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ESTIMATES OF SOLUTIONS FOR PARABOLIC DIFFERENTIAL AND DIFFERENCE FUNCTIONAL EQUATIONS AND APPLICATIONS

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Abstract. The theorems on the estimates of solutions for nonlinear second-order partial differential functional equations of parabolic type with Dirichlet's condition and for suitable implicit finite difference functional schemes are proved. The proofs are based on the comparison technique. The convergent and stable difference method is considered without the assumption of the global generalized Perron condition posed on the functional variable but with the local one only. It is a consequence of our estimates theorems. In particular, these results cover quasi-linear equations. However, such equations are also treated separately. The functional dependence is of the Volterra type.

Keywords: parabolic differential and discrete functional equations, estimate of solution, implicit difference method.

Mathematics Subject Classification: 35R10, 35B30, 65M12, 65M06.

1. INTRODUCTION

The aim of the paper is to prove theorems on the estimates of solutions for nonlinear second-order partial differential functional equations of parabolic type with Dirichlet's condition and for generated by them implicit finite difference functional schemes. We also give the applications of the results. More precisely, we prove the theorem on the convergence of a difference method to a classical solution for the differential functional problem, which by the given estimates, may be treated in the subspace $C(\Omega, R) \subset C(\Omega, \mathbb{R})$, where $R \subset \mathbb{R}$ is an interval. It is a new idea in area of nonlinear implicit difference methods which was studied for explicit methods by K. Kropielnicka and L. Sapa [14]. This considerably extends the class of problems which are solvable by the described method. Therefore, the Lipschitz, Perron or generalized Perron conditions posed on f with respect to z need not be global, in $C(\Omega, \mathbb{R})$, as in the papers due to M. Malec, Cz. Mączka, W. Voigt, M. Rosati and L. Sapa [15–19,24,25],

Z. Kamont, H. Leszczyński and K. Kropielnicka [9–13], but in $C(\Omega, R)$ only. In particular, equations with the polynomial right-hand side are admitted (see the examples in Section 8). Our results can be extended to weakly coupled systems. Let us stress that unlike [14, 25], the Courant-Friedrichs-Levy condition on the steps of a mesh is omitted (see Remark 6.4).

We now formulate the differential functional problem. Let functions $f : \Delta \to \mathbb{R}$ and $\varphi : E_0 \cup \partial_0 E \to \mathbb{R}$ be given (the relevant sets are defined in Section 2.1). Consider a nonlinear second-order partial differential functional equation of parabolic type of the form

$$\partial_t z(t,x) = f(t,x,z,\partial_x z(t,x), \partial_{xx} z(t,x)) \tag{1.1}$$

with the *initial condition* and the *boundary condition of the Dirichlet type*

$$z(t,x) = \varphi(t,x) \text{ on } E_0 \cup \partial_0 E, \tag{1.2}$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$, $\partial_{xx} z = \left[\partial_{x_i x_j} z\right]_{i,j=1}^n$. The equation may be nonlinear with respect to second derivatives. Such an equation is called strongly nonlinear. The functional dependence is of the Volterra type (e.g., delays or Volterra type integrals).

Let $a_{ij} : \Delta^A \to \mathbb{R}$ and $F : \Delta^F \to \mathbb{R}$, $i,j = 1,\ldots,n$, be given functions (see Section 2.1). If we assume that each a_{ij} is non-positive or non-negative in Δ^A, then our results, in particular, cover a quasi-linear differential functional equation of the form

$$\partial_t z(t,x) = \sum_{i,j=1}^n a_{ij}(t,x,z)\, \partial_{x_i x_j} z(t,x) + F(t,x,z,\partial_x z(t,x)). \tag{1.3}$$

To omit this condition, another scheme is also studied.

We assume the existence of a classical solution of problems (1.1), (1.2) and (1.3), (1.2). Theorems on the existence and uniqueness of such solutions for some special parabolic differential functional equations with different boundary conditions can be found in [3–5, 7, 21, 29] and the references therein.

The equation

$$\partial_t z(t,x) = \sum_{i=1}^n \partial_{x_i x_i} a(z(t,x)) + \sum_{i=1}^n \partial_{x_i} b(z(t,x)) + c(z(t,x)), \tag{1.4}$$

where $a, b, c : \mathbb{R} \to \mathbb{R}$ are given functions, is a special case of (1.1) and (1.3). In applications, the second-order term on the right-hand side of (1.4) corresponds to a diffusive or dispersive process, the first-order term represents a convective or advective phenomenon, while the last term corresponds to a reactive process, sorption, source or sink. The unknown usually represents a nonnegative biological, medical, physical or chemical variable such as density, saturation or concentration. A lot of equations of type (1.4) with the polynomial right-hand side is described in [1, 6, 8, 20]. It is for example the generalized Fisher equation

$$\partial_t z(t,x) = \partial_{x_1 x_1} z(t,x) + \beta z(t,x) \left[1 - (z(t,x))^\delta\right], \tag{1.5}$$

$n = 1$, $\beta \in \mathbb{R}$, $\delta > 0$, and the reaction-diffusion equation

$$\partial_t z\,(t,x) = \sum_{i=1}^{n} \partial_{x_i x_i}\,(z\,(t,x))^m - \beta\,(z\,(t,x))^\delta\,, \tag{1.6}$$

$m > 0$, $\delta > 0$, $\beta \in \mathbb{R}$ (see Examples 8.1, 8.2). Equation (1.5) for $\beta = \delta = 1$ (the Fisher equation) is the archetypical deterministic model for the spread of an advantageous gene in a population of diploid individuals living in a one-dimensional habitat. Equation (1.6) is a simple and widely used model for various physical, chemical and biological problems involving diffusion with a source or with absorption, as for instance in modelling filtration in porous media, transport of thermal energy in a plasma, flow of a chemically reacting fluid from a flat surface, evolution of populations, etc. Equations of type (1.4) which can be also covered by our theorems are the Newell-Whitehead equation, the Zeldovich equation, the KPP equation, the Nagumo equation, the Huxley equation and others considered in [6, 20].

The results concerning numerical methods, differential functional and difference functional inequalities or the uniqueness theory, appearing in the papers of P. Besala, G. Paszek [2], C.V. Pao [22], R. Redheffer, W. Walter [23, 28], J. Szarski [26, 27] and numerous others, do not apply to nonlinear equations and quasi-linear equations with such a general functional dependence as in our paper.

The paper is organized in the following way. In Section 2 notation is introduced and some definitions are formulated. Section 3 deals with the estimates of solutions for problems (1.1), (1.2) and (1.3), (1.2). Two next sections are concerned with the estimates of solutions for some auxiliaries discrete functional equations and for implicit difference functional schemes generated by problem (1.1), (1.2), respectively. In Section 6 the convergence of the difference method for (1.1), (1.2) is studied. In Section 7 the modified difference method for (1.3), (1.2) is considered. Finally, in the last section examples illustrating our results and numerical experiments are presented.

2. NOTATION AND DEFINITIONS

2.1. SETS AND FUNCTION SPACES, PARABOLICITY

Let $T > 0$, $X = (X_1, \ldots, X_n)$, $\tau_0 \geq 0$, $\tau = (\tau_1, \ldots, \tau_n)$, where $X_i > 0$, $\tau_i \geq 0$ for $i = 1, \ldots, n$, be given. Define

$$E = [0,T] \times (-X, X) \subset \mathbb{R}^{1+n}, \tag{2.1}$$

$$E_0 = [-\tau_0, 0] \times [-X-\tau, X+\tau] \subset \mathbb{R}^{1+n},$$

$$\partial_0 E = [0,T] \times ([-X-\tau, X+\tau] \setminus (-X,X)) \subset \mathbb{R}^{1+n}.$$

Let, moreover,

$$\Omega = E \cup E_0 \cup \partial_0 E, \tag{2.2}$$

$$\Omega_t = \Omega \cap ([-\tau_0, t] \times \mathbb{R}^n)\,, \quad t \in [0,T]\,.$$

Denote by $M_{n\times n}$ the class of all $n\times n$ symmetric real matrices. Define the sets

$$\Delta = E\times C(\Omega,\mathbb{R})\times\mathbb{R}^n\times M_{n\times n}, \tag{2.3}$$

$$\Delta^A = E\times C(\Omega,\mathbb{R}), \quad \Delta^F = E\times C(\Omega,\mathbb{R})\times\mathbb{R}^n,$$

$$\Delta^* = E\times C(\Omega,R)\times\mathbb{R}^n\times M_{n\times n},$$

$$\Delta^{A*} = E\times C(\Omega,R), \quad \Delta^{F*} = E\times C(\Omega,R)\times\mathbb{R}^n,$$

where $R\subset\mathbb{R}$ is a fixed interval and $C(\Omega,R)=\{z:\Omega\to R\}\cap C(\Omega,\mathbb{R})$.

Equation (1.1) is said to be parabolic in *Walter's sense* if for any two matrices $q,\widetilde{q}\in M_{n\times n}$ there is

$$q\le\widetilde{q}\Rightarrow f(t,x,z,p,q)\le f(t,x,z,p,\widetilde{q})$$

for $(t,x)\in E$, $z\in C(\Omega,\mathbb{R})$, $p\in\mathbb{R}^n$, where the inequality $q\le\widetilde{q}$ means that the matrix $\widetilde{q}-q$ is positive defined (see [28, §23]). To adopt this definition to equation (1.3) it is enough to replace the right-hand side of the above implication by the inequality $\sum_{i,j=1}^n a_{ij}(t,x,z)(q_{ij}-\widetilde{q}_{ij})\le 0$.

