_2(p)$ there exists a uniquely determined $\mathbf{f}_S \in \mathcal{H}_P$ satisfying

$$f_1(p) + f_2(p) = (\mathbf{f}_S, \mathbf{h}_S(p))_{\mathcal{H}_S} \text{ on } E.$$

Then, $\mathbf{f}_S$ may be considered as a sum of $\mathbf{f}_1$ and $\mathbf{f}_2$ through these transforms and so, we shall introduce the notation

$$\mathbf{f}_S = \mathbf{f}_1[+]\mathbf{f}_2.$$

This sum for the members $\mathbf{f}_1 \in \mathcal{H}_1$ and $\mathbf{f}_2 \in \mathcal{H}_2$ is introduced through the three transforms induced by $\{\mathcal{H}_j,\ E,\ \mathbf{h}_j\}$ $(j = 1, 2)$ and $\{\mathcal{H}_S,\ E,\ \mathbf{h}_S\}$.

The operator $\mathbf{f}_1[+]\mathbf{f}_2$ is expressible in terms of $\mathbf{f}_1$ and $\mathbf{f}_2$ by the inversion formula

$$(\mathbf{f}_1, \mathbf{h}_1(p))_{\mathcal{H}_1} + (\mathbf{f}_2, \mathbf{h}_2(p))_{\mathcal{H}_2} \longrightarrow \mathbf{f}_1[+]\mathbf{f}_2$$

in the sense (P2) from $H_{K_1+K_2}(E)$ onto $\mathcal{H}_S$. Then, from (P2) and (2.5) we have the following triangle inequality.

Proposition 2.4. *We have the inequality*

$$\|\mathbf{f}_1[+]\mathbf{f}_2\|^2_{\mathcal{H}_S} \le \|\mathbf{f}_1\|^2_{\mathcal{H}_1} + \|\mathbf{f}_2\|^2_{\mathcal{H}_2}.$$

If $\{\mathbf{h}_j(p)\colon p \in E\}$ are complete in $\mathcal{H}_j$ $(j = 1, 2)$, then $\mathcal{H}_j$ and $H_{K_j}(E)$ are isometrical. By using the isometric mappings induced by the Hilbert space valued functions $\mathbf{h}_j$ $(j = 1, 2)$ and $\mathbf{h}_S$, we can introduce the sum space of $\mathcal{H}_1$ and $\mathcal{H}_2$ in the form

$$\mathcal{H}_1[+]\mathcal{H}_2$$

through these transforms.

2.2. EXAMPLE

In order to see the typical example for the operators $F_1[\times]F_2$, by non-negative integrable functions ρ_j, not zero identically, we shall consider the following reproducing kernels and the integral transforms. For $j = 1, 2$, we define K_j by

$$K_j(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(i(x - y) \cdot t)\rho_j(t)\, dt.$$

We now consider the induced integral transforms $L_j\colon L_2(\mathbb{R};\rho_j) \to H_{K_j}$ by

$$(L_jF)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} F(x)\rho_j(x)\exp(-it\cdot x)\,dx.$$

Then, for the reproducing kernel Hilbert space H_{K_j} admitting the kernel K_j, we have the isometric identities:

$$\|f_j\|^2_{H_{K_j}} = \frac{1}{2\pi}\int_{\mathbb{R}} |F_j(t)|^2\rho_j(t)dt, \tag{2.8}$$

respectively. It follows from the Fubini theorem that

$$K_1(x,y)K_2(x,y) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} \exp(i(x-y)\cdot t)(\rho_1 * \rho_2)(t)\,dt.$$

The same can be said for $L_1F_1 \cdot L_2F_2$ as follows:

$$(L_1F_1)(x)(L_2F_2)(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} \exp(-ix\cdot t)(F_1\rho_1) * (F_2\rho_2)(t)\,dt.$$

By the property of the product kernel space $H_{K_1K_2}$, we have

$$\|L_1F_1 \cdot L_2F_2\|_{H_{K_1K_2}} \le \|L_1F_1\|_{H_{K_1}} \cdot \|L_2F_2\|_{H_{K_2}}.$$

If we write out in full both sides, we obtain the next result.

