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ON THE STRUCTURE OF CERTAIN NONTRANSITIVE DIFFEOMORPHISM GROUPS ON OPEN MANIFOLDS

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Abstract. It is shown that in some generic cases the identity component of the group of leaf preserving diffeomorphisms (with not necessarily compact support) on a foliated open manifold is perfect. Next, it is proved that it is also bounded, i.e. bounded with respect to any bi-invariant metric. It follows that the group is uniformly perfect as well.

Keywords: foliated manifold, bounded group, conjugation-invariant norm, group of diffeomorphisms, commutator, perfectness, uniform perfectness.

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1. INTRODUCTION

Let us recall that a group G is called *perfect* if $G = [G, G]$, where the commutator subgroup is generated by all commutators $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$, $g_1, g_2 \in G$. In terms of homology of groups this means that $H_1(G) = G/[G, G] = 0$.

Let $(M, \mathcal{F})$ be a foliated manifold. We say that a diffeomorphism $f : M \to M$ is *leaf preserving* (resp. *foliation preserving*) if $f(L_x) = L_x$ (resp. $f(L_x) = L_{f(x)}$) for all $x \in M$, where L_x is the leaf of $\mathcal{F}$ passing through x. By $\mathrm{Diff}^\infty_c(M, \mathcal{F})$ we denote the group of leaf preserving diffeomorphisms of M which are isotopic to the identity through compactly supported isotopies of leaf preserving diffeomorphisms.

The following result is due to Rybicki [13] (and independently Tsuboi [18]).

Theorem 1.1. *Let $(M, \mathcal{F})$ be a foliated smooth manifold. Then $\mathrm{Diff}^\infty_c(M, \mathcal{F})$ is perfect.*

The proof of Theorem 1.1 modifies arguments of Herman and Thurston (cf. Rybicki [13]). If the foliation $\mathcal{F}$ is trivial, i.e. $\mathcal{F} = \{M\}$, then Theorem 1.1 reduces to a classical result due to Thurston [17] stating that the group $\mathrm{Diff}^\infty_c(M)$ of C^∞-diffeomorphisms of

M compactly isotopic to the identity is simple. In the case of C^r-diffeomorphisms with r large and finite it is very likely that Theorem 1.1 is no longer true. See a discussion on this problem in [7].

Observe that the group $\mathrm{Diff}^\infty_c(M,\mathcal{F})$ is locally contractible (cf. [14]) and, consequently, connected.

For any manifold M let $\mathrm{Diff}^r(M)$ be the group of all C^r-diffeomorphisms on M which are isotopic to the identity, $r = 0, 1, \dots, \infty$. We have the following result by McDuff [12] on the structure of diffeomorphism or homeomorphism groups of an open manifold.

Theorem 1.2. *Let M be a C^r-smooth manifold such that $M = \mathrm{Int}\, Q$, where Q is a compact manifold with boundary, $r = 0, 1, \dots, \infty$. Then $\mathrm{Diff}^r(M)$ is a perfect group.*

Recently, a complete proof of Theorem 1.2 was presented by Schweitzer in [16].

Obviously the group $\mathrm{Diff}^r(M)$ is not simple since it admits $\mathrm{Diff}^r_c(M)$ as a nontrivial normal subgroup. It is worth observing that the proof of McDuff's theorem is completely different from those in the compact case (see [3, 5, 8–10, 17]). Especially the problem of perfectness of the group $\mathrm{Diff}^r_c(M)$ for $r = \dim M + 1$ is still open. But there are arguments (cf. [7, 11]) that the group $\mathrm{Diff}^{n+1}_c(M)$ would not be perfect.

Let M be a connected compact manifold of dimension $n \geq 1$. We will deal with the product $(n+k)$-dimensional manifold $M^{(k)} = M \times \mathbb{R}^k$ endowed with the k-dimensional product foliation $\mathcal{F}_k = \{\{\mathrm{pt}\} \times \mathbb{R}^k\}$, $k \geq 1$.