The *maximum norms* in $\mathbb{R}^n$ and $M_{n\times n}$ are denoted by $\|\cdot\|$, while in the *space of continuous functions* $C(A,\mathbb{R})$, $A\subset\Omega$ a compact subset, by $\|\cdot\|_A$.

For a fixed $t\in[0,T]$,

$$\|z\|_{\Omega_t}=\max\left\{\left|z\left(\widetilde{t},x\right)\right|:\ \left(\widetilde{t},x\right)\in\Omega_t\right\} \tag{2.4}$$

is a seminorm in $C(\Omega,\mathbb{R})$, where $z\in C(\Omega,\mathbb{R})$.

2.2. DISCRETIZATION, DIFFERENCE AND INTERPOLATING OPERATORS

We use vectorial inequalities to mean that the same inequalities hold between the corresponding components. We write $x\diamond y=(x_1y_1,\ldots,x_ny_n)$ for $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$. Define a mesh on the set Ω in the following way. Let $(h_0,h')=h$, $h'=(h_1,\ldots,h_n)$, stand for the steps of the mesh. Denote by H the set of all h such that there exist $N_0\in\mathbb{Z}$ and $N=(N_1,\ldots,N_n)\in\mathbb{N}^n$ with the properties: $N_0h_0=\tau_0$, $N\diamond h'=X+\tau$. Obviously, $H\neq\emptyset$ and there are $K_0\in\mathbb{N}$ and $K=(K_1,\ldots,K_n)\in\mathbb{Z}^n$ such that $K_0h_0\le T<(K_0+1)h_0$, $K\diamond h'<X\le(K+1)\diamond h'$. For $h\in H$ and $(\mu,m)\in\mathbb{Z}^{1+n}$, $m=(m_1,\ldots,m_n)$, we define nodal points $\left(t^{(\mu)},x^{(m)}\right)$, $x^{(m)}=\left(x_1^{(m_1)},\ldots,x_n^{(m_n)}\right)$, in the following way

$$t^{(\mu)}=\mu h_0,\quad x^{(m)}=m\diamond h'.$$

For $h\in H$, we put

$$R_h^{1+n}=\left\{\left(t^{(\mu)},x^{(m)}\right):\ (\mu,m)\in\mathbb{Z}^{1+n}\right\}. \tag{2.5}$$

Define the discrete sets

$$E_h=E\cap R_h^{1+n}, \tag{2.6}$$

$$E_{0.h} = E_0 \cap R_h^{1+n}, \quad \partial_0 E_h = \partial_0 E \cap R_h^{1+n},$$

$$\Omega_h = E_h \cup E_{0.h} \cup \partial_0 E_h,$$

$$\Omega_{h.\mu} = \Omega_h \cap \left(\left[-\tau_0, t^{(\mu)} \right] \times \mathbb{R}^n \right), \quad \mu = 0, \ldots, K_0.$$

Let, moreover,

$$E_h^+ = \left\{ \left(t^{(\mu)}, x^{(m)} \right) \in E_h : \ 0 \le \mu \le K_0 - 1 \right\}, \tag{2.7}$$

$$I_h = \left\{ t^{(\mu)} : \ 0 \le \mu \le K_0 \right\}, \quad I_h^+ = \left\{ t^{(\mu)} : \ 0 \le \mu \le K_0 - 1 \right\}. \tag{2.8}$$

Write $\chi = 1 + 2n^2$ and

$$\Lambda = \{ \lambda = (\lambda_1, \ldots, \lambda_n) : \ \lambda_i \in \{-1, 0, 1\}, \ i = 1, \ldots, n, \ |\lambda| \le 2 \}, \tag{2.9}$$

$$\Lambda' = \Lambda \backslash \{0\},$$

where $|\lambda| = |\lambda_1| + \ldots + |\lambda_n|$.

Note that χ is the number of elements of Λ. Let $\psi : \Lambda \to \{1, \ldots, \chi\}$ be a function such that $\psi(\lambda) \ne \psi\left(\overline{\lambda}\right)$ for $\lambda \ne \overline{\lambda}$. We assume that $\prec$ is an order in Λ defined in the following way: $\lambda \prec \overline{\lambda}$ if $\psi(\lambda) \le \psi\left(\overline{\lambda}\right)$. Elements of the space $\mathbb{R}^\chi$ we denote by $\xi = \left(\xi^{(\lambda)}\right)_{\lambda \in \Lambda}$. For a function $z \in F(\Omega_h, \mathbb{R})$ and a point $\left(t^{(\mu)}, x^{(m)}\right) \in E_h$ we put $z_{\langle \mu, m \rangle} = \left(z^{(\mu, m+\lambda)}\right)_{\lambda \in \Lambda}$.

For a *mesh function* $z : \Omega_h \supset A_h \to \mathbb{R}$ and a point $\left(t^{(\mu)}, x^{(m)}\right) \in A_h$, we put $z^{(\mu,m)} = z\left(t^{(\mu)}, x^{(m)}\right)$, $|z|^{(\mu,m)} = |z^{(\mu,m)}|$. We denote the space of all such functions by $F(A_h, \mathbb{R})$ and call it the *space of mesh functions*. In $F(A_h, \mathbb{R})$, we introduce the *maximum norm*

$$\|z\|_{A_h} = \max \left\{ \left| z^{(\mu,m)} \right| : \ \left(t^{(\mu)}, x^{(m)} \right) \in A_h \right\}, \tag{2.10}$$

where $z \in F(A_h, \mathbb{R})$. The symbol $z|_{A_h}$ stands for the restriction of z to A_h. Analogously, $F(A_h, R) = \{z : A_h \to R\} \cap F(A_h, \mathbb{R})$, where $R \subset \mathbb{R}$ is a fixed interval.

For a fixed $\mu \in \{0, 1, \ldots, K_0\}$,

$$\|z\|_{\Omega_{h.\mu}} = \max \left\{ \left| z^{(\widetilde{\mu},m)} \right| : \ \left(t^{(\widetilde{\mu})}, x^{(m)} \right) \in \Omega_{h.\mu} \right\} \tag{2.11}$$

is a seminorm in the space $F(\Omega_h, \mathbb{R})$, where $z \in F(\Omega_h, \mathbb{R})$.

For a function $z : \ I_h \supset A_h \to \mathbb{R}_+$, we put $z^{(\mu)} = z\left(t^{(\mu)}\right)$, $t^{(\mu)} \in A_h$, where $\mathbb{R}_+ = [0, +\infty)$.

Write

$$\Gamma = \{(i,j) : \ 1 \le i, j \le n, \ i \ne j\}$$

and suppose that $\Gamma_+, \Gamma_- \subset \Gamma$ are such that $\Gamma_+ \cup \Gamma_- = \Gamma$, $\Gamma_+ \cap \Gamma_- = \emptyset$ (in particular, it may happen that $\Gamma_+ = \emptyset$ or $\Gamma_- = \emptyset$). We assume that $(i,j) \in \Gamma_+$ when $(j,i) \in \Gamma_+$ and $(i,j) \in \Gamma_-$ when $(j,i) \in \Gamma_-$.

Let $z \in F(\Omega_h, \mathbb{R})$ and $\left(t^{(\mu)}, x^{(m)}\right) \in E_h$. Set

$$\delta_i^+ z^{(\mu,m)} = \frac{1}{h_i}\left[z^{(\mu,m+e_i)} - z^{(\mu,m)}\right], \tag{2.12}$$

$$\delta_i^- z^{(\mu,m)} = \frac{1}{h_i}\left[z^{(\mu,m)} - z^{(\mu,m-e_i)}\right],$$

where $e_i = (0,\ldots,0,1,0,\ldots,0)$ with 1 in the ith entry, $i = 1,\ldots,n$. We apply the *difference quotients* δ_0, $\delta = (\delta_1,\ldots,\delta_n)$, $\delta^{(2)} = [\delta_{ij}]_{i,j=1}^n$ given by

$$\delta_0 z^{(\mu,m)} = \frac{1}{h_0}\left[z^{(\mu+1,m)} - z^{(\mu,m)}\right], \tag{2.13}$$

$$\delta_i z^{(\mu,m)} = \frac{1}{2}\left[\delta_i^+ z^{(\mu,m)} + \delta_i^- z^{(\mu,m)}\right] \quad \text{for} \quad i = 1,\ldots,n,$$

$$\delta_{ii} z^{(\mu,m)} = \delta_i^+ \delta_i^- z^{(\mu,m)} \quad \text{for} \quad i = 1,\ldots,n,$$

$$\delta_{ij} z^{(\mu,m)} = \frac{1}{2}\left[\delta_i^+ \delta_j^- z^{(\mu,m)} + \delta_i^- \delta_j^+ z^{(\mu,m)}\right] \quad \text{for} \quad (i,j) \in \Gamma_-,$$

$$\delta_{ij} z^{(\mu,m)} = \frac{1}{2}\left[\delta_i^+ \delta_j^+ z^{(\mu,m)} + \delta_i^- \delta_j^- z^{(\mu,m)}\right] \quad \text{for} \quad (i,j) \in \Gamma_+.$$

We use these operators to approximate derivatives in equations (1.1) and (1.3).

