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INTEGRABILITY OF TRIGONOMETRIC SERIES WITH GENERALIZED SEMI-CONVEX COEFFICIENTS

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Abstract. In this paper we deal with cosine and sine trigonometric series with generalized semi-convex coefficients. Integrability conditions for them are obtained.

Keywords: trigonometric series, generalized semi-convex sequences.

Mathematics Subject Classification: 42A20, 42A32.

1. INTRODUCTION

There are many known classical theorems pertaining to the integrability of cosine and sine trigonometric series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx, \tag{1.1}$$

$$\sum_{k=1}^{\infty}a_k\sin kx. \tag{1.2}$$

The first results regarding the integrability of the above series considered the cases of monotone, quasi-monotone or coefficients of bounded variation.

Later on, several papers have been written (see for example the classical result of A.N. Kolmogorov [4] or recently papers of J. Nemeth [5], S. Tikhonov [6], L. Leindler [7], and references therein) on the integrability of the series (1.1) and (1.2) when the sequence $\{a_k\}$ is a null-sequence and convex or quasi-convex, i.e. $\triangle^2 a_k\geq 0$ or

$$\sum_{k=1}^{\infty}(k+1)|\triangle^2 a_k|<\infty, \tag{1.3}$$

where $\triangle^2 a_k=\triangle\left(\triangle a_k\right)$, $\triangle a_k=a_k-a_{k+1}$.

We shall consider the series (1.1) and (1.2) whose coefficients tend to zero, and fulfill any condition that provides for their pointwise convergence on $(0,\pi)$. Let us denote their sums by $f(x)$ and $g(x)$ respectively.

If the coefficients a_k are quasi-convex, it is well-known that f is an integrable function on $[0,\pi]$ (see [4]), and the following expression is valid

$$\int_0^{\pi} |f(x)|dx \leq \pi \sum_{k=1}^{\infty}(k+1)|\triangle^2 a_k|.$$

Among others S.A. Telyakovskiĭ [10] has proved two theorems providing sufficient conditions under which the sum functions of the series (1.1) and (1.2) will be integrable functions. In his theorems the sequence of coefficients a_k belongs to the class of null-sequences of bounded variation of second order.

A null-sequences is said to be of bounded variation of second order if the condition

$$\sum_{k=1}^{\infty} |\triangle^2 a_k| < \infty \tag{1.4}$$

is satisfied.

Telyakovskiĭ 's theorems read as follows.

Theorem 1.1. *If $a_k \to 0$ as $k \to \infty$ and (1.4) is satisfied, then series (1.1) converges pointwise on $(0,\pi]$, uniformly on $[\varepsilon,\pi]$ for every $\varepsilon > 0$, and for $1 \leq \ell \leq m$, the sum function $f(x)$ satisfies*

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell} |f(x)|dx = O\left(\frac{m+1-\ell}{m}\sum_{k=0}^{\ell-1}\frac{(k+1)}{\ell}|\triangle a_k|\right) + \\ + O\left(\sum_{k=\ell}^{\infty}\min(k+1-\ell,\, m+1-\ell)|\triangle^2 a_k|\right).\end{aligned} \tag{1.5}$$

Theorem 1.2. *If $a_k \to 0$ as $k \to \infty$ and (1.4) is satisfied, then series (1.2) converges pointwise on $(0,\pi]$, uniformly on $[\varepsilon,\pi]$ for every $\varepsilon > 0$, and for $1 \leq \ell \leq m$ the sum function $f(x)$ satisfies*

$$\int_{\pi/(m+1)}^{\pi/\ell} |g(x)|dx = \sum_{k=\ell}^{m} |a_k| u_k + O\left(A_{\ell,m}\right), \tag{1.6}$$

where

$$u_k := \log \frac{\sin\frac{\pi}{2k}}{\sin\frac{\pi}{2(k+1)}},$$

and

$$A_{\ell,m} := \frac{m+1-\ell}{m}\sum_{k=1}^{\ell-1}\frac{k^2}{\ell^2}|\triangle a_k| + \sum_{k=\ell}^{\infty}\min(k+1-\ell,\, m+1-\ell)|\triangle^2 a_k|.$$

The following definition is introduced in [1].