By $G^k = \mathrm{Diff}^\infty(M^{(k)}, \mathcal{F}_k)$ (resp. $G^k_c = \mathrm{Diff}^\infty_c(M^{(k)}, \mathcal{F}_k)$) we denote the group of all leaf preserving C^∞-diffeomorphisms which can be joined with the identity by smooth isotopies (resp. compactly supported smooth isotopies) of leaf preserving C^∞-diffeomorphisms.

Our first goal is the following result being an analogue of Theorem 1.1.

Theorem 1.3. *Let M be a connected compact manifold. Then the group G^k is perfect.*

Of course G^k_c is also perfect in view of Theorem 1.1. On the other hand, Theorem 1.3 is probably no longer true for finite $r > 0$, similarly Theorem 1.1.

The next results are related with the notion of boundedness. Recall that a group is called *bounded* if it is bounded with respect to any bi-invariant metric.

Theorem 1.4. *The group G^k is bounded.*

We will also prove (see Theorem 3.3) that the subgroup G^k_c is bounded. On the other hand, it might be very difficult to prove for an arbitrary smooth foliated manifold that the group $\mathrm{Diff}^\infty_c(M,\mathcal{F})$ is bounded. A possible result would depend on the topology of M and of the leaves of foliation.

The problem of the boundedness of groups of diffeomorphisms is closely related to the problem of uniform perfectness (cf. [4]). Group G is called *uniformly perfect* if G is perfect and there exists a positive integer N such that any $g \in G$ can be expressed as a product of at most N commutators of elements of G. If G is perfect then $\mathrm{cl}_G(g)$, the *commutator length* of $g \in G$, is the smallest N such that g can be represented by a product of N commutators. For any perfect group G denote by cld_G

the *commutator length diameter* of G, i.e. $\mathrm{cld}_G := \sup_{g\in G} \mathrm{cl}_G(g)$. Then G is uniformly perfect iff $\mathrm{cld}_G < \infty$.

Recently Burago, Ivanov and Polterovich [4] and, independently, Tsuboi [19] proved basic results concerning the uniform perfectness of diffeomorphism groups of many manifolds. In contrast to the problem of perfectness these results depend essentially on the topology of the underlying manifold. These results generalize older ones, e.g. [2]. See also Rybicki [15] and Abe-Fukui [1].

The following result is a consequence of Theorem 1.4 and Lemma 2.4.

Corollary 1.5. *Groups G^k_c and G^k are uniformly perfect. Moreover* $\mathrm{cld}_{G^k_c} \leq 2$ *and* $\mathrm{cld}_{G^k} \leq 4$.

In this paper we use several ideas from the non-foliated case (see Rybicki [15]).

2. THE PROOF OF THEOREM 1.3

We have the following version of Isotopy Extension Theorem.

Theorem 2.1. *Let g_t be an isotopy in G^k with $g_0 = \mathrm{Id}$ and let $N \times K \subset M \times \mathbb{R}^k$ be a compact set. Then for any open neighbourhood U of $\bigcup_{t\in[0,1]} g_t(N \times K)$ there exists an isotopy f_t in G^k such that $f_t = g_t$ on $N \times K$ and* $\mathrm{supp}(f_t) \subset U$. *Moreover, if* $\mathrm{supp}(g_t) \subset N \times \mathbb{R}^k$ *then we may choose f_t with* $\mathrm{supp}(f_t) \subset N \times \mathbb{R}^k$.

Proof. The proof is analogous to the usual (non-foliated) case (cf. [6]). □

See that the above theorem is still valid for M noncompact.

In the sequel for an open set U we denote by G^k_U the subgroup of G^k containing elements which can be joined with the identity through isotopies in G^k compactly supported in U.

Recall that a subset $V \subseteq N$ of an open manifold N is called a *neighbourhood of infinity* if $\overline{N \setminus V}$ is compact. We say that N is *one-ended* if every neighbourhood of infinity contains connected neighbourhood of infinity. Note that $M^{(k)} = M \times \mathbb{R}^k$ is one-ended for $k \geq 2$. By a *translation system* on $M^{(k)}$ we understand a family $\{V_i\}_{i=0}^{\infty}$ of closed neighbourhoods of infinity such that $V_{i+1} \subset \mathrm{Int}\, V_i$ and $\bigcap_{i=0}^{\infty} V_i = \emptyset$.