We say that an operator $G_h : F(\Omega_h, \mathbb{R}) \to C(\Omega, \mathbb{R})$ is an *interpolating operator* if it has the properties:

(1) for all $z \in C^{1,2}(\Omega, \mathbb{R})$

$$\lim_{h\to 0} \|G_h[Z] - z\|_\Omega = 0,$$

where $Z := z|_{\Omega_h}$ is the restriction of z to Ω_h,

(2) for all $z, \overline{z} \in F(\Omega_h, \mathbb{R})$

$$\|G_h[z] - G_h[\overline{z}]\|_{\Omega_{t^{(\mu)}}} \leq \|z - \overline{z}\|_{\Omega_{h.\mu}}, \quad \mu = 0,\ldots,K_0.$$

We apply these operators to approximate the functional term in equations (1.1) and (1.3). An example of G_h is the well-known linear operator T_h introduced in [10].

3. ESTIMATES OF SOLUTIONS FOR THE DIFFERENTIAL FUNCTIONAL PROBLEMS

In this section we give a theorem concerning the estimates of solutions for the differential functional problems (1.1), (1.2) and (1.3), (1.2).

We need the following assumptions on the functions f, φ.

Assumption H$[f, \varphi]$

(H_1) There is a function $\sigma_0 : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ for which the following properties hold:

(1) σ_0 is continuous;
(2) the maximal solution $\widetilde{\omega}$ of the Cauchy problem

$$\omega^{'}(t) = \sigma_0(t, \omega(t)), \quad \omega(0) = \|\varphi\|_{E_0} \quad \text{is defined on} \quad [0, T]; \tag{3.1}$$

(3) for each $(t, x) \in E$ and $z \in C(\Omega, \mathbb{R})$

$$|f(t, x, z, 0, 0)| \leq \sigma_0\left(t, \|z\|_{\Omega_t}\right). \tag{3.2}$$

Theorem 3.1. *Let Assumption* H$[f, \varphi]$ *be satisfied and let* (1.1) *be parabolic. If* $u \in C^{1,2}(\Omega, \mathbb{R})$ *is a solution of* (1.1), (1.2) *and*

$$|\varphi(t, x)| \leq \widetilde{\omega}(t) \quad on \quad \partial_0 E, \tag{3.3}$$

then

$$|u(t, x)| \leq \widetilde{\omega}(t) \quad on \quad \Omega, \tag{3.4}$$

where $\widetilde{\omega}$ *is the maximal solution of* (3.1).

The proof of a more general version of Theorem 3.1 is given in [14]. So called parabolic solutions of (1.1) are considered there.

Remark 3.2. It follows from the monotonicity of $\widetilde{\omega}$ that if we put $\omega(0) = \|\varphi\|_{E_0 \cup \partial_0 E}$ in (3.1), then (3.3) holds. But, by the ordinary differential inequalities, the estimate in (3.4) is worse than for $\omega(0) = \|\varphi\|_{E_0}$.

Remark 3.3. Replacing (3.2) by

$$|F(t, x, z, 0)| \leq \sigma_0\left(t, \|z\|_{\Omega_t}\right) \tag{3.5}$$

we immediately obtain a version of Theorem 3.1 for problem (1.3), (1.2).

4. DISCRETE FUNCTIONAL EQUATIONS

We consider an implicit discrete functional equation of the Volterra type with the initial boundary condition. Next we give two theorems on the existence and uniqueness and on the estimate of a solution of this problem, respectively. They will be applied in the existence, uniqueness and estimate proofs of a solution for the implicit difference functional scheme (5.1) generated by the differential functional problem (1.1), (1.2) in the next section.

Suppose that a functional $F_h : E_h^+ \times F(\Omega_h, \mathbb{R}) \times \mathbb{R}^\chi \to \mathbb{R}$ is given. For $\left(t^{(\mu)}, x^{(m)}, z, \xi\right) \in E_h^+ \times F(\Omega_h, \mathbb{R}) \times \mathbb{R}^\chi$, we write $F_h[z, \xi]^{(\mu,m)} = F_h\left(t^{(\mu)}, x^{(m)}, z, \xi\right)$. Given $\varphi_h \in F(E_{0.h} \cup \partial_0 E_h, \mathbb{R})$, we consider the discrete functional equation

$$z^{(\mu+1,m)} = F_h\left[z, z_{\langle \mu+1,m \rangle}\right]^{(\mu,m)} \tag{4.1}$$

with the initial boundary condition

$$z^{(\mu,m)} = \varphi_h^{(\mu,m)} \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h. \tag{4.2}$$

Note that the numbers $z^{(\mu+1,m+\lambda)}$, $\lambda\in\Lambda$, appear in $z_{\langle\mu+1,m\rangle}$ so (4.1), (4.2) is an implicit problem.

We say that the functional F_h satisfies the Volterra condition if $\left(t^{(\mu)},x^{(m)}\right)\in E_h^+$, $\xi\in\mathbb{R}^\chi$, $z,\overline{z}\in F(\Omega_h,\mathbb{R})$, $z|_{\Omega_{h.\mu}}=\overline{z}|_{\Omega_{h.\mu}}$, then $F_h[z,\xi]^{(\mu,m)}=F_h[\overline{z},\xi]^{(\mu,m)}$. Observe that the Volterra condition states that the value of F_h at $\left(t^{(\mu)},x^{(m)},z,\xi\right)$ depends on $\left(t^{(\mu)},x^{(m)},\xi\right)$ and the restriction of the function z to the set $\Omega_{h.\mu}$ only. However, this well-known condition does not imply the existence of a solution for (4.1), (4.2) so we give a suitable theorem.

The following assumptions on F_h will be needed.

Assumption H[F_h]

(H_1) F_h is of the Volterra type, $h\in H$.
(H_2) There exist partial derivatives $\left(\partial_{\xi^{(\lambda)}}F_h\right)_{\lambda\in\Lambda}$ on $E_h^+\times F(\Omega_h,\mathbb{R})\times\mathbb{R}^\chi$, and $\partial_{\xi^{(0)}}F_h[z,\cdot]^{(\mu,m)}$ is bounded for each $\left(t^{(\mu)},x^{(m)},z\right)\in E_h^+\times F(\Omega_h,\mathbb{R})$.
(H_3) The conditions

$$\partial_{\xi^{(\lambda)}}F_h[z,\xi]^{(\mu,m)}\geq 0,\quad \lambda\in\Lambda',\tag{4.3}$$

$$\sum_{\lambda\in\Lambda}\partial_{\xi^{(\lambda)}}F_h[z,\xi]^{(\mu,m)}=0\tag{4.4}$$

are satisfied at each $\left(t^{(\mu)},x^{(m)},z,\xi\right)\in E_h^+\times F(\Omega_h,\mathbb{R})\times\mathbb{R}^\chi$.

Theorem 4.1. *If Assumption* H[F_h] *is satisfied, then there exists exactly one solution* $v\in F(\Omega_h,\mathbb{R})$ *of problem* (4.1), (4.2).

The proof of Theorem 4.1 is given in [24] and is omitted.

Suppose that $v\in F(\Omega_h,\mathbb{R})$ is the solution of (4.1), (4.2). Define $\widetilde{m}(\mu)\in\mathbb{Z}^n$, $\mu=0,\dots,K_0$, as follows

$$|v|^{(\mu,\widetilde{m}(\mu))}=\max\left\{|v|^{(\mu,m)}:\ \left(t^{(\mu)},x^{(m)}\right)\in\Omega_h\right\}.\tag{4.5}$$

Theorem 4.2. *Suppose that Assumption* H[F_h] *is satisfied and*

(1) $\sigma_h: I_h^+\times\mathbb{R}_+\to\mathbb{R}_+$ *is nondecreasing with respect to the second variable and if* $\left(t^{(\mu)},x^{(\widetilde{m}(\mu+1))}\right)\in E_h^+$, *then*

$$\left|F_h\left[v,v_{\langle\mu+1,\widetilde{m}(\mu+1)\rangle}\right]^{(\mu,\widetilde{m}(\mu+1))}\right|\leq\sigma_h\left(t^{(\mu)},\|v\|_{\Omega_{h.\mu}}\right),\tag{4.6}$$

$\mu=0,\dots,K_0-1$, *where* $v\in F(\Omega_h,\mathbb{R})$ *is the solution of* (4.1), (4.2),

(2) $\beta: I_h\to\mathbb{R}_+$ *is nondecreasing and satisfies the recurrent inequality*

$$\beta^{(\mu+1)}\geq\sigma_h\left(t^{(\mu)},\beta^{(\mu)}\right),\quad \mu=0,\dots,K_0-1,\tag{4.7}$$

and $\beta^{(0)}\geq\|\varphi_h\|_{E_{0.h}\cup\partial_0E_h}$.

Then

$$\|v\|_{\Omega_{h.\mu}}\leq\beta^{(\mu)},\quad \mu=0,\dots,K_0.\tag{4.8}$$

Proof. We prove assertion (4.8) by induction on μ.

It follows from the initial condition (4.2) that inequality (4.8) is satisfied for $\mu = 0$.