A sequence $\{a_k\}$ is of bounded variation of integer order $p \geq 0$ if

$$\sum_{k=1}^{\infty} |\triangle^p a_k| < \infty, \tag{1.7}$$

where $\triangle^p a_k = \triangle\left(\triangle^{p-1} a_k\right) = \triangle^{p-1} a_k - \triangle^{p-1} a_{k+1}$, and we agree with $\triangle^0 a_k = a_k$.

Also in the same paper an example is given to show that (1.7) is an effective generalization (in the sense that every null-sequence of bounded variation of $p-$th order is also a null-sequence of bounded variation of $(p+1)-$th order, but the converse is not true) of the null sequences of bounded variation. In a similar direction, using condition (1.7) for $p \geq 2$, instead of (1.4), the present author (see [8], similar results can also be found in [9]) has proved some estimates of the form (1.5) which generalized some previous results of Telyakovskiĭ [10].

Throughout the paper we assume that $a_0 = 0$. The following notion is given in [3].

A sequence $\{a_k\}$ is said to be *generalized semi-convex* if $a_k \to 0$ as $k \to \infty$, and

$$\sum_{k=1}^{\infty} k^{\alpha} |\triangle^{\alpha+1} a_{k-1} + \triangle^{\alpha+1} a_k| < \infty \quad (\alpha = 1, 2, \dots). \tag{1.8}$$

For $\alpha = 1$ the above notion agrees with the one introduced in [2] (the semi-convex sequence). It should be noticed here that every quasi-convex null sequence is semi-convex.

The main goal of the present paper is to prove some estimates of the form (1.5) and (1.6) considering the series (1.1) and (1.2) whose coefficients tend to zero and are generalized semi-convex. In fact, we shall verify analogous results with those of Telyakovskiĭ assuming only that the null-sequence $\{a_k\}$ is a generalized semi-convex sequence with $\alpha = 2$.

For two positive functions $g(u)$ and $h(u)$ we write $g(u) = O\left(h(u)\right)$, $u \to 0$, if there is a positive constant A, such that $g(u) \leq A h(u)$ in a neighborhood of the point $u = 0$. The constant A could generally, but not necessarily be the same at each occurrence.

2. MAIN RESULTS

First we shall prove the following theorem.

Theorem 2.1. *If $a_k \to 0$ as $k \to \infty$ and (1.8) is satisfied with $\alpha = 2$, then series (1.1) converges pointwise on $(0, \pi)$, uniformly on $[\varepsilon, \pi - \varepsilon]$ for every $\varepsilon > 0$, and for*

$1 \leq \ell \leq m$ *the sum function* $f(x)$ *satisfies*

$$\int_{\pi/(m+1)}^{\pi/\ell} |f(x)|dx = \sum_{k=\ell}^{m} d_k |\triangle a_{k-1} + \triangle a_k| + \\ + O\Bigg(\frac{m+1-\ell}{m} \sum_{k=1}^{\ell-1} \frac{k^2}{\ell} |\triangle^2 a_{k-1} + \triangle^2 a_k| + \\ + \sum_{k=\ell}^{\infty} \min(k+1-\ell,\, m+1-\ell) k^2 |\triangle^3 a_{k-1} + \triangle^3 a_k| \Bigg), \tag{2.1}$$

where

$$d_k = \frac{\sin \frac{\pi}{k(k+1)}}{2 \sin \frac{\pi}{2(k+1)} \sin \frac{\pi}{2k}}.$$