Proposition 2.5. *Let ρ_1, ρ_2 be two non-negative integrable functions that are not zero identically, on $\mathbb{R}$. If $F_1, F_2\colon \mathbb{R} \to [0,\infty]$ are measurable functions, then we have*

$$\int_{\mathbb{R}} \frac{1}{(\rho_1 * \rho_2)(t)} \left| \int_{\mathbb{R}} F_1(\xi)\rho_1(\xi)F_2(t-\xi)\rho_2(t-\xi)\,d\xi \right|^2 dt \le$$
$$\le \int_{\mathbb{R}} |F_1(t)|^2\rho_1(t)dt \cdot \int_{\mathbb{R}} |F_2(t)|^2\rho_2(t)\,dt. \tag{2.9}$$

In this case, we have the explicit representation

$$(F_1[\times]F_2)(t) = \frac{((F_1\rho_1) * (F_2\rho_2))(t)}{(\rho_1 * \rho_2)(t)}.$$

Similarly, for the sum case, we have

$$(F_1[+]F_2)(t) = \frac{F_1(t)\rho_1(t) + F_2(t)\rho_2(t)}{\rho_1(t) + \rho_2(t)}.$$

So, in the weighted Fourier transform case, our new product and sum mean the weighted convolution and sum.

Proposition 2.5 was expanded in various directions with applications to inverse problems and partial differential equations through L_p $(p > 1)$ versions and converse inequalities. See, for example, [6–8, 19–29].

2.3. APPLICATION TO INTEGRAL EQUATIONS

We shall assume that the above linear transforms

$$L_j\colon \mathcal{H}_j \ni \mathbf{f}_j \longmapsto f_j \in H_{K_j}(E)$$

and

$$L\colon \mathcal{H}_P \ni \mathbf{f}_P \longmapsto f_P \in H_{K_1K_2}(E)$$

are isometrical. Within this framework, we are now going to consider the integral equation

$$\mathbf{f}_1[\times]\mathbf{f}_2^{(1)} + \mathbf{f}_1[\times]\mathbf{f}_2^{(2)} = \mathbf{g} \tag{2.10}$$

for $\mathbf{f}_1 \in \mathcal{H}_1$, $\mathbf{f}_2^{(1)}, \mathbf{f}_2^{(2)} \in \mathcal{H}_2$ and $\mathbf{g} \in \mathcal{H}_P$.

At this point, the reader may recall the Fredholm integral equations of the second kind, as a prototype example. Then, by taking the transform L, we obtain

$$L_1\mathbf{f}_1(L_2\mathbf{f}_2^{(1)} + L_2\mathbf{f}_2^{(2)}) = g(p),$$

and so

$$f_1(p)\left(f_2^{(1)}(p) + f_2^{(2)}(p)\right) = g(p) \tag{2.11}$$

on the functions on E. Then, for given $\mathbf{f}_2^{(1)}, \mathbf{f}_2^{(2)} \in \mathcal{H}_2$ and $\mathbf{g} \in \mathcal{H}_P$, when we solve the equation, we wish to be able to consider the following representation in some reasonable mathematical sense:

$$\mathbf{f}_1 = L_1^{-1}\left(\frac{g(p)}{f_2^{(1)}(p) + f_2^{(2)}(p)}\right).$$

Here, the essential problem, however, rises in the solvability question. Namely, how to obtain the solution of the equation (2.11)

$$\frac{g(p)}{f_2^{(1)}(p) + f_2^{(2)}(p)}.$$

This important problem may be solved effectively by using the Tikhonov regularization.

3. TIKHONOV REGULARIZATION

3.1. GENERAL FRAMEWORK

Let E be an arbitrary set, and let H_K be a reproducing kernel Hilbert space admitting the reproducing kernel $K(p, q)$ on E. For any Hilbert space $\mathcal{H}$ we consider a bounded linear operator L from H_K into $\mathcal{H}$. In general, in order to solve the operator equation $Lf = \mathbf{d}$, we would be interested in the best approximation problem of finding

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} \tag{3.1}$$

for a vector $\mathbf{d}$ in $\mathcal{H}$. This problem, itself, leads to the concept of the Moore-Penrose generalized inverse and by using the theory of reproducing kernels, the theory is well-formulated, see [25, 30]. However, the just mentioned extremal problem is complicated in many senses. So, we shall consider the Tikhonov regularization within our framework.