Assume (4.8) for a fixed μ, $0 \leq \mu \leq K_0 - 1$, we prove it for $\mu + 1$. Let $\left(t^{(\mu)}, x^{(\widetilde{m}(\mu+1))}\right) \in E_h^+$. The assumptions of the theorem and the induction assumption lead to the inequalities

$$\begin{aligned}\left|v^{(\mu+1,\widetilde{m}(\mu+1))}\right| &= \left|F_h\left[v, v_{\langle\mu+1,\widetilde{m}(\mu+1)\rangle}\right]^{(\mu,\widetilde{m}(\mu+1))}\right| \leq \\ &\leq \sigma_h\left(t^{(\mu)}, \|v\|_{\Omega_{h.\mu}}\right) \leq \sigma_h\left(t^{(\mu)}, \beta^{(\mu)}\right) \leq \beta^{(\mu+1)}.\end{aligned} \tag{4.9}$$

In the case $\left(t^{(\mu)}, x^{(\widetilde{m}(\mu+1))}\right) \in \partial_0 E_h$, the boundary condition (4.2) and the monotonicity of β imply

$$\left|v^{(\mu+1,\widetilde{m}(\mu+1))}\right| \leq \|\varphi_h\|_{E_{0.h}\cup\partial_0 E_h} \leq \beta^{(0)} \leq \beta^{(\mu+1)}. \tag{4.10}$$

Hence, the proof is complete by induction. □

Remark 4.3. Let the assumptions of Theorem 4.2 be satisfied with

$$\sigma_h(t, y) = (1 + Lh_0)\, y, \quad (t, y) \in I_h^+ \times \mathbb{R}_+,$$

where $L \geq 0$. Then

$$\|v\|_{\Omega_{h.\mu}} \leq (1 + Lh_0)^\mu \|\varphi_h\|_{E_{0.h}\cup\partial_0 E_h} \leq \exp(LT)\, \|\varphi_h\|_{E_{0.h}\cup\partial_0 E_h} \tag{4.11}$$

for $\mu = 0, \ldots, K_0$. These estimates may be obtained by solving the initial comparison problem

$$\begin{cases}\beta^{(\mu+1)} = (1 + Lh_0)\,\beta^{(\mu)}, & \mu = 0, \ldots, K_0 - 1,\\ \beta^{(0)} = \|\varphi_h\|_{E_{0.h}\cup\partial_0 E_h}\end{cases} \tag{4.12}$$

(see assumption (2)). Moreover, it follows from the proof of Theorem 4.2 that if

$$\left|v^{(\mu,m)}\right| \leq (1 + Lh_0)^\mu \|\varphi_h\|_{E_{0.h}} \quad \text{on} \quad \partial_0 E_h, \tag{4.13}$$

then, by putting $\beta^{(0)} = \|\varphi_h\|_{E_{0.h}}$ in (4.12), the norm $\|\varphi_h\|_{E_{0.h}\cup\partial_0 E_h}$ in (4.11) can be replaced by $\|\varphi_h\|_{E_{0.h}}$ and the estimate is better.

Remark 4.4. It follows from the proof of Theorem 4.2 that if we assume (4.6) for any $\left(t^{(\mu)}, x^{(m)}\right) \in E_h^+$ and $z \in F(\Omega_h, \mathbb{R})$ (see Theorem 4.1 in [14]), then Theorem 4.2 will be also true. But such a version will not be useful in the proof of Theorem 5.3.

5. ESTIMATES OF SOLUTIONS FOR IMPLICIT DIFFERENCE FUNCTIONAL SCHEMES

We define an *implicit finite difference functional scheme* which will be applied to approximate a classical solution of the differential functional problem (1.1), (1.2). It is the system of algebraic equations

$$\begin{cases}\delta_0 z^{(\mu,m)} = f\left(t^{(\mu)}, x^{(m)}, G_h[z], \delta z^{(\mu+1,m)}, \delta^{(2)} z^{(\mu+1,m)}\right),\\ z^{(\mu,m)} = \varphi_h^{(\mu,m)} \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h,\end{cases} \tag{5.1}$$

where $\varphi_h \in F(E_{0.h} \cup \partial_0 E_h, \mathbb{R})$ is a given function, G_h is a given interpolating operator and $z \in F(\Omega_h, \mathbb{R})$.

We say that f satisfies the Volterra condition if $(t,x) \in E$ and $z, \overline{z} \in C(\Omega, \mathbb{R})$, $z|_{\Omega_t} = \overline{z}|_{\Omega_t}$, then $f(t,x,z,p,q) = f(t,x,\overline{z},p,q)$ for $p \in \mathbb{R}^n$, $q \in M_{n\times n}$. Note that the Volterra condition states that the value of f at (t,x,z,p,q) depends on (t,x,p,q) and the restriction of the function z to the set Ω_t only. However, this well-known condition does not imply the existence of a solution for (5.1) and further assumptions are needed. We give a theorem on the existence, uniqueness and on the estimate of a solution for (5.1).

We need the following assumptions on the functions f, φ_h, the interpolating operator G_h and the steps h of the mesh Ω_h.

Assumption F[f, φ_h, G_h]

(F_1) f of variables $(t,x,z,p,q) \in \Delta$ is of the Volterra type.
(F_2) There exist partial derivatives $\partial_p f = (\partial_{p_1} f, \ldots, \partial_{p_n} f)$, $\partial_q f = \left[\partial_{q_{ij}} f\right]_{i,j=1}^{n}$ on Δ, and $\partial_{p_i} f$, $\partial_{q_{ij}} f$, $i,j = 1,\ldots,n$ are bounded on Δ.
(F_3) The matrix $\partial_q f$ is symmetric and

$$\partial_{q_{ij}} f(P) \geq 0 \quad \text{and} \quad \partial_{q_{ij}} f(P) \not\equiv 0 \quad \text{for} \quad (i,j) \in \Gamma_+,$$

$$\partial_{q_{ij}} f(P) \leq 0 \quad \text{for} \quad (i,j) \in \Gamma_-$$

at each $P \in \Delta$.
(F_4) There is a function $\sigma : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ for which the following properties hold:
(1) σ is nondecreasing with respect to both variables;
(2) the maximal solution $\widetilde{\omega}(\cdot; h)$ of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t)), \quad \omega(0) = \|\varphi_h\|_{E_{0.h} \cup \partial_0 E_h} \tag{5.2}$$

is defined on $[0,T]$;
(3) for each $(t,x) \in E$ and $z \in C(\Omega, \mathbb{R})$

$$|f(t,x,z,0,0)| \leq \sigma\left(t, \|z\|_{\Omega_t}\right). \tag{5.3}$$

(F_5) For each $z, \overline{z} \in F(\Omega_h, \mathbb{R})$ if $z|_{\Omega_{h.\mu}} = \overline{z}|_{\Omega_{h.\mu}}$, then $G_h[z]|_{\Omega_{t(\mu)}} = G_h[\overline{z}]|_{\Omega_{t(\mu)}}$, $\mu = 0,\ldots,K_0$, and for each $z \in F(\Omega_h, \mathbb{R})$

$$\|G_h[z]\|_{\Omega_{t(\mu)}} \leq \|z\|_{\Omega_{h.\mu}} \quad \text{for} \quad \mu = 0,\ldots,K_0. \tag{5.4}$$

Remark 5.1. It is required in assumption (F_3) that for each $(i,j) \in \Gamma$ the function $g_{ij}(P) = \operatorname{sign} \partial_{q_{ij}} f(P)$, $P \in \Delta$, is constant on Δ. This assumption can be also considered as a definition of the sets Γ_+ and Γ_-. Moreover, simple calculations show that assumption (F_5) is true for $G_h = T_h$.

Assumption S[f, h]

(S_1) The steps $h = \left(h_0, h'\right) \in H$ are such that

$$-\frac{h_i}{2}\left|\partial_{p_i} f(P)\right| + \partial_{q_{ii}} f(P) - h_i \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_j}\left|\partial_{q_{ij}} f(P)\right| \geq 0 \tag{5.5}$$

at each $P \in \Delta$, $i = 1, \ldots, n$.

(S_2) There is $c_0 > 0$ such that $h_i h_j^{-1} \leq c_0$ for $i, j = 1, \ldots, n$.

We begin with a useful lemma.

For $\xi \in \mathbb{R}^{\chi}$, $\xi = \left(\xi^{(\lambda)}\right)_{\lambda\in\Lambda}$, we put

$$\delta_i^+ \xi^{(0)} = \frac{1}{h_i}\left[\xi^{(e_i)} - \xi^{(0)}\right], \quad \delta_i^- \xi^{(0)} = \frac{1}{h_i}\left[\xi^{(0)} - \xi^{(-e_i)}\right], \tag{5.6}$$

$i = 1, \ldots, n$. The expressions

$$\delta\xi^{(0)} = \left(\delta_1\xi^{(0)}, \ldots, \delta_n\xi^{(0)}\right), \quad \delta^{(2)}\xi^{(0)} = \left[\delta_{ij}\xi^{(0)}\right]_{i,j=1}^{n}$$

are defined in the following way

$$\delta_i\xi^{(0)} = \frac{1}{2}\left[\delta_i^+\xi^{(0)} + \delta_i^-\xi^{(0)}\right] \quad \text{for} \quad i = 1, \ldots, n, \tag{5.7}$$

$$\delta_{ii}\xi^{(0)} = \delta_i^+\delta_i^-\xi^{(0)} \quad \text{for} \quad i = 1, \ldots, n,$$

$$\delta_{ij}\xi^{(0)} = \frac{1}{2}\left[\delta_i^+\delta_j^-\xi^{(0)} + \delta_i^-\delta_j^+\xi^{(0)}\right] \quad \text{for} \quad (i,j) \in \Gamma_-,$$

$$\delta_{ij}\xi^{(0)} = \frac{1}{2}\left[\delta_i^+\delta_j^+\xi^{(0)} + \delta_i^-\delta_j^-\xi^{(0)}\right] \quad \text{for} \quad (i,j) \in \Gamma_+.$$

Consider the functional $F_h : E_h^+ \times F(\Omega_h, \mathbb{R}) \times \mathbb{R}^{\chi} \to \mathbb{R}$ defined by

$$F_h[z,\xi]^{(\mu,m)} = z^{(\mu,m)} + h_0 f\left(t^{(\mu)}, x^{(m)}, G_h[z], \delta\xi^{(0)}, \delta^{(2)}\xi^{(0)}\right). \tag{5.8}$$

Note that

$$F_h\left[z, z_{\langle\mu+1,m\rangle}\right]^{(\mu,m)} = z^{(\mu,m)} + h_0 f\left(t^{(\mu)}, x^{(m)}, G_h[z], \delta z^{(\mu+1,m)}, \delta^{(2)} z^{(\mu+1,m)}\right).$$

Lemma 5.2. *Let Assumptions* $\mathrm{F}[f,\varphi_h,G_h]$ *and* $\mathrm{S}[f,h]$ *hold. Then the functional* F_h *defined by* (5.8) *satisfies Assumption* $\mathrm{H}[F_h]$.