Fix $R > 0$ and $V = V_R = M \times \{x \in \mathbb{R}^k : |x| > R\}$. Then V is a neighbourhood of infinity of $M^{(k)}$. For $0 < R_1 < R_2 \leq \infty$ we set $A^k(R_1, R_2) = \{x \in \mathbb{R}^k : R_1 < |x| < R_2\}$ and $\overline{A^k}(R_1, R_2) = \overline{A^k(R_1, R_2)}$.

In the case $k = 1$ the manifold $M^{(k)}$ has two ends. We will omit the proof in this case as it is analogous to the case $k \geq 2$.

Proposition 2.2. *We have the following properties:*

1. *For any $g \in G^k$ there exist $f \in G^k_c$ and $h \in G^k_V$ such that $g = fh$.*
2. *For every $g \in G^k_V$ there is a sequence*

$$R < a_1 < \bar{a}_1 < \bar{b}_1 < b_1 < a_2 < \ldots < a_i < \bar{a}_i < \bar{b}_i < b_i < \ldots$$

tending increasingly to infinity and $h \in G^k_V$ such that:

1) $h = g$ *on* $\bigcup_{i=1}^{\infty} M \times \overline{A^k}(\bar{a}_i, \bar{b}_i)$,
2) $\operatorname{supp}(h) \subset \bigcup_{i=1}^{\infty} M \times A^k(a_i, b_i)$,
3) *If* $h = h_1 h_2 \ldots$ *such that* $\operatorname{supp}(h_i) \subset M \times A^k(a_i, b_i)$ *then* $h_i \in G^k_{M\times A^k(a_i,b_i)}$ *for all* $i \geq 1$,
4) $h = \mathrm{Id}$ *on* $V_{R'}$ *for some* $R' > 0$ *if* g *has compact support.*

3. *For any sequence* $\{\alpha_i\}_{i\in\mathbb{N}}$ *with* $\alpha_i \in (R, \infty)$ *tending increasingly to infinity there exists a* C^∞*-mapping* $[0, \infty) \ni t \mapsto f_t \in G^k_V$ *with* $f_0 = \mathrm{Id}$, $f_j = (f_1)^j$ *for* $j = 2, 3, \ldots$ *such that* $f_1(V_i) = V_{i+1}$, $i \in \mathbb{N}$, *for the translation system* $\{V_i\}_{i\in\mathbb{N}}$, *where* $V_i = M \times \{x \in \mathbb{R}^k : |x| \geq \alpha_i\}$, $i \in \mathbb{N}$.

Proof. Let $g \in G^k$ and let g_t be an isotopy from Id to g in G^k. We take U a neighbourhood of $\bigcup_{t\in[0,1]} g_t(M \times C)$ where $C = \{x \in \mathbb{R}^k : |x| \leq R\}$. From Theorem 2.1 for $N = M$ there exists an isotopy f_t in G^k with $f_t = g_t$ on $M \times C$ and $\operatorname{supp}(f_t) \subset U$. We set $f = f_1$ and $h = f^{-1}g$. Then $f \in G^k_c$ and $h \in G^k_V$ since $h = \mathrm{Id}$ on $M \times C = M^{(k)} \setminus V$.

To show (2) we proceed by induction. Assume that we have chosen sequence

$$R < a_1 < \bar{a}_1 < \bar{b}_1 < b_1 < \ldots < a_i < \bar{a}_i < \bar{b}_i < b_i$$

and isotopies $h^1_t, \ldots, h^i_t$ in G^k with $\operatorname{supp}(h^j_t) \subset M \times A^k(a_j, b_j)$ and $h^j_t = g_t$ on $M \times \overline{A^k}(\bar{a}_j, \bar{b}_j)$, $j = 1, \ldots, i$. Then we may choose $b_i < a_{i+1} < \bar{a}_{i+1} < \bar{b}_{i+1} < b_{i+1}$ such that $\bigcup_{t\in[0,1]} g_t(M \times \overline{A^k}(\bar{a}_{i+1}, \bar{b}_{i+1}))$ is disjoint with $M \times \overline{A^k}(R, b_i)$ and is contained in $M \times A^k(a_{i+1}, b_{i+1})$. From Theorem 2.1 there exists an isotopy h^{i+1}_t in G^k such that $h^{i+1}_t = g_t$ on $M \times \overline{A^k}(\bar{a}_{i+1}, \bar{b}_{i+1})$ and $\operatorname{supp}(h^{i+1}_t) \subset M \times A^k(a_{i+1}, b_{i+1})$. Taking $h_t = \prod_{i=1}^{\infty} h^i_t$ we obtain (2).