We set, for a small $\lambda > 0$,

$$K_L(\cdot, p; \lambda) = \frac{1}{L^*L + \lambda I} K(\cdot, p),$$

where L^* denotes the adjoint operator of L. Then, by introducing the inner product $(f, g)_{H_K(L;\lambda)} = \lambda(f, g)_{H_K} + (Lf, Lg)_{\mathcal{H}}$, we construct the Hilbert space $H_K(L; \lambda)$ comprising all the functions of H_K. This space, of course, admits a reproducing kernel. Furthermore, we directly obtain the following result proposition.

Proposition 3.1. *The extremal function $f_{\mathbf{d},\lambda}(p)$ in the Tikhonov regularization*

$$\inf_{f \in H_K} \{\lambda \|f\|^2_{H_K} + \|\mathbf{d} - Lf\|^2_{\mathcal{H}}\}$$

exists. Additionally, there is a unique element for which the corresponding minimum is attained and it is represented in terms of the kernel $K_L(p, q; \lambda)$ as follows:

$$f_{\mathbf{d},\lambda}(p) = (\mathbf{d}, LK_L(\cdot, p; \lambda))_{\mathcal{H}} \,. \tag{3.2}$$

Here, the kernel $K_L(p, q; \lambda)$ is the reproducing kernel for the Hilbert space $H_K(L; \lambda)$ and it is determined as the unique solution $\widetilde{K}(p, q; \lambda)$ of the equation

$$\widetilde{K}(p, q; \lambda) + \frac{1}{\lambda}(L\widetilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\lambda} K(p, q)$$

with $\widetilde{K}_q = \widetilde{K}(\cdot, q; \lambda) \in H_K$ for $q \in E$ and $K_p = K(\cdot, p) \in H_K$ for $p \in E$.

The next proposition gives the inversion for errorness data.

Proposition 3.2. *Suppose that $\lambda\colon (0, 1) \to (0, \infty)$ is a function of δ such that*

$$\lim_{\delta \downarrow 0} \left(\lambda(\delta) + \frac{\delta^2}{\lambda(\delta)} \right) = 0.$$

Let $D : (0, 1) \to \mathcal{H}$ be a function such that $\|D(\delta) - \mathbf{d}\|_{\mathcal{H}} \leq \delta$ for all $\delta \in (0, 1)$. If $\mathbf{d}$ is contained in the range set of the Moore-Penrose inverse, then $\lim_{\delta \downarrow 0} f_{D(\delta),\lambda(\delta)} = f_{\mathbf{d}}$.

When $\mathbf{d}$ contains error or noise we need its error estimate. For this error estimate, we are able to invoke the next general result.

Proposition 3.3 ([3]). *We have*

$$|f_{\mathbf{d},\lambda}(p)| \leq \frac{1}{\sqrt{2\lambda}}\sqrt{K(p,p)}\;\|\mathbf{d}\|_{\mathcal{H}}.$$

3.2. SOLUTIONS OF INTEGRAL EQUATIONS

Now we shall consider the linear mapping from $H_{K_1}(E)$, for fixed $f_2^{(1)}, f_2^{(2)} \in H_{K_2}(E)$,

$$\varphi(f_1) = f_1(p)\left(f_2^{(1)}(p) + f_2^{(2)}(p)\right)$$

into $H_{K_1K_2}(E)$. Note that this mapping φ is bounded. Indeed,

$$\|\varphi(f_1)\|^2_{H_{K_1K_2}(E)} \leq \|f_1\|^2_{H_{K_1}(E)}\left(\|f_2^{(1)}\|^2_{H_{K_2}(E)} + \|f_2^{(2)}\|^2_{H_{K_2}(E)}\right).$$

So, we can consider the Tikhonov functional:

$$\inf_{f_1\in H_{K_1}(E)}\left\{\lambda\|f_1\|^2_{H_{K_1}(E)} + \|g-\varphi(f_1)\|^2_{H_{K_1K_2}(E)}\right\}.$$

The extremal function $f_{1,\lambda}$ exists uniquely and we have, if (2.10) has the Moore-Penrose generalized inverse $f_1(p)$, $\lim_{\lambda\to 0} f_{1,\lambda}(p) = f_1(p)$ on E uniformly where $K_1(p,p)$ is bounded. Furthermore, its convergence is also in the sense of the norm of $H_{K_1}(E)$. Sometimes we can take $\lambda = 0$ and in this case we can represent the solution in some direct form.