The proof of the above lemma is analogous to that of Lemma 4.6 in [12] and is therefore omitted.

Theorem 5.3. *If Assumptions* $\mathrm{F}[f,\varphi_h,G_h]$, $\mathrm{S}[f,h]$ *hold, then:*

(i) *there exists the unique solution* $v \in F(\Omega_h, \mathbb{R})$ *of* (5.1),

(ii) *the following estimate*

$$\|v\|_{\Omega_{h.\mu}} \leq \widetilde{\omega}\left(t^{(\mu)}; h\right) \leq \widetilde{\omega}(T; h) \quad \textit{for} \quad 0 \leq \mu \leq K_0 \tag{5.9}$$

is true, where $\widetilde{\omega}(\cdot; h)$ *is the maximal solution of* (5.2).

Proof. Let $F_h : E_h^+ \times F(\Omega_h, \mathbb{R}) \times \mathbb{R}^\chi \to \mathbb{R}$ be defined by (5.8).

The existence of the unique solution $v \in F(\Omega_h, \mathbb{R})$ of (5.1) follows from Theorem 4.1 and Lemma 5.2.

To prove (ii) we apply Theorem 4.2 and Lemma 5.2. Obviously, v satisfies problem (4.1), (4.2). Suppose that $\left(t^{(\mu)}, x^{(\widetilde{m}(\mu+1))}\right) \in E_h^+$ for some $\mu = 0, \ldots, K_0-1$ (see (4.5)). We prove that

$$\left| F_h \left[v, v_{\langle \mu+1, \widetilde{m}(\mu+1) \rangle} \right]^{(\mu, \widetilde{m}(\mu+1))} \right| \leq \|v\|_{\Omega_{h.\mu}} + h_0 \sigma \left(t^{(\mu)}, \|v\|_{\Omega_{h.\mu}} \right). \tag{5.10}$$

Note that

$$\begin{aligned} &v^{(\mu+1, \widetilde{m}(\mu+1))} = v^{(\mu, \widetilde{m}(\mu+1))} + \\ &+ h_0 f\left(t^{(\mu)}, x^{(\widetilde{m}(\mu+1))}, G_h[v], \delta v^{(\mu+1, \widetilde{m}(\mu+1))}, \delta^{(2)} v^{(\mu+1, \widetilde{m}(\mu+1))} \right) - \\ &- h_0 f\left(t^{(\mu)}, x^{(\widetilde{m}(\mu+1))}, G_h[v], 0, 0 \right) + h_0 f\left(t^{(\mu)}, x^{(\widetilde{m}(\mu+1))}, G_h[v], 0, 0 \right). \end{aligned} \tag{5.11}$$

It follows from Assumption F$[f, \varphi_h, G_h]$ and the mean value theorem that

$$\begin{aligned} v^{(\mu+1, \widetilde{m}(\mu+1))} = v^{(\mu, \widetilde{m}(\mu+1))} &+ h_0 f\left(t^{(\mu)}, x^{(\widetilde{m}(\mu+1))}, G_h[v], 0, 0 \right) + \\ &+ h_0 \sum_{i=1}^{n} \partial_{p_i} f\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) \delta_i v^{(\mu+1, \widetilde{m}(\mu+1))} + \\ &+ h_0 \sum_{i,j=1}^{n} \partial_{q_{ij}} f\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) \delta_{ij} v^{(\mu+1, \widetilde{m}(\mu+1))}, \end{aligned} \tag{5.12}$$

where $P^{(\mu, \widetilde{m}(\mu+1))}$ is an intermediate point. Write

$$\begin{aligned} S^{(0)}\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) &= -2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \partial_{q_{ii}} f\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) + \\ &\quad + h_0 \sum_{(i,j) \in \Gamma} \frac{1}{h_i h_j} \left| \partial_{q_{ij}} f\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) \right|, \\ S_+^{(i)}\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) &= \frac{h_0}{2h_i} \partial_{p_i} f\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) + \frac{h_0}{h_i^2} \partial_{q_{ii}} f\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) - \\ &\quad - h_0 \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{h_i h_j} \left| \partial_{q_{ij}} f\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) \right|, \\ S_-^{(i)}\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) &= -\frac{h_0}{2h_i} \partial_{p_i} f\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) + \frac{h_0}{h_i^2} \partial_{q_{ii}} f\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) - \\ &\quad - h_0 \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{h_i h_j} \left| \partial_{q_{ij}} f\left(P^{(\mu, \widetilde{m}(\mu+1))} \right) \right|, \end{aligned}$$

where $i = 1, \dots, n$. Note that assumptions (F_3) and (S_1) imply

$$S^{(0)}\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) \le 0, \quad S_+^{(i)}\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) \ge 0, \quad S_-^{(i)}\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) \ge 0,$$

$i = 1, \dots, n$ and

$$S^{(0)}\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) + \sum_{i=1}^{n} S_+^{(i)}\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) + \sum_{i=1}^{n} S_-^{(i)}\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) +$$

$$+h_0 \sum_{(i,j)\in\Gamma_+} \frac{1}{h_i h_j} \partial_{q_{ij}} f\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) - h_0 \sum_{(i,j)\in\Gamma_-} \frac{1}{h_i h_j} \partial_{q_{ij}} f\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) = 0.$$

After grouping the expressions in (5.12) appropriately, in view of assumptions (F_3) (F_4), (F_5), (S_1), the definitions of the difference operators and the relations above, we get

$$\begin{aligned}
&\left|v^{(\mu+1,\widetilde{m}(\mu+1))}\right| \left[1 - S^{(0)}\left(P^{(\mu,\widetilde{m}(\mu+1))}\right)\right] \le \|v\|_{\Omega_{h.\mu}} + h_0 \sigma\left(t^{(\mu)}, \|v\|_{\Omega_{h.\mu}}\right) + \\
&+ \sum_{i=1}^{n} S_+^{(i)}\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) \left|v^{(\mu+1,\widetilde{m}(\mu+1)+e_i)}\right| + \\
&+ \sum_{i=1}^{n} S_-^{(i)}\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) \left|v^{(\mu+1,\widetilde{m}(\mu+1)-e_i)}\right| + \\
&+h_0 \sum_{(i,j)\in\Gamma_+} \frac{1}{2h_i h_j} \partial_{q_{ij}} f\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) \times \\
&\qquad \times \left[\left|v^{(\mu+1,\widetilde{m}(\mu+1)+e_i+e_j)}\right| + \left|v^{(\mu+1,\widetilde{m}(\mu+1)-e_i-e_j)}\right|\right] - \\
&-h_0 \sum_{(i,j)\in\Gamma_-} \frac{1}{2h_i h_j} \partial_{q_{ij}} f\left(P^{(\mu,\widetilde{m}(\mu+1))}\right) \times \\
&\qquad \times \left[\left|v^{(\mu+1,\widetilde{m}(\mu+1)+e_i-e_j)}\right| + \left|v^{(\mu+1,\widetilde{m}(\mu+1)-e_i+e_j)}\right|\right].
\end{aligned} \tag{5.13}$$

By (5.13),

$$\left|v^{(\mu+1,\widetilde{m}(\mu+1))}\right| \le \|v\|_{\Omega_{h.\mu}} + h_0 \sigma\left(t^{(\mu)}, \|v\|_{\Omega_{h.\mu}}\right) \tag{5.14}$$

and hence (5.10) is true.

Denote by $\eta : I_h \to \mathbb{R}_+$ the solution of the initial comparison difference problem

$$\begin{cases} \eta^{(\mu+1)} = \eta^{(\mu)} + h_0 \sigma\left(t^{(\mu)}, \eta^{(\mu)}\right), \ \mu = 0, \dots, K_0 - 1, \\ \eta^{(0)} = \|\varphi_h\|_{E_{0.h} \cup \partial_0 E_h}. \end{cases} \tag{5.15}$$

It follows from Theorem 4.2 that

$$\|v\|_{\Omega_{h.\mu}} \le \eta^{(\mu)}, \quad \mu = 0, \dots, K_0. \tag{5.16}$$

It can be easily prove by induction that

$$\eta^{(\mu)} \leq \widetilde{\omega}\left(t^{(\mu)};h\right) \leq \widetilde{\omega}\left(T;h\right) \quad \text{for} \quad \mu = 0,\ldots,K_0. \tag{5.17}$$

The proof is complete. □

6. DIFFERENCE METHOD

We give a theorem about the convergence of a sequence of solutions for the implicit finite difference functional schemes (5.1) to a solution for the differential functional problem (1.1), (1.2).