Now let $\{\alpha_i\}_{i\in\mathbb{N}}$ be as in (3). We take an isotopy $\tau : [0,1] \to \mathrm{Diff}^\infty(\overline{A^k}(R, \infty))$ with $\tau_0 = \mathrm{Id}$ and $\tau_1(\overline{A^k}(\alpha_i, \alpha_{i+1})) = \overline{A^k}(\alpha_{i+1}, \alpha_{i+2})$, $i \in \mathbb{N}$. Continuing by induction let $\tau : [i, i+1] \to \mathrm{Diff}^\infty(\overline{A^k}(R, \infty))$ be an isotopy from τ_j to $\tau_{j+1} = (\tau_1)^{j+1}$. We get $\tau = \bigcup_{i=0}^{\infty} \tau|_{[j,j+1]} : [0, \infty) \to \mathrm{Diff}^\infty(\overline{A^k}(R, \infty))$ where τ is smoothed on neighbourhoods of $j = 1, 2, \ldots$ if necessary. We set $f_t = \mathrm{Id}_M \times \tau_t$. □

Remark 2.3. The condition $h_i \in G^k_{M\times A^k(a_i,b_i)}$ in Proposition 2.2 (3) means that the diffeomorphisms h_i do not admit any twistings.

Lemma 2.4. *Any* $g \in G^k_V$ *can be written as a product of two commutators of elements of* G^k_V.

Proof. Let $g \in G^k_V$. Choose a sequence

$$R < a_1 < \bar{a}_1 < \bar{b}_1 < b_1 < \ldots < a_i < \bar{a}_i < \bar{b}_i < b_i < \ldots$$

and $h \in G^k_V$ as in Proposition 2.2 (2).

Put $\bar{h} = h^{-1}g$, that is $g = h\bar{h}$. It suffices to show that h is a commutator of elements in G^k_V (by the same way it is true for $\bar{h}$).

Choose arbitrary $\alpha_0 \in (R, a_1)$ and $\alpha_i \in (b_i, a_{i+1})$ for $i = 1, 2, \ldots$. In view of Proposition 2.2 (3) there exists an isotopy $[0, \infty) \ni t \mapsto f_t \in G^k$ with $f_0 = \mathrm{Id}$,

$f_j = (f_1)^j$, $j = 1, 2, \ldots$, and such that $f_1(V_i) = V_{i+1}$ for $i \in \mathbb{N}$, where $V_i = M \times \{x \in \mathbb{R}^k : |x| \geq \alpha_i\}$.

Now define $\tilde{h} \in G^k$ as follows. Set $\tilde{h} = h$ on $M \times \overline{A^k}(R, \alpha_1)$ and $\tilde{h} = h(fhf^{-1}) \ldots (f_i h f_i^{-1})$ on $M \times \overline{A^k}(R, \alpha_{i+1})$ for $i \geq 1$. Since we have $f_j = (f_1)^j$ then $\tilde{h}|_{[R,\alpha_i)}$ is a consistent family of functions, and $\tilde{h} = \bigcup_{i=1}^{\infty} \tilde{h}|_{[R,\alpha_i)}$ is a local homeomorphism. It is easily checked that $\tilde{h}$ is a bijection.

By definition we have the equality $\tilde{h} = hf\tilde{h}f^{-1}$. Hence $h = [\tilde{h}, f]$ as claimed. □

Remark 2.5. Notice, that decomposition $g = h\bar{h}$ is necessary in the proof above. In fact, if we define $\tilde{h}$ directly from g instead from h then we cannot ensure that the resulting $\tilde{h}$ is surjective. If we define $\tilde{h}$ by using $\tilde{g}_i = (f^i g f^{-i}) \ldots (fgf^{-1})g$ on $M \times \overline{A^k}(R, \alpha_i)$, $i = 1, 2, \ldots$ then the family $\tilde{g}_i$ is inconsistent and we cannot glue-up $\tilde{g}_i$.

