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ON THE UNIQUENESS OF MINIMAL PROJECTIONS IN BANACH SPACES

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Abstract. Let X be a uniformly convex Banach space with a continuous semi-inner product. We investigate the relation of orthogonality in X and generalized projections acting on X. We prove uniqueness of orthogonal and co-orthogonal projections.

Keywords: minimal projection, orthogonal projection, co-orthogonal projection, uniqueness of norm-one projection.

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1. INTRODUCTION

In the theory of operators on a Hilbert space most of the terminology and techniques are developed by use of the inner-product. It is known that a Banach space can be represented as a semi-inner product space with a more general axiom system than that of a Hilbert space (see [10]). Hence, in a Banach space we can define orthogonality and transversality relations. A natural consequence of these relations are an orthogonal set $M^{\perp}$ and a transversal set $M^{\top}$ for a set M. In a Hilbert space X we have $M^{\perp} = M^{\top}$ and $X = M \oplus M^{\perp}$ for a closed subspace M of X. It turns out that there holds the decomposition theorem on a uniformly convex Banach space with a continuous semi-inner product. This result is presented in detail in Theorem 2.5. However, $M^{\perp}$ is not always a linear subspace of X. If it were, the space X would have to be isomorphic to some Hilbert space by the Lindenstrauss-Tzafriri theorem [9], but this is not always true. In Theorem 3.10 we give conditions for $M^{\perp}$ to be a subspace of X. If the set $M^{\perp}$ is a subspace of the space X, then M is one-co-complemented and the converse of this statement is also true. In this paper a new definition of a generalized projection is given. The inspiration for this was the metric projection. The main result in this article is Theorem 3.5. We show that for a closed subspace M in a uniformly convex Banach space with a continuous semi-inner product there exists at most one homogenous generalized projection $P : X \to M$ satisfying the Lipschitz condition with the constant equal to one.

2. AUXILIARY RESULTS

To apply Hilbert space type methods to the theory of Banach spaces, G. Lumer [10] constructed a semi-inner product (s.i.p.) on a complex linear space X as a complex function $[\cdot,\cdot]$ on $X \times X$ with the following properties:

$$[\alpha x+\beta y, z] = \alpha[x,z]+\beta[y,z],\ x,y,z\in X,\ \alpha,\beta\in\mathbb{C}, \tag{2.1}$$

$$[x,\lambda y] = \overline{\lambda}[x,y],\ x,y\in X,\quad \lambda\in\mathbb{C}. \tag{2.2}$$

$$[x,x]>0 \quad \text{for } x\neq 0, \tag{2.3}$$

$$|[x,y]|^2 \leq [x,x][y,y],\ x,y\in X. \tag{2.4}$$

$(X,[\cdot,\cdot])$ is called a complex space with semi-inner product.

The importance of a semi-inner product space (s.i.p.s.) is that every normed space can be represented as a semi-inner product space so that the theory of operators on a Banach space can be represented by Hilbert space type arguments.

Theorem 2.1 ([4], [10]). *A semi-inner product space $(X,[\cdot,\cdot])$ is a normed linear space with the norm*

$$\|x\| = [x,x]^{1/2},\ x\in X.$$

Every normed linear space can be made into a semi-inner product space (in general, in infinitely many different ways).

In a normed space X we set

$$S = \{x\in X : \|x\|=1\}.$$

We introduce additional properties of the semi-inner product that will help us to carry over Hilbert space type arguments to the case of a Banach space. Note that a semi-inner product is continuous with respect to the first component. A very convenient property of a s.i.p. is continuity with respect to the second variable.

A s.i.p.s. X is called a *continuous s.i.p. space* when a semi-inner product satisfies the following additional condition:
for every $x,y\in S$,

$$\mathrm{Re}[y,x+\lambda y]\to \mathrm{Re}[y,x] \text{ for all real } \lambda\to 0. \tag{2.5}$$

The space X is a *uniformly continuous s.i.p.s.* if the above limit (2.5) is approached uniformly for all $(x,y)\in S\times S$.

Define a relation on a s.i.p. space which may be called an orthogonality relation. Let $x,y\in X$. We say that x is *normal* to y and y is *transversal* to x if $[y,x]=0$. A vector $x\in X$ is normal to a subspace N and N is transversal to x if x is normal to all vectors from N.