Let $U := u|_{\Omega_h} \in F\left(\Omega_h, R\right)$ be the restriction of a solution $u \in C^{1,2}\left(\Omega, R\right)$ for (1.1), (1.2) to the mesh Ω_h and let $v \in F\left(\Omega_h, R\right)$ be a solution for (5.1), where $R \subset \mathbb{R}$ is an interval independent of $h \in H$. We say that the difference method (5.1) is *uniformly convergent* if

$$\lim_{h\to 0} \|U - v\|_{\Omega_h} = 0.$$

An important question is how to assign the interval R. The answer to this question is the following lemma.

Lemma 6.1. *If Assumptions* H$[f,\varphi]$, F$[f,\varphi_h,G_h]$, S$[f,h]$ *and condition* (3.3) (*see Remark 3.2*) *are fulfilled and there is a constant* $A \geq 0$ *such that*

$$\|\varphi_h\|_{E_{0.h}\cup\partial_0 E_h} \leq A, \quad h \in H \tag{6.1}$$

and the maximal solution $\overline{\omega}$ *of the Cauchy problem*

$$\omega^{'}\left(t\right) = \sigma\left(t,\omega\left(t\right)\right), \quad \omega\left(0\right) = A \tag{6.2}$$

is defined on $[0,T]$*, then*

$$R = \left[-\omega^*\left(T\right), \omega^*\left(T\right)\right], \tag{6.3}$$

where $\omega^*\left(T\right) = \max\{\widetilde{\omega}\left(T\right), \overline{\omega}\left(T\right)\}$ (*see* H$[f,\varphi]$, (F_4)).

Proof. It follows from (5.2), (6.1), (6.2) and the ordinary differential inequalities that $\widetilde{\omega}\left(T;h\right) \leq \overline{\omega}\left(T\right)$, $h \in H$. Hence, Theorems 3.1 and 5.3 give (6.3). □

Assumption F^***[**f,u,G_h**]**

(F_1^*) There are functions $\sigma_1 : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$, $\rho_1 : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that:
  1. σ_1 is continuous and nondecreasing with respect to both variables; moreover, $\sigma_1\left(t,0\right) = 0$ for $t \in [0,T]$;
  2. ρ_1 is nondecreasing with respect to both variables;
  3. for each $c \geq 0$ and $\varepsilon, \varepsilon_0 \geq 0$, the maximal solution of the Cauchy problem

$$\omega^{'}\left(t\right) = c\sigma_1\left(t,\omega\left(t\right)\right) + \varepsilon, \quad \omega\left(0\right) = \varepsilon_0 \tag{6.4}$$

is defined on $[0,T]$ and the function $\widetilde{\omega}(t) = 0$ for $t \in [0,T]$ is the maximal solution of (6.4) for each $c \geq 0$ and $\varepsilon, \varepsilon_0 = 0$;

(4) the *generalized Perron type estimate*

$$|f(t,x,z,p,q) - f(t,x,\overline{z},p,q)| \leq \rho_1(\|p\|,\|q\|)\,\sigma_1\left(t, \|z-\overline{z}\|_{\Omega_t}\right) \tag{6.5}$$

holds on Δ^*.

(F_2^*) $u \in C^{1,2}(\Omega,\mathbb{R})$ is a solution of (1.1), (1.2).

(F_3^*) $G_h[z] \in C(\Omega, R)$ for each $z \in F(\Omega_h, R)$.

Remark 6.2. The generalized Perron condition (6.5) is assumed to be satisfied on Δ^* not on Δ, so it is the local one. Moreover, it follows from simple calculations that (F_3^*) holds for $G_h = T_h$.

Theorem 6.3. *Let the assumptions of Lemma 6.1 and Assumption* $\mathrm{F}^*[f,u,G_h]$ *hold. Moreover, suppose that there is a function* $\gamma_0 : H \to \mathbb{R}_+$ *such that*

$$\left|\varphi^{(\mu,m)} - \varphi_h^{(\mu,m)}\right| \leq \gamma_0(h) \quad \textit{on} \;\; E_{0.h} \cup \partial_0 E_h \quad \textit{and} \quad \lim_{h\to 0} \gamma_0(h) = 0. \tag{6.6}$$

Under these assumptions there is a function $\alpha : H \to \mathbb{R}_+$ *such that*

$$\|U - v\|_{\Omega_{h.\mu}} \leq \alpha(h) \quad \textit{for} \;\; 0 \leq \mu \leq K_0 \quad \textit{and} \quad \lim_{h\to 0} \alpha(h) = 0. \tag{6.7}$$

Proof. It follows from Lemma 6.1 that $u \in C^{1,2}(\Omega, R)$ and $v \in F(\Omega_h, R)$, where R is given by (6.3). The proof of Theorem 6.3 for $R = \mathbb{R}$ is given in [24] (see also [12]). For our R we have to replace $F(\Omega_h,\mathbb{R})$ by $F(\Omega_h, R)$ only in that proof. □

Remark 6.4. Observe that we do not assume in Theorem 6.3 the Courant-Friedrichs--Levy condition

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \partial_{q_{ii}} f(P) + h_0 \sum_{(i,j)\in\Gamma} \frac{1}{h_i h_j} \left|\partial_{q_{ij}} f(P)\right| \geq 0, \tag{6.8}$$

$P \in \Delta$, which is typical in explicit methods (see [14, 25]).

Remark 6.5. Suppose that the assumptions of Theorem 6.3 hold, f is Lipschitz continuous with respect to z, p, q and $G_h = T_h$. It follows from the properties of the difference quotients and T_h that if $u \in C^{2,3}(\Omega,\mathbb{R})$ and $\alpha_0 = O(\|h\|)$, then $U - v = O(\|h\|)$. But if $u \in C^{2,4}(\Omega,\mathbb{R})$ and $\alpha_0 = O\left(h_0 + \|h'\|^2\right)$, then $U - v = O\left(h_0 + \|h'\|^2\right)$.

Remark 6.6. Suppose that the assumptions of Theorem 6.3 are satisfied and, moreover, there is a constant $\overline{c} > 0$ such that

$$\left\|\delta w^{(\mu,m)}\right\|, \; \left\|\delta^{(2)} w^{(\mu,m)}\right\| \leq \overline{c} \;\; \text{on} \;\; E_h \tag{6.9}$$

for all solutions $w \in F(\Omega_h, R)$ of perturbed finite difference functional schemes of (5.1). It follows from an analysis of the proof of this theorem that the presented difference method is stable. It is enough to replace U by w. If $\rho_1 = \text{const}$, then condition (6.9) can be omitted.

Remark 6.7. All the results can be extended to weakly coupled differential functional systems. One part of each system may be strongly nonlinear and the other quasi-linear. This is a new result even in the case of systems without functional terms. For simplicity we consider one equation only.

7. QUASI-LINEAR EQUATION

We are interested in the numerical approximation of a classical solution of problem (1.3), (1.2).

Now, we put

$$f(t,x,z,p,q)=\sum_{i,j=1}^{n} a_{ij}(t,x,z)\,q_{ij}+F(t,x,z,p) \tag{7.1}$$

for $(t,x,z,p,q)\in\Delta$, and consider the implicit difference functional scheme (5.1) with this f for problems (1.3), (1.2). Assuming

$$|F(t,x,z,p)-F(t,x,\overline{z},p)|\leq \rho_2(\|p\|)\,\sigma_1\left(t,\|z-\overline{z}\|_{\Omega_t}\right), \tag{7.2}$$

$$|a_{ij}(t,x,z)-a_{ij}(t,x,\overline{z})|\leq \sigma_1\left(t,\|z-\overline{z}\|_{\Omega_t}\right), \tag{7.3}$$

$i,j=1,\ldots,n$, respectively on Δ^{F*} and Δ^{A*}, we may put $\rho_1(y_1,y_2)=n^2y_2+\rho_2(y_1)$ for $y_1,y_2\in\mathbb{R}_+$ in (F_1^*).

If we apply Theorems 5.3, 6.3 then we need, in particular, the following assumption on the matrix $A=[a_{ij}]_{i,j=1}^{n}$: for each $(i,j)\in\Gamma$, the function

$$\widetilde{a}_{ij}(t,x,z)=\operatorname{sign}\,a_{ij}(t,x,z)\quad\text{for}\quad (t,x,z)\in\Delta^A$$

is constant (see (F_3)). In [24] it is shown, for $R=\mathbb{R}$, that the condition that the coefficients a_{ij} are of the same sign in Δ^A can be omitted if we modify the difference operator $\delta^{(2)}$. More precisely, we consider the scheme (5.1) with δ_0, δ, δ_{ii}, $i=1,\ldots,n$, given in Section 2, and we define δ_{ij}, $i,j=1,\ldots,n$, $i\neq j$, by

$$\begin{aligned}\delta_{ij}z^{(\mu+1,m)}&=\frac{1}{2}\left[\delta_i^+\delta_j^-z^{(\mu+1,m)}+\delta_i^-\delta_j^+z^{(\mu+1,m)}\right]\ \text{if}\ a_{ij}\left(t^{(\mu)},x^{(m)},G_h[z]\right)<0,\\ \delta_{ij}z^{(\mu+1,m)}&=\frac{1}{2}\left[\delta_i^+\delta_j^+z^{(\mu+1,m)}+\delta_i^-\delta_j^-z^{(\mu+1,m)}\right]\ \text{if}\ a_{ij}\left(t^{(\mu)},x^{(m)},G_h[z]\right)\geq 0,\end{aligned} \tag{7.4}$$

where $z\in F(\Omega_h,\mathbb{R})$, $\left(t^{(\mu)},x^{(m)}\right)\in E_h$. Observe that the finite difference functional scheme (5.1) with f given by (7.1) and δ_{ij} by (7.4) depends on the sign of a_{ij} at $\left(t^{(\mu)},x^{(m)},G_h[z]\right)$ and this sign need not be the same in Δ^A.