In order to use the representation (3.2), by setting $H(p,q;\lambda) = L_1K_{L_1}(p,q,\lambda)$ that is needed in (3.2) for the representation of the approximate fractional functions, we have the functional equation

$$\lambda H(p,q;\lambda) + (H(\cdot,q;\lambda), L_1K_1(\cdot,p))_{H_{K_1K_2}} = L_1K_1(p,q). \tag{3.3}$$

In the very difficult case of the numerical and real inversion formula of the Laplace transform, in some cases of (3.3), Fujiwara gave solutions with $\lambda = 10^{-400}$ and 600 digits precision. So, in this method, we will be able to give the approximate fractional functions by using Proposition 3.2 and (3.3), numerically, for many cases containing the present situation.

In [4], we examined the details for the solutions of (2.11) in the very general situation of Fourier Analysis. The equation is depending on the function f_2 and so, the problem is very difficult. Due to this fact, the method in the image identification in the sense of (P1)–(P3) is almost impossible. However, surprisingly enough, solutions in the sense of the Moore-Penrose generalized inverses in correspondence with (3.1), in the framework of reproducing kernel Hilbert spaces, can be represented analytically and explicitly by using Fourier integrals. That will mean, of course, the intuitive solutions of the equation can be represented analytically and, similarly, when the solutions exist in (2.11).

4. GENERAL INTEGRAL EQUATIONS

First, we recall a prototype example: The integral equation with the mixed Toeplitz-Hankel kernel is as follows:

$$\lambda\varphi(x) + \int_E \big[k_1(x-y) + k_2(x+y)\big]\varphi(y)dy = f(x), \tag{4.1}$$

where k_1, k_2 are called the Toeplitz, Hankel kernel respectively, and the domain E is one of the following: the finite interval $E = (0, a)$, half of axis $E = (0, +\infty)$, or whole space $E = (-\infty, +\infty)$. Unless the case $E = (-\infty, +\infty)$, the domain of k_1 and that of k_2 are not identical; for example, in case $E = (0, a)$, the domain of k_1 is $(-a, a)$ and that of k_2 is $(0, 2a)$.

Equation (4.1) was posed a long time ago, and it is an interesting subject in the theory of integral equations as it has many applications. Actually, this equation has interested mathematicians, and is still an open problem in general cases (see $[1, 2, 5, 9, 17, 31, 33]$).

In order to consider the integral equation (4.1), we set

$$\Omega(t;\rho) = \rho_1 * (2\pi + \rho_2) + \int_{\mathbb{R}} \rho_1(\xi - t)\rho_3(\xi)d\xi.$$

We consider the integral transforms and the kernels for $j = 1, 2, 3$ in § 2.2.

Now, we shall consider the algebraic equation, with the non-linear operator φ_{f_2,f_3} equation, from H_{K_1}, for fixed $f_j \in H_{K_j} (j = 2, 3)$

$$(\varphi_{f_2,f_3}(f_1))(x) = \alpha f_1(x) + f_1(x)f_2(x) + \overline{f_1(x)}f_3(x) = g(x) \tag{4.2}$$

for a function g of a function space that is determined naturally as the image space of the operator φ_{f_2,f_3}. Then, we obtain the identity

$$\begin{gathered}(\varphi_{f_2,f_3}(F_1))(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} \exp(ix\cdot t)\cdot \\ \cdot((F_1\rho_1) * (2\pi\alpha + (F_2\rho_2)) + ((F_1\rho_1) * *(F_3\rho_3)))(t)\,dt,\end{gathered}$$

for

$$((F_1\rho_1) * *(F_3\rho_3))(t) = \int_{\mathbb{R}} F_1(\xi)\rho_1(\xi)F_3(t+\xi)\rho_3(t+\xi)d\xi.$$