Proof. For n-th partial sums $S_n^c(x)$ of series (1.1) we have ($a_0 = 0$)

$$\begin{aligned} S_n^c(x) &= \frac{1}{2\sin x} \sum_{k=1}^{n} 2a_k \cos kx \sin x = \\ &= \frac{1}{2\sin x} \sum_{k=1}^{n-1} (a_{k-1} - a_{k+1}) \sin kx + a_{n-1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x} = \\ &= \frac{1}{2\sin x} \sum_{k=1}^{n} (\triangle a_{k-1} + \triangle a_k) \sin kx + a_{n-1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x}. \end{aligned}$$

Applying the summation by parts to the last equality twice we obtain

$$\begin{aligned} S_n^c(x) &= \frac{1}{2\sin x} \sum_{k=1}^{n-1} (\triangle^2 a_{k-1} + \triangle^2 a_k) \widetilde{D}_k(x) + \\ &+ (\triangle a_{n-1} + \triangle a_n) \frac{\widetilde{D}_n(x)}{2\sin x} + a_{n-1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x} = \\ &= \frac{1}{2\sin x} \sum_{k=1}^{n-2} (\triangle^3 a_{k-1} + \triangle^3 a_k) \widetilde{F}_k(x) + (\triangle^2 a_{n-2} + \triangle^2 a_{n-1}) \widetilde{F}_{n-1}(x) + \\ &+ (\triangle a_{n-1} + \triangle a_n) \frac{\widetilde{D}_n(x)}{2\sin x} + a_{n-1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x}, \end{aligned} \tag{2.2}$$

where $\widetilde{D}_n(x)$ and $\widetilde{F}_n(x)$ are the conjugate Dirichlet's and the conjugate Fejer's kernels defined respectively by

$$\widetilde{D}_n(x) = \sum_{j=1}^{n} \sin jx, \quad \widetilde{F}_n(x) = \sum_{j=1}^{n} \widetilde{D}_j(x).$$

Using the assumption that $a_n \to 0$ as $n \to \infty$, the known estimate $\widetilde{D}_n(x) = O(1/x)$ for $x \in (0, \pi]$, and the fact that the function $1/\sin x$ is bounded on $[\varepsilon, \pi - \varepsilon]$, $\varepsilon > 0$, we obtain

$$\left|(\triangle a_{n-1} + \triangle a_n)\frac{\widetilde{D}_n(x)}{2\sin x}\right| \leq A(\varepsilon)|a_{n-1} - a_{n+1}| \to 0 \quad \text{as} \quad n \to \infty,$$

where $A(\varepsilon)$ is a positive constant that depends only on ε.

Therefore, from above and $\widetilde{F}_k(x) \leq k^2$ it is obvious that the series (1.1) converges uniformly on $[\varepsilon, \pi - \varepsilon]$, $\varepsilon > 0$, and from (2.2) the following representation holds

$$f(x) = \frac{1}{2\sin x}\sum_{k=1}^{\infty}(\triangle^2 a_{k-1} + \triangle^2 a_k)\widetilde{D}_k(x), \quad x \in (0, \pi). \tag{2.3}$$

Let $i \geq 2$ be an integer and $x \in \left(\frac{\pi}{i+1}, \frac{\pi}{i}\right] \cup \left(\frac{\pi}{2}, \pi\right)$. Using the equality

$$\widetilde{D}_k(x) = \frac{1}{2}\cot\frac{x}{2} - \frac{\cos\left(k + \frac{1}{2}\right)x}{2\sin\frac{x}{2}},$$