Remark 7.1. It follows from the proofs that Theorems 5.3 and 6.3 are true for the modified difference method above.

8. EXAMPLES

To illustrate the class of problems which can be covered by our estimates theorems and numerical methods, we consider three examples.

Put $n = 1$. Let $E = [0,1] \times (-1,1)$, $E_0 = \{0\} \times [-1,1]$, $\partial_0 E = [0,1] \times ([-1,1] \setminus (-1,1))$.

Example 8.1. Consider the Fisher equation

$$\partial_t z(t,x) = \partial_{xx} z(t,x) + z(t,x)\left[1 - z(t,x)\right] \tag{8.1}$$

with the initial-boundary condition

$$z(t,x) = \frac{1}{2}t^2 \quad \text{on} \quad E_0 \cup \partial_0 E. \tag{8.2}$$

The constant functions $u_0(t,x) \equiv 0$ and $v_0(t,x) \equiv 1$ are, respectively, a lower and an upper solution of problem (8.1), (8.2). It follows from Theorem 4.1 in [5] (see also Theorem 2.1 in [4]) that this problem has the unique solution $u \in C^{1,2}(\Omega, \mathbb{R})$ and $u(t,x) \in [0,1]$ for $(t,x) \in \Omega$. Note that we can put for instance $\sigma_0(t,r) = r(r+1)$, $t \in [0,1]$, $r \in \mathbb{R}_+$, in Theorem 3.1 (see also Remark 3.2) and obtain $|u(t,x)| \leq e^t \left(3 - e^t\right)^{-1} \leq e\left(3 - e\right)^{-1}$ for $(t,x) \in \Omega$. Putting $\varphi_h = \varphi|_{\Omega_h}$, $G_h = T_h$ and $\sigma = \sigma_0$ in Theorem 5.3 we have $\left|v^{(\mu,m)}\right| \leq e^{t^{(\mu)}}\left(3 - e^{t^{(\mu)}}\right)^{-1} \leq e\left(3-e\right)^{-1}$ for $\left(t^{(\mu)}, x^{(m)}\right) \in \Omega_h$, where v is the solution of the implicit difference scheme (5.1) for (8.1), (8.2). Hence, we can put $R = \left[-e\left(3-e\right)^{-1}, e\left(3-e\right)^{-1}\right]$ in Assumption $\mathrm{F}^*[f, u, G_h]$. Note that this assumption is not fulfilled for $R = \mathbb{R}$, because $f(t,x,z,p,q) = q_{11} + z(t,x)\left[1 - z(t,x)\right]$ does not fulfill the generalized Perron condition on $C(\Omega, \mathbb{R})$. Put $h_0 = h_1 = 10^{-1}$. Note that the Courant-Friedrichs-Levy condition (6.8) for such steps is not satisfied. For each $t^{(\mu)}$ we use the method of an inverse matrix to solve the implicit difference scheme. Let v_{min}, v_{max} be the smallest and largest values, respectively, of v at time $t^{(\mu)}$ (Tab. 1).

Table 1. Values of v_{min}, v_{max}

$t^{(\mu)}$	v_{min}	v_{max}
0.1	$4.28 \cdot 10^{-4}$	$5.00 \cdot 10^{-3}$
0.2	$2.44 \cdot 10^{-3}$	$2.00 \cdot 10^{-2}$
0.3	$7.61 \cdot 10^{-3}$	$4.50 \cdot 10^{-2}$
0.4	$1.75 \cdot 10^{-2}$	$8.00 \cdot 10^{-2}$
0.5	$3.37 \cdot 10^{-2}$	$1.25 \cdot 10^{-1}$
0.6	$5.76 \cdot 10^{-2}$	$1.80 \cdot 10^{-1}$
0.7	$9.01 \cdot 10^{-2}$	$2.45 \cdot 10^{-1}$
0.8	$1.32 \cdot 10^{-1}$	$3.20 \cdot 10^{-1}$
0.9	$1.84 \cdot 10^{-1}$	$4.05 \cdot 10^{-1}$
1.0	$2.46 \cdot 10^{-1}$	$5.00 \cdot 10^{-1}$

Put $n = 2$. Let $E = [0,1] \times (-1,1)^2$, $E_0 = \{0\} \times [-1,1]^2$, $\partial_0 E = [0,1] \times \left([-1,1]^2 \setminus (-1,1)^2\right)$.

Example 8.2. Consider the porous media equation with absorption

$$\partial_t z(t,x,y) = \partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) - [z(t,x,y)]^2 \tag{8.3}$$

with the initial-boundary condition

$$z(t,x,y) = \frac{1}{2}t^2 \quad \text{on} \quad E_0 \cup \partial_0 E. \tag{8.4}$$

The constant functions $u_0(t,x,y) \equiv 0$ and $v_0(t,x,y) \equiv \frac{1}{2}$ are, respectively, a lower and an upper solution of problem (8.3), (8.4). It follows from Theorem 4.1 in [5] (see also Theorem 2.1 in [4]) that this problem has the unique solution $u \in C^{1,2}(\Omega,\mathbb{R})$ and $u(t,x,y) \in \left[0,\frac{1}{2}\right]$ for $(t,x,y) \in \Omega$. Note that we can put for instance $\sigma_0(t,r) = r^2$, $t \in [0,1]$, $r \in \mathbb{R}_+$, in Theorem 3.1 (see also Remark 3.2) and obtain $|u(t,x,y)| \le (2-t)^{-1} \le 1$ for $(t,x,y) \in \Omega$. Putting $\varphi_h = \varphi|_{\Omega_h}$, $G_h = T_h$ and $\sigma = \sigma_0$ in Theorem 5.3 we have $\left|v^{(\mu,m)}\right| \le \left(2 - t^{(\mu)}\right)^{-1} \le 1$ for $\left(t^{(\mu)}, x^{(m)}\right) \in \Omega_h$, where v is the solution of the implicit difference scheme (5.1) for (8.3), (8.4). Hence, we can put $R = [-1,1]$ in Assumption $\mathrm{F}^*[f,u,G_h]$. Note that this assumption is not fulfilled for $R = \mathbb{R}$, because $f(t,x,y,z,p,q) = q_{11} + q_{22} - [z(t,x,y)]^2$ does not fulfill the generalized Perron condition on $C(\Omega,\mathbb{R})$. Put $h_0 = h_1 = h_2 = 10^{-1}$. Note that the Courant-Friedrichs-Levy condition (6.8) for such steps is not satisfied. For each $t^{(\mu)}$ we use the method of an inverse matrix to solve the implicit difference scheme. Let v_{min}, v_{max} be the smallest and largest values, respectively, of v at time $t^{(\mu)}$ (Tab. 2).

Table 2. Values of v_{min}, v_{max}

$t^{(\mu)}$	v_{min}	v_{max}
0.1	$7.27 \cdot 10^{-4}$	$5.00 \cdot 10^{-3}$
0.2	$3.93 \cdot 10^{-3}$	$2.00 \cdot 10^{-2}$
0.3	$1.15 \cdot 10^{-2}$	$4.50 \cdot 10^{-2}$
0.4	$2.53 \cdot 10^{-2}$	$8.00 \cdot 10^{-2}$
0.5	$4.63 \cdot 10^{-2}$	$1.25 \cdot 10^{-1}$
0.6	$7.52 \cdot 10^{-2}$	$1.80 \cdot 10^{-1}$
0.7	$1.12 \cdot 10^{-1}$	$2.45 \cdot 10^{-1}$
0.8	$1.58 \cdot 10^{-1}$	$3.20 \cdot 10^{-1}$
0.9	$2.12 \cdot 10^{-1}$	$4.05 \cdot 10^{-1}$
1.0	$2.74 \cdot 10^{-1}$	$5.00 \cdot 10^{-1}$

Example 8.3. Consider the strongly nonlinear with a quasi-linear term differential integral equation with deviated variables

$$\begin{aligned}\partial_t z(t,x,y) = {} & \arctan[\partial_{xx} z(t,x,y) + \partial_{xy} z(t,x,y) + \partial_{yy} z(t,x,y)] + \\ & + [2 + \cos z(t,x,y)]\,[\partial_{xx} z(t,x,y) + \partial_{xy} z(t,x,y) + \\ & + \partial_{yy} z(t,x,y)] + [\sin z(t,x,y)]\,\partial_x z(t,x,y) + \\ & + \frac{1}{8}\int_0^t \int_{-1}^1 \int_{-1}^1 z(s,a,b)\,db\,da\,ds + [z(0.5t,0,0)]^2 + g(t,x,y)\end{aligned} \tag{8.5}$$

with the initial-boundary condition

$$z(t,x,y) = 0.01 \sin t \cos(x+y) \quad \text{on} \quad E_0 \cup \partial_0 E, \tag{8.6}$$

where

$$\begin{aligned} g(t,x,y) = {} & \arctan[0.03 \sin t \cos(x+y)] + \\ & + 0.01 [\cos t + 6 \sin t + 3 \sin t \cos(0.01 \sin t \cos(x+y))] \cos(x+y) + \\ & + 0.01 \sin t \sin(0.01 \sin t \cos(x+y)) \sin(x+y) + 0.005 \sin^2 1 (\cos t - 1) - \\ & - 0.0001 \sin^2 0.5t. \end{aligned}$$