For a normed space, R.C. James [6] studied the orthogonality relation (in the sense of Birkhoff) defined as follows:

A vector x is *orthogonal* to y in the sense of Birkhoff if

$$\|x + \lambda y\| \geq \|x\| \text{ for all } \lambda \in \mathbb{C}.$$

It is worth noting that orthogonality in the sense of Birkhoff is very close to the concept of an element of best approximation. It was shown that in a continuous s.i.p.s. an orthogonality relation is equivalent to a Birkhoff orthogonality relation (see [4]).

Theorem 2.2 ([4])**.** *In a continuous s.i.p.s. x is normal to y if and only if x is orthogonal to y in the sense of Birkhoff.*

Since a s.i.p. is not commutative, this orthogonality relation is not symmetric, i.e. if x is normal to y, then y is not neccessarily normal to x. So, for a subset M of X we define an *orthogonal set* by

$$M^{\perp} = \{x \in X \ : \ \forall\, y \in M \ \ [y,x] = 0 \ \}$$

and a *transversal set* by

$$M^{\top} = \{x \in X \ : \ \forall\, y \in M \ \ [x,y] = 0 \ \}\,.$$

It is easy to see that

$$X^{\perp} = X^{\top} = \{0\}\,, \tag{2.6}$$

$$M \cap M^{\perp} = \{0\}, \tag{2.7}$$

$$M \cap M^{\top} = \{0\}. \tag{2.8}$$

2.1. THE DECOMPOSITION THEOREM

To extend Hilbert space type arguments to the theory of decomposition we need to impose an additional structure on a s.i.p. chiefly to guarantee the existence of normal vectors to closed subspaces.

A normed space is *uniformly convex* if given $\varepsilon \in (0,2]$, there exists $\delta(\varepsilon) > 0$ such that for $x, y \in S$, $\|x - y\| > \varepsilon$ implies $\|x + y\|/2 \leq 1 - \delta(\varepsilon)$.

Recall the notion of strict convexity. A normed space is *strictly convex* if whenever $\|x\| + \|y\| = \|x + y\|$, where $x, y \neq 0$, then $y = \lambda x$ for some real $\lambda > 0$.

It is well known that uniform convexity implies strict convexity. The following two lemmas will help us to characterize a strictly convex space by the structure of the semi-inner product. We will also need them for further considerations. Note also that for linearly dependent elements we have equality in the Schwarz inequality.

Lemma 2.3 ([4])**.** *A s.i.p.s. is strictly convex if and only if whenever $[x,y] = \|x\|\|y\|$, where $x, y \neq 0$, then $y = \lambda x$ for some real $\lambda > 0$.*

Lemma 2.4 ([4])**.** *Let X be a strictly convex space with a semi-inner product. Let $y, z \in X$. If $[x,y] = [x,z]$ for all $x \in X$, then $y = z$.*

Let M and M' be subsets of a linear space X. We say that $X = M \oplus M'$ if and only if for $x \in X$ there exist unique elements $x_M \in M$, $x_{M'} \in M'$ such that $x = x_M + x_{M'}$ and $M \cap M' = \{0\}$.

We will prove that in a uniformly convex Banach space with a continuous semi-inner product we have

$$X = M \oplus M^{\perp}$$

for a closed subspace M of X.

Theorem 2.5. *Let X be a uniformly convex Banach space with a continuous semi-inner product. Let M be a closed subspace of X. Then each $x \in X$ can be uniquely decomposed in the form $x = y + z$ with $y \in M$ and $z \in M^{\perp}$.*

Proof. It is well known that, in a uniformly convex Banach space, for a closed subspace M and a vector $x \notin M$, there exists a unique nonzero vector $y \in M$ such that

$$\|x - y\| = d(x, M) = \inf\{\|x - y'\| : y' \in M\}.$$

Let us set $z = x - y$. Then z is normal to M.