from (2.3) we have

$$\begin{aligned} f(x) &= \frac{1}{2\sin x}\sum_{k=1}^{i-1}(\triangle^2 a_{k-1} + \triangle^2 a_k)\widetilde{D}_k(x) + \\ &\quad + \frac{1}{2\sin x}\sum_{k=i}^{\infty}(\triangle^2 a_{k-1} + \triangle^2 a_k)\left(\frac{1}{2}\cot\frac{x}{2} - \frac{\cos\left(k + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}\right) = \\ &= \frac{\triangle a_{i-1} + \triangle a_i}{4\sin^2\frac{x}{2}} + \frac{1}{2\sin x}\sum_{k=1}^{i-1}(\triangle^2 a_{k-1} + \triangle^2 a_k)\widetilde{D}_k(x) + \\ &\quad + \frac{1}{2\sin x}\sum_{k=i}^{\infty}(\triangle^2 a_{k-1} + \triangle^2 a_k)\varphi_k(x) = \\ &= f_0(x) + f_1(x) + f_2(x), \end{aligned} \tag{2.4}$$

where

$$\varphi_k(x) := -\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} + \sum_{j=1}^{k}\sin jx.$$

Firstly, let us estimate the integral of the second term of (2.4). Indeed, since $1/\sin x$ is a bounded function on $[\varepsilon, \pi-\varepsilon]$, and $\left|\widetilde{D}_k(x)\right| \le k^2 x$, then we have

$$\begin{aligned}
\int\limits_{\pi/(m+1)}^{\pi/\ell} |f_1(x)|dx &\le \sum_{i=\ell}^{m} \int\limits_{\pi/(i+1)}^{\pi/i} \sum_{k=1}^{i-1} |\triangle^2 a_{k-1} + \triangle^2 a_k| \left|\frac{\widetilde{D}_k(x)}{2\sin x}\right| dx \le \\
&\le \sum_{i=\ell}^{m} \sum_{k=1}^{\ell-1} |\triangle^2 a_{k-1} + \triangle^2 a_k| \int\limits_{\pi/(i+1)}^{\pi/\ell} \left|\frac{\widetilde{D}_k(x)}{2\sin x}\right| dx + \\
&\quad + \sum_{i=\ell+1}^{m} \sum_{k=\ell}^{i-1} |\triangle^2 a_{k-1} + \triangle^2 a_k| \int\limits_{\pi/(i+1)}^{\pi/i} \left|\frac{\widetilde{D}_k(x)}{2\sin x}\right| dx \le \\
&\le \sum_{k=1}^{\ell-1} k^2 |\triangle^2 a_{k-1} + \triangle^2 a_k| \int\limits_{\pi/(m+1)}^{\pi/\ell} x dx + \\
&\quad + \sum_{k=\ell}^{m-1} k^2 |\triangle^2 a_{k-1} + \triangle^2 a_k| \int\limits_{\pi/(m+1)}^{\pi/(k+1)} x dx \le \\
&\le \pi^2 \frac{m+1-\ell}{m} \sum_{k=1}^{\ell-1} \frac{k^2}{\ell^2} |\triangle^2 a_{k-1} + \triangle^2 a_k| + \\
&\quad + \pi^2 \sum_{k=\ell}^{m} \sum_{j=k}^{\infty} j^2 |\triangle^3 a_{j-1} + \triangle^3 a_j|.
\end{aligned} \tag{2.5}$$

Now we estimate the integral of the third term of the right-hand side of (2.4). By applying the summation by parts again, we obtain

$$\begin{aligned}
f_2(x) &= \frac{1}{2\sin x} \lim_{p\to\infty} \sum_{k=i}^{p} (\triangle^2 a_{k-1} + \triangle^2 a_k) \varphi_k(x) = \\
&= \frac{1}{2\sin x} \lim_{p\to\infty} \Bigg\{ \sum_{k=i}^{p-1} (\triangle^3 a_{k-1} + \triangle^3 a_k) \psi_k(x) + \\
&\quad + (\triangle^2 a_{p-1} + \triangle^2 a_p) \psi_n(x) - (\triangle^2 a_{i-1} + \triangle^2 a_i) \psi_{i-1}(x) \Bigg\} = \\
&= \frac{1}{2\sin x} \sum_{k=i}^{\infty} (\triangle^3 a_{k-1} + \triangle^3 a_k) \big\{ \psi_k(x) - \psi_{i-1}(x) \big\},
\end{aligned}$$

where

$$\psi_n(x) = \sum_{j=1}^{n} \varphi_j(x) = -\frac{\sin{(n+1)}\,x}{4\sin^2 \frac{x}{2}}.$$