The function $u(t,x,y) = 0.01 \sin t \cos(x+y)$ is an analytic solution of problem (8.5), (8.6). Obviously, $|u(t,x,y)| \le 0.01$ for $(t,x,y) \in \Omega$. Note that we can put for instance $\sigma_0(t,r) = 3\left(r + \frac{1}{4}\right)^2$, $t \in [0,1]$, $r \in \mathbb{R}_+$, in Theorem 3.1 (see also Remark 3.2) and obtain $|u(t,x,y)| \le \frac{1}{4}\left(\frac{4}{26} + 3t\right)\left(\frac{100}{26} - 3t\right)^{-1} \le \frac{41}{44}$ for $(t,x,y) \in \Omega$. Putting $\varphi_h = \varphi|_{\Omega_h}$, $G_h = T_h$ and $\sigma = \sigma_0$ in Theorem 5.3 we have $\left|v^{(\mu,m)}\right| \le \frac{1}{4}\left(\frac{4}{26} + 3t^{(\mu)}\right)\left(\frac{100}{26} - 3t^{(\mu)}\right)^{-1} \le \frac{41}{44}$ for $\left(t^{(\mu)}, x^{(m)}\right) \in \Omega_h$, where v is the solution of the implicit difference functional scheme (5.1) for (8.5), (8.6). Hence, we can put $R = \left[-\frac{41}{44}, \frac{41}{44}\right]$ in Assumption $\mathrm{F}^*[f,u,G_h]$. Note that this assumption is not fulfilled for $R = \mathbb{R}$, because $f(t,x,y,z,p,q) = \arctan\left(q_{11} + \frac{1}{2}q_{12} + \frac{1}{2}q_{21} + q_{22}\right) + [2 + \cos z(t,x,y)] \left(q_{11} + \frac{1}{2}q_{12} + \frac{1}{2}q_{21} + q_{22}\right) + [\sin z(t,x,y)]\, p_1 + \frac{1}{8}\int_0^t \int_{-1}^1 \int_{-1}^1 z(s,a,b)\, db\, da\, ds + [z(0.5t,0,0)]^2 + g(t,x,y)$ does not fulfill the generalized Perron condition on $C(\Omega,\mathbb{R})$. Put $h_0 = h_1 = h_2 = 10^{-1}$. For each $t^{(\mu)}$ we use one hundred iterations of the Newton method to solve the implicit difference functional scheme. Let v_{min}, v_{max} be the smallest and largest values, respectively, of v at time $t^{(\mu)}$ (Tab. 3). Moreover, let ε_{max}, ε_{mean} be the largest and mean values, respectively, of the errors $|U - v|$ of the difference method (5.1) at time $t^{(\mu)}$ (Tab. 4).

Table 3. Values of v_{min}, v_{max}

$t^{(\mu)}$	v_{min}	v_{max}
0.1	$-4.15 \cdot 10^{-4}$	$9.98 \cdot 10^{-4}$
0.2	$-8.26 \cdot 10^{-4}$	$1.98 \cdot 10^{-3}$
0.3	$-1.22 \cdot 10^{-3}$	$2.95 \cdot 10^{-3}$
0.4	$-1.62 \cdot 10^{-3}$	$3.89 \cdot 10^{-3}$
0.5	$-1.99 \cdot 10^{-3}$	$4.79 \cdot 10^{-3}$
0.6	$-2.34 \cdot 10^{-3}$	$5.64 \cdot 10^{-3}$
0.7	$-2.68 \cdot 10^{-3}$	$6.44 \cdot 10^{-3}$
0.8	$-2.98 \cdot 10^{-3}$	$7.17 \cdot 10^{-3}$
0.9	$-3.25 \cdot 10^{-3}$	$7.83 \cdot 10^{-3}$
1.0	$-3.50 \cdot 10^{-3}$	$8.41 \cdot 10^{-3}$

Table 4. Errors of the difference method

$t^{(\mu)}$	ε_{max}	ε_{mean}
0.1	$4.82 \cdot 10^{-4}$	$2.04 \cdot 10^{-4}$
0.2	$6.55 \cdot 10^{-4}$	$2.69 \cdot 10^{-4}$
0.3	$7.04 \cdot 10^{-4}$	$2.86 \cdot 10^{-4}$
0.4	$7.05 \cdot 10^{-4}$	$2.85 \cdot 10^{-4}$
0.5	$6.84 \cdot 10^{-4}$	$2.75 \cdot 10^{-4}$
0.6	$6.50 \cdot 10^{-4}$	$2.62 \cdot 10^{-4}$
0.7	$6.08 \cdot 10^{-4}$	$2.45 \cdot 10^{-4}$
0.8	$5.60 \cdot 10^{-4}$	$2.25 \cdot 10^{-4}$
0.9	$5.06 \cdot 10^{-4}$	$2.03 \cdot 10^{-4}$
1.0	$4.46 \cdot 10^{-4}$	$1.79 \cdot 10^{-4}$

Note that the Courant-Friedrichs-Levy condition (6.8) for such steps is not satisfied and the explicit method given in [14] is not convergent. In fact, the errors ε_{max}, ε_{mean} of that method exceeded 10^{11} and 10^{10}, respectively.

The results shown in the tables are consistent with our mathematical analysis. The tables of errors are typical of difference methods.

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REFERENCES

[1] U.G. Abdulla, *On the Dirichlet problem for reaction-diffusion equations in non-smooth domains*, Nonlinear Anal. **47** (2001), 765–776.

[2] P. Besala, G. Paszek, *Differential-functional inequalities of parabolic type in unbounded regions*, Ann. Polon. Math. **38** (1980), 217–228.

[3] S. Brzychczy, *Existence and uniqueness of solutions of infinite systems of semilinear parabolic differential-functional equations in arbitrary domains in ordered Banach spaces*, Math. Comput. Modelling **36** (2002), 1183–1192.

[4] S. Brzychczy, *Monotone iterative methods for infinite systems of reaction-diffusion--convection equations with functional dependence*, Opuscula Math. **25** (2005), 29–99.

[5] S. Brzychczy, *Infinite Systems of Parabolic Differential-Functional Equations*, AGH University of Science and Technology Press, Cracow, 2006.

[6] B.H. Gilding, R. Kersner, *Memorandum No. 1585*, Faculty of Mathematical Sciences, University of Twente, 2001.

[7] J. Hale, L. Verdyun, *Introduction to Functional Differential Equations*, Springer, 1993.

[8] H.N.A. Ismail, A.A.A. Rabboh, *A restrictive Padé approximation for the solution of the generalized Fisher and Burger-Fisher equations*, Appl. Math. Comput. **154** (2004), 203–210.

[9] Z. Kamont, H. Leszczyński, *Stability of difference equations generated by parabolic differential-functional problems*, Rend. Mat. Appl. (7) **16** (1996), 265–287.

[10] Z. Kamont, *Hyperbolic Functional Differential Inequalities and Applications*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1999.

[11] Z. Kamont, *Numerical approximations of difference functional equations and applications*, Opuscula Math. **25** (2005), 109–130.

[12] Z. Kamont, K. Kropielnicka, *Implicit difference functional inequalities and applications*, J. Math. Inequal. **2** (2008), 407–427.

[13] K. Kropielnicka, *Implicit difference methods for quasi-linear parabolic functional differential problems of the Dirichlet type*, Appl. Math. **35** (2008), 155–175.

[14] K. Kropielnicka, L. Sapa, *Estimate of solutions for differential and difference functional equations with applications to difference methods*, Appl. Math. Comput. **217** (2011), 6206–6218.

[15] M. Malec, *Sur une famille biparamétrique de schémas des différences finies pour un système d'équations paraboliques aux dérivées mixtes et avec des conditions aux limites du type de Neumann*, Ann. Polon. Math. **32** (1976), 33–42.

[16] M. Malec, *Sur une méthode des differences finies pour une équation non linéaire différentielle fonctionnelle aux dérivées mixtes*, Ann. Polon. Math. **36** (1979), 1–10.

[17] M. Malec, Cz. Mączka, W. Voigt, *Weak difference-functional inequalities and their application to the difference analogue of non-linear parabolic differential-functional equations*, Numer. Math. **11** (1983), 69–79.

[18] M. Malec, M. Rosati, *A convergent scheme for non-linear systems of differential functional equations of parabolic type*, Rend. Mat. Appl. (7) **3** (1983), 211–227.

[19] M. Malec, L. Sapa, *A finite difference method for nonlinear parabolic-elliptic systems of second order partial differential equations*, Opuscula Math. **27** (2007), 259–289.

[20] J.D. Murray, *Mathematical Biology*, Springer, Berlin, 1993.

[21] C.V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York-London, 1992.

[22] C.V. Pao, *Finite difference reaction-diffusion systems with coupled boundary conditions and time delays*, J. Math. Anal. Appl. **272** (2002), 407–434.

[23] R. Redheffer, W. Walter, *Comparison theorems for parabolic functional inequalities*, Pacific J. Math. **85** (1979), 447–470.

[24] L. Sapa, *Implicit difference methods for differential functional parabolic equations with Dirichlet's condition*, to appear.

[25] L. Sapa, *A finite difference method for quasi-linear and nonlinear differential functional parabolic equations with Dirichlet's condition*, Ann. Polon. Math. **93** (2008), 113–133.

[26] J. Szarski, *Differential Inequalities*, Monograph, PWN - Polish Scientific Publishers, Warsow, 1965.

[27] J. Szarski, *Strong maximum principle for non-linear parabolic differential-functional inequalities in arbitrary domains*, Ann. Polon. Math. **31** (1975), 197–203.

[28] W. Walter, *Differential and Integral Inequalities*, Monograph, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[29] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer, 1996.

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