In order to prove the uniqueness of the representation $x = y + z$ we assume that $x = y_1 + z_1 = y_2 + z_2$, where $y_1, y_2 \in M$ and $z_1, z_2 \in M^{\perp}$. It follows that $z_1 - z_2 = y_1 - y_2 \in M$. If $z_1 - z_2 \in M \cap M^{\perp}$, then $z_1 - z_2 = 0$ and $y_1 = y_2$. If $z_1 - z_2 \notin M^{\perp}$, then

$$0 = [z_1 - z_2, z_1] = [z_1, z_1] - [z_2, z_1] \geq \|z_1\|^2 - \|z_1\| \, \|z_2\|,$$
$$0 = [z_2 - z_1, z_2] = [z_2, z_2] - [z_1, z_2] \geq \|z_2\|^2 - \|z_1\| \, \|z_2\|.$$

Therefore,

$$\|z_1\| = \|z_2\| \text{ and } \|z_1\| \, \|z_2\| = [z_1, z_2].$$

By the strict convexity of X, we obtain $z_1 = z_2$. This implies that $y_1 = y_2$. □

In all that follows, we assume that X is a uniformly convex Banach space with a continuous semi-inner product and M is a proper closed subspace of X.
In this case, we can define a metric projection $P_m : X \to M$ such that $P_m(x)$ is an element that best approximates $x \in X$ with respect to M, i.e.

$$\|x - P_m(x)\| = \operatorname{dist}(x, M).$$

In a Hilbert space we have $M^{\perp} = M^{\top}$. The following theorem shows the relationship between an orthogonal set and a transversal set.

Theorem 2.6. *Let X be a uniformly convex Banach space with a continuous semi-inner product. Let M be a closed subspace of X. Then*

$$M \subset \left(M^{\top}\right)^{\perp},$$
$$\left(M^{\perp}\right)^{\top} = M.$$

Proof. If $x \in M$, then $[y, x] = 0$ for $y \in M^{\top}$, hence

$$M \subset \left(M^{\top}\right)^{\perp}.$$

If $x \in M$, then we have $[x, y] = 0$ for $y \in M^{\perp}$, i.e. $x \in (M^{\perp})^{\top}$.

Conversely, suppose that $x \in (M^{\perp})^{\top}$, i.e. $[x, y] = 0$ for $y \in M^{\perp}$. By Theorem 2.5, there exist $x_1 \in M$ and $x_2 \in M^{\perp}$ such that $x = x_1 + x_2$. Then $[x_1 + x_2, y] = 0$. Hence $[x_2, y] = 0$ for all $y \in M^{\perp}$. Setting $y = x_2$, we deduce that $x_2 = 0$. Therefore, $x = x_1 \in M$. □

It should be noted that the set $M^{\top}$ is a closed subspace of X. According to Theorem 2.2 $M^{\perp}$ is a closed subset (but not necessarily a subspace) of X.

Example 2.7. Let $X = l_p$, $1 < p < \infty$. Let us equip l_p with a semi-inner product given by

$$[y, x] = \begin{cases} \|x\|_p^{2-p} \displaystyle\sum_{k=1}^{\infty} y_k \overline{x_k} |x_k|^{p-2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where $\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$.

(i) Let $M = span\{e_1, e_2, \ldots, e_n\}$, where $n \in \mathbb{N}$ and $(e_i)_{i=1}^n$ are elements from the standard basis in l_p. Then v is orthogonal to M if and only if $v(i) = 0$ for $i = 1, 2, \ldots, n$. Note that in this case $M^{\perp}$ is a linear subspace of l_p.

(ii) Take $n \in \mathbb{N}$, $n > 1$. Let $M = span\{e\}$, where $e(i) = 1$ for $i = 1, 2, \ldots, n$ and $e(i) = 0$ otherwise. Then v is orthogonal to M if and only if $\sum_{i=1}^n |v(i)|^{p-2}\overline{v(i)} = 0$. In this case $M^{\perp}$ is not a linear subspace of l_p.

3. ORTHOGONAL PROJECTIONS IN BANACH SPACES

3.1. GENERALIZED PROJECTIONS

Let X be a Banach space and M be a subspace of X. An operator $P : X \to M$ is called a *generalized projection* if it satisfies the following conditions:

(P1) P is continuous;
(P2) $\ker P = \{x - Px : x \in X\}$;
(P3) $X = \ker P \oplus M$;
(P4) For every $x \in X$, we set $P(x) = x_M$, where $x = x_{\ker P} + x_M$, $x_{\ker P} \in \ker P$, $x_M \in M$.

An inspiration to define a generalized projection was the metric projection P_m that satisfies the conditions (P1)-(P4). The continuous property of the metric projection is a consequence of the assumptions on the space X (see [5]). It is easy to show that every continuous linear projection is a generalized projection.

Note that if there exists a projection $P : X \to M$, then M is closed. Moreover, a projection P is linear if and only if $\ker P$ is a subspace of X. Furthermore, every linear and continuous projection has properties from (P1) to (P4).