Thus, since

$$\Big|\frac{\psi_k(x)-\psi_{i-1}(x)}{2\sin x}\Big| = O\left(\frac{1}{x^2}\right),$$

we have

$$\begin{aligned}\int\limits_{\pi/(m+1)}^{\pi/\ell} |f_2(x)|dx &\le A\sum_{i=\ell}^{m}\int\limits_{\pi/(i+1)}^{\pi/i}\sum_{k=i}^{\infty}|\triangle^3 a_{k-1}+\triangle^3 a_k|\frac{dx}{x^2} \le \\ &\le A\sum_{i=\ell}^{m}\sum_{k=i}^{\infty}k^2|\triangle^3 a_{k-1}+\triangle^3 a_k|.\end{aligned} \tag{2.6}$$

The last term in (2.5) can be rewritten as

$$\begin{aligned}&\sum_{i=\ell}^{m}\sum_{k=i}^{\infty}k^2|\triangle^3 a_{k-1}+\triangle^3 a_k| = \\ &=\sum_{i=\ell}^{m}\sum_{k=i}^{m}k^2|\triangle^3 a_{k-1}+\triangle^3 a_k| + \sum_{i=\ell}^{m}\sum_{k=m+1}^{\infty}k^2|\triangle^3 a_{k-1}+\triangle^3 a_k| = \\ &=\sum_{k=\ell}^{m}(k+1-\ell)k^2|\triangle^3 a_{k-1}+\triangle^3 a_k| + (m+1-\ell)\sum_{k=m+1}^{\infty}k^2|\triangle^3 a_{k-1}+\triangle^3 a_k|.\end{aligned} \tag{2.7}$$

Finally we note that

$$\int\limits_{\pi/(m+1)}^{\pi/\ell} |f_0(x)|dx = \sum_{i=\ell}^{m}|\triangle a_{i-1}+\triangle a_i|\int\limits_{\pi/(i+1)}^{\pi/i}\frac{dx}{4\sin^2\frac{x}{2}} = \sum_{i=\ell}^{m}d_i|\triangle a_{i-1}+\triangle a_i|, \tag{2.8}$$

where

$$\int\limits_{\pi/(i+1)}^{\pi/i}\frac{dx}{4\sin^2\frac{x}{2}} = \frac{\sin\frac{\pi}{i(i+1)}}{2\sin\frac{\pi}{2(i+1)}\sin\frac{\pi}{2i}} = d_i.$$

The proof of (2.1) follows from (2.8), (2.5), (2.6) and (2.7). □

Now we shall give a similar result that pertains to the sine series.

Theorem 2.2. *If $a_k \to 0$ as $k\to\infty$ and* (1.8) *is satisfied with $\alpha = 2$, then series* (1.2) *converges pointwise on $(0,\pi)$, uniformly on $[\varepsilon,\pi-\varepsilon]$ for every $\varepsilon > 0$, and for $1\le \ell\le m$ the sum function $g(x)$ satisfies*

$$\begin{aligned}\int\limits_{\pi/(m+1)}^{\pi/\ell}|g(x)|dx = O\Big(&\frac{m+1-\ell}{m}\sum_{k=0}^{\ell-1}\frac{(k+1)^2}{\ell}|\triangle^2 a_{k-1}+\triangle^2 a_k| + \\ &+\sum_{k=\ell}^{\infty}\min(k+1-\ell,\, m+1-\ell)k^2|\triangle^3 a_{k-1}+\triangle^3 a_k|\Big).\end{aligned}$$

Proof. The proof of Theorem 2.2 is similar to that of Theorem 2.1. Therefore, we have omitted it. □

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