Here, we must assume that F_1 are real valued functions, otherwise, here, we must put the complex conjugate. Then, the integral transform (4.2) is non-linear for F_1 functions. Following the operator φ_{f_2,f_3}, we shall consider the identity

$$\begin{aligned}\mathbf{K}(x,y) &:= K_1(x,y) + K_1(x,y)K_2(x,y) + \overline{K_1(x,y)}K_3(x,y) = \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}} \exp(i(x-y)\cdot t)\cdot\Omega(t;\rho)\,dt. \end{aligned} \tag{4.3}$$

Then, by the structure of the reproducing kernel Hilbert spaces of sum and product, we see that the image g of the nonlinear operator φ_{f_2,f_3} belongs to the reproducing kernel Hilbert space $H_{\mathbf{K}}$ defined by (4.3) and furthermore, we obtain the inequality

$$\|g\|^2_{H_{\mathbf{K}}} \leq \|f_1\|^2_{H_{K_1}} \left(|\alpha|^2 + \|f_2\|^2_{H_{K_2}} + \|f_3\|^2_{H_{K_3}}\right); \tag{4.4}$$

meanwhile, in the t space, we obtain the convolution inequality

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{\Omega(t;\rho)} \left\{ \left| \Big(((F_1\rho_1) * (2\pi\alpha + (F_2\rho_2)) + \big((F_1\rho_1) ** (F_3\rho_3)\big) \Big)(t) \right|^2 \right\} dt \leq \\ &\leq \int_{\mathbb{R}} |F_1(t)|^2 \rho_1(t)\, dt \cdot \left(2\pi|\alpha|^2 + \int_{\mathbb{R}} |F_2(t)|^2 \rho_2(t)\, dt + \int_{\mathbb{R}} |F_3(t)|^2 \rho_3(t)\, dt \right). \end{aligned} \tag{4.5}$$

Now, we wish to solve the convolution equation of the type

$$((F_1\rho_1) * (2\pi\alpha + (F_2\rho_2)) + (F_1\rho_1) ** (F_3\rho_3))(t) = \tilde{G}(t), \tag{4.6}$$

that is the form (4.1), by setting $F_j\rho_j = F_j$ and that was transformed to the algebraic equation (4.6).

The fundamental inequality (4.5) means that the operator φ_{f_2,f_2} is bounded on H_{K_1} into the space $H_{\mathbf{K}}$, however the mapping is nonlinear. We need a bounded linear operator from some reproducing kernel Hilbert space into a Hilbert space for applying Proposition 2.3. In order to introduce a suitable reproducing kernel Hilbert space, we shall assume that ρ_1 is positive continuous on the support $[a,b]$ of ρ_1 with $-\infty \leq a < b \leq +\infty$. In particular, note that $F_1 \in L_1(a,b)$. We shall consider the reproducing kernel Hilbert space $H_{K_{\rho_1}}$. We define the positive definite quadratic form function

$$K_{\rho_1}(t,\tau) = \int_a^{\min(t,\tau)} \frac{1}{\rho_1(\xi)} d\xi.$$

Then the reproducing kernel Hilbert space $H_{K_{\rho_1}}$ is composed of functions f, absolutely continuous, $f(a) = 0$ with the inner product

$$(f_1, f_2)_{H_{K_{\rho_1}}} = \int_a^b f_1'(t) f_2'(t) \rho_1(t) dt,$$

as we see directly ([25]). Then, we see that for F_1 satisfying (3.3), when we set

$$f_{\rho_1}(t) = \int_a^t F_1(\xi) d\xi,$$

we obtain the isometric identity

$$\|f_{\rho_1}\|^2_{H_{K_{\rho_1}}} = \int_{\mathbb{R}} |F_1(t)|^2 \rho_1(t) dt = \frac{1}{2\pi} \|f_1\|^2_{H_{K_1}},$$

in (3.3). When we consider the operator (4.6) from this reproducing kernel Hilbert space $H_{K_{\rho_1}}$ into the Hilbert space $L_2(\Omega_\rho)$ comprising the functions F with norm squares

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}} |F(t)|^2 \frac{1}{\Omega(t;\rho)} dt < \infty,$$

it is a bounded linear operator as we see from (4.5).

For some general integral equations (4.1) we can write down solutions of (4.1) with long arguments and calculations. However, its representation is involved and so the Tikhonov regularization method would be suitable and practical.

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