3.2. ORTHOGONAL PROJECTIONS

Let $P : X \to M$ be a generalized projection. We say that P is *orthogonal* if $(\ker P)^{\perp} = M$.

The following theorem holds.

Theorem 3.1. *Let M be a closed subspace of a uniformly convex Banach space X with a continuous semi-inner product. Let $P : X \to M$ be a projection (not necessarily linear) satisfying conditions* (P2)-(P4). *If P is homogeneous and*

$$\|P(x) - P(y)\| \leq \|x - y\| \text{ for all } x, y \in X,$$

then P is orthogonal.

Proof. Note that $P(0) = 0$, hence for $x \in X$ we have

$$\|P(x)\| \leq \|x\|. \tag{3.1}$$

Moreover, if $y - P(y) \in \ker P$, then $\lambda(y - P(y)) \in \ker P$. Indeed, using the homogeneity of P we obtain

$$P(\lambda(y - P(y))) = \lambda P(y - P(y)) = 0.$$

We shall show that $(\ker P)^{\perp} = M$. Setting x equal to $P(x) + \lambda(y - P(y))$ in (3.1) we obtain

$$\|P(P(x) + \lambda(y - P(y)))\| \leq \|P(x) + \lambda(y - P(y))\|,$$

hence

$$\|P(x)\| \leq \|P(x) + \lambda(y - P(y))\|$$

by virtue of Theorem 2.2, which is equivalent to the fact that $P(x)$ is orthogonal to every $z \in \ker P$.

Conversely, suppose that $x \in (\ker P)^{\perp}$. Then $[z, x] = 0$ for $z \in \ker P$. Hence $[x - P(x), x] = 0$ and

$$\|x\|^2 = [x - P(x) + P(x), x] = [x - P(x), x] + [P(x), x] \leq \|x\|\|P(x)\| \leq \|x\|^2.$$

By assumptions, it follows that $\|x\| = \|P(x)\|$ and $\|P(x)\|\|x\| = [P(x), x]$. By the strict convexity of X, we obtain $P(x) = x$, and so $x \in M$. □

Now we can conclude that in a uniformly convex Banach space with a continuous semi-inner product every orthogonal projection is linear and sastisfies the Lipschitz condition.

Theorem 3.2. *Assume that X is a uniformly convex Banach space with a continuous semi-inner product and M is a closed subspace of X. Let $P \colon X \to M$ be a generalized projection. If P is orthogonal, then P is linear and Lipschitz continuous with the constant equal to one.*

Proof. Note that $P(x_1 + x_2) - P(x_1) - P(x_2) \in M$ for $x_1, x_2 \in X$. Let

$$y = P(x_1 + x_2) - P(x_1) - P(x_2).$$

Then

$$\begin{aligned}\|P(x_1 + x_2) - P(x_1) - P(x_2)\|^2 &= [P(x_1 + x_2) - P(x_1) - P(x_2), y] = \\ &= -[(x_1 + x_2) - P(x_1 + x_2), y] + [x_1 - P(x_1), y] + [x_2 - P(x_2), y] = 0.\end{aligned}$$

Therefore, $P(x_1 + x_2) = P(x_1) + P(x_2)$. Now, let $y = P(\alpha x) - \alpha P(x)$. Then

$$\begin{aligned}\|P(\alpha x) - \alpha P(x)\|^2 &= [P(\alpha x) - \alpha P(x), y] = \\ &= -[\alpha x - P(\alpha x), y] + \alpha[x - P(x), y] = 0,\end{aligned}$$

hence $P(\alpha x) = \alpha P(x)$ for $x \in X$ and a scalar α.

We next show that $\|P\| = 1$. Let $x \in X$. Then $Px - x \in \ker P$ and

$$\|Px\|^2 = [Px, Px] = [Px - x + x, Px] = [Px - x, Px] + [x, Px] = [x, Px].$$

Using (2.4) we get

$$\|Px\| \leq \|x\|,$$

hence $\|P\| = 1$. □

Lemma 3.3. *Let $P\colon X \to M$ be an orthogonal projection. Then P is a unique orthogonal projection.*

Proof. Let P_i be an orthogonal projection ($i = 1, 2$). Hence $(\ker P_i)^\perp = M$ ($i = 1, 2$). Then $P_1x - P_2x \in M$ and

$$\begin{aligned}\|P_1x - P_2x\|^2 &= [P_1x - P_2x, P_1x - P_2x] = \\ &= [P_1x - x + x - P_2x, P_1x - P_2x] = \\ &= [P_1x - x, P_1x - P_2x] + [x - P_2x, P_1x - P_2x] = 0.\end{aligned}$$

Consequently, we conclude that $P_1x = P_2x$, which completes the proof. □

Lewicki and Skrzypek proved that the minimal projection onto a symmetric subspace of a smooth Banach space is unique (see [8, Theorem 2.9]). Now, we show an analogous theorem in a uniformly convex Banach space X with a continuous s.i.p. In its proof we use the structure of a semi-inner product.

Theorem 3.4. *Let X be a uniformly convex Banach space with continuous semi-inner product. Let M be a closed subspace of X. If there exists a linear projection $P : X \to M$ such that $\|P\| = 1$, then P is unique.*

Proof. Suppose that there exist linear projections P_1, P_2 such that $\|P_1\| = \|P_2\| = 1$. Then according to Theorem 3.1 they are orthogonal and hence $P_1 = P_2$ by Lemma 3.3. □

A stronger result is given below. Its proof is similar to those of Theorem 3.4, so we omit it.

Theorem 3.5. *Let M be a closed subspace of a uniformly convex Banach space X with continuous semi-inner product. If there exists a homogeneous projection $P : X \to M$ satisfying* (P1)-(P4) *such that*

$$\|P(x) - P(y)\| \leq \|x - y\| \text{ for all } x, y \in X,$$

then P is unique.

A linear subspace M is *one-complemented* if there exists a linear projection $P : X \to M$ such that $\|P\| = 1$.

Remark 3.6. Let M be a subspace of X such that $\dim M = 1$. Then from the Hahn-Banach theorem there exists a linear projection such that $\|P\| = 1$. Therefore, P is an orthogonal projection and M is one-complemented.

In this paper we give a necessary and sufficient condition for the set $M^{\perp}$ to be a subspace of X. We also show when the equality

$$M = \left(M^{\top}\right)^{\perp} \tag{3.2}$$

holds.

Theorem 3.7. *Let X be a uniformly convex Banach space with continuous semi-inner product and M be a closed subspace of X. Then M is one-complemented if and only if there exists a closed subspace V of X such that $V^{\perp} = M$.*

Proof. Let M be one-complemented, hence there exists a linear, continuous projection $P : X \to M$ such that $\|P\| = 1$. By virtue of Theorem 3.1, P is an orthogonal projection, thus $(\ker P)^{\perp} = M$. Setting $V = \ker P$ we complete the first part of the proof.

Conversely, suppose that there exists a closed subspace V such that $V^{\perp} = M$. Then $X = V \oplus V^{\perp} = V \oplus M$. We define an orthogonal projection $P_V : X \to M$ such that

$$P_V x = P_V(x_V + x_M) = x_M,$$

where $x_V \in V,\ x_M \in M$. This finishes the proof. □

The following theorem gives a characterization of one-complemented spaces.

Theorem 3.8. *A subspace M of a uniformly convex Banach space X with continuous semi-inner product is one-complemented if and only if*

$$M = (M^{\top})^{\perp}. \tag{3.3}$$

Moreover, if (3.3) *holds, then a projection $P\colon X \to M$ given by*

$$P\left(x_M + x_{M^{\top}}\right) = x_M, \quad x_M \in M,\ x_{M^{\top}} \in M^{\top}, \tag{3.4}$$

is the only projection with the norm equal to one.

Proof. From Theorem 3.7 we deduce that exists a closed subspace V of X such that

$$V^{\perp} = M. \tag{3.5}$$

Hence

$$V = \left(V^{\perp}\right)^{\top} = M^{\top}. \tag{3.6}$$

From (3.5) and (3.6) we get $M = V^{\perp} = \left(M^{\top}\right)^{\perp}$. By Theorem 2.5, we deduce that

$$X = V \oplus V^{\perp} = M \oplus M^{\top}.$$

Conversely, let $M = (M^{\top})^{\perp}$. Hence

$$X = (M^{\top})^{\perp} \oplus M^{\top} = M \oplus M^{\top}. \tag{3.7}$$

From (3.7) it easy to see that a linear projection $P \colon X \to M$ given by the formula (3.4) is orthogonal. □

3.3. CO-ORTHOGONAL PROJECTIONS

A projection P is called *co-orthogonal* if $M^{\perp} = \ker P$. Note that not every co-orthogonal projection is linear, for example a metric projection.

We start with the following theorem.

Theorem 3.9. *Let M be a closed proper subspace of a uniformly convex Banach space X with a continuous semi-inner product. Let $P \colon X \to M$ be a linear projection. Then the following conditions are equivalent:*

(i) *P is co-orthogonal,*
(ii) $\|Id - P\| = 1$.

Proof. Suppose that the linear projection $P \colon X \to M$ is co-orthogonal.

Let $x \in X$. Then $x - Px \in \ker P$ and

$$\begin{aligned}\|x - Px\|^2 = [x - Px, x - Px] = [x, x - Px] - [Px, x - Px] =\\ = [x, x - Px] \le \|x\|\|x - Px\|.\end{aligned}$$

Therefore, we have

$$\|x - Px\| \le \|x\|,$$

hence $\|Id - P\| = 1$.

Conversely, suppose that for each $x \in X$ we get

$$\|x - Px\| \le \|x\|. \tag{3.8}$$

We now show that $\ker P = M^{\perp}$. Setting x equal to $x - Px + \lambda Py$ in (3.8) we obtain

$$\|x - Px + \lambda Py - P(x - Px + \lambda Py)\| \le \|x - Px + \lambda Py\|,$$

hence

$$\|x - Px\| \leq \|x - Px + \lambda Py\|$$

by virtue of Theorem 2.2, which is equivalent to $x - Px$ is orthogonal to every $z \in M$.

On the other hand, suppose that $x \in M^{\perp}$. Then $[z, x] = 0$ for $z \in M$. Hence $[Px, x] = 0$ and

$$\|x\|^2 = [x - Px, x].$$

Therefore,

$$\|x\|^2 = [x - Px, x] \leq \|x - Px\|\|x\| \leq \|x\|^2.$$

By assumption it follows $\|x - Px\| = \|x\|$. By Lemma 2.4, we obtain that $x - Px = x$, therefore $x \in \ker P$. □

Let us now characterize the linearity of the set of $M^{\perp}$. We present the following theorem.

Theorem 3.10. *Let M be a closed proper subspace of a uniformly convex Banach space X with a continuous semi-inner product. Then the following conditions are equivalent:*

(i) *the set $M^{\perp}$ is a linear space,*
(ii) *there exists a linear projection $P \colon X \to M$ such that $\|Id - P\| = 1$.*

Proof. If $M^{\perp}$ is a linear subspace, we get $X = M \oplus M^{\perp}$. Then it easy to see that linear projection $P \colon X \to M$ given by the formula

$$Px = P\left(x_M + x_{M^{\perp}}\right) = x_M, \ x \in X \tag{3.9}$$

is co-orthogonal.
Conversely, if a linear projection $P \colon X \to M$ is co-orthogonal, then $\ker P = M^{\perp}$. □

Finally, we will prove the following lemma.

Lemma 3.11 ([8]). *Let $P \colon X \to M$ be a co-orthogonal linear projection. Then P is a unique co-orthogonal linear projection.*

Proof. Let P_i be a co-orthogonal projection, hence $\ker P_i = M^{\perp}$ $(i = 1, 2)$. Then $P_1x - P_2x \in M$ and $x - P_1x \in M^{\perp}$, $x - P_2x \in M^{\perp}$. Since $M^{\perp}$ is a subspace of X, then $P_1x - P_2x \in M^{\perp}$. According to (2.7) we conclude $P_1x = P_2x$, which completes the proof. □

Let M be a closed proper subspace of a uniformly convex Banach space X with a continuous semi-inner product.
We say that M is *one-co-complemented* if there exists a linear projection $P \colon X \to M$ such that $\|Id - P\| = 1$.

From the above discussion we obtain the following result.

Theorem 3.12. *Let M be a closed proper subspace of a uniformly convex Banach space X with a continuous semi-inner product. Then M is one-co-complemented if and only if $M^{\perp}$ is a vector space. Moreover, if $M^{\perp}$ is a linear space, then a projection $P\colon X \to M$ given by*

$$P\left(x_M + x_{M^{\perp}}\right) = x_M,$$

is the only projection which satisfies the equality $\|Id - P\| = 1$.

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