Proof of Theorem 1.3. To show that G^k is perfect it is enough to show that $G^k \subset [G^k, G^k]$. Let $g \in G^k$. In view of Proposition 2.2 (1) $g = fh$, such that $f \in G^k_c$ and $h \in G^k_V$ for a neighbourhood of infinity V.

Due to Theorem 1.1 diffeomorphism f can be expressed as a product of commutators and by Lemma 2.4 we have $h = [h_1, h_2][h_3, h_4]$ for some $h_1, h_2, h_3, h_4 \in G^k_V$. Hence g is a product of commutators of elements from G^k, so $g \in [G^k, G^k]$. □

Remark 2.6. Let us observe that it is rather hopeless to obtain the above theorem for any foliated open manifold as it was done in Theorem 1.1 for the compactly supported case. We conjecture, however, that the compactness assumption on M in Theorem 1.3 can be omitted.

We also have the following theorem.

Theorem 2.7. *Let* $\mathrm{Diff}^{\infty}_{[c]}(M^{(k)}, \mathcal{F}_k)$ *be the subgroup of* G^k *consisting of all elements* $f \in G^k$ *such that* $\overline{\{x \in M : f|_{L_x} \neq \mathrm{Id}\}}$ *is a compact subset of* M*. Then* $\mathrm{Diff}^{\infty}_{[c]}(M^{(k)}, \mathcal{F}_k)$ *is perfect.*

Proof. We proceed as in the proof of Theorem 1.3 but we need some modifications of proofs of Proposition 2.2 and Lemma 2.4.

First, let g_t be an isotopy in $\mathrm{Diff}^{\infty}_{[c]}(M^{(k)}, \mathcal{F}_k)$ and $\mathrm{supp}(g_t) \subset N \times \mathbb{R}^k$. From Theorem 2.1 we may assume that all constructed mappings in Proposition 2.2 (1), (2) have supports contained in $N \times \mathbb{R}^k$.

Next, to prove (3) we define τ as above and we fix $\xi : M \to \mathbb{R}$ of class C^∞ such that $\xi = 1$ on N and $\xi = 0$ over a compact set. Define $[0, \infty) \ni t \mapsto f_t \in G^k_V \cap \mathrm{Diff}^{\infty}_{[c]}(M^{(k)}, \mathcal{F}_k)$, where $V = M \times \{x \in \mathbb{R}^k : |x| > R\}$, by the formula

$$f_t = \mathrm{Id}_M \times (\xi \cdot (\tau_t - \mathrm{Id}) + \mathrm{Id})$$

i.e. $f_t(x, y) = (x, \xi(x) \cdot (\tau_t(y) - y) + y)$ for $(x, y) \in M \times \mathbb{R}^k$. Here $\cdot$ means multiplication in $\mathbb{R}^k$. We get $f_0 = \mathrm{Id}$ and $f_j = (f_1)^j$ on $N \times \overline{A^k}(R, \infty)$, $j \geq 1$.

Now as in the proof of Lemma 2.4 we get for $g \in G^k_V \cap \mathrm{Diff}^{\infty}_{[c]}(M^{(k)}, \mathcal{F}_k)$ with $\mathrm{supp}(g) \subset N \times \mathbb{R}^k$ diffeomorphisms $h \in G^k_V$ and $\bar{h} = h^{-1}g \in G^k_c$ such that $\mathrm{supp}(h), \mathrm{supp}(\bar{h}) \subset N \times \mathbb{R}^k$. We will proceed with h.

Let $\{\alpha_i\}$ be the sequence used in the construction of τ. We define $\tilde{h} \in \mathrm{Diff}^\infty_{[c]}(M^{(k)}, \mathcal{F}_k)$ by setting $\tilde{h} = h$ on $N \times \overline{A^k}(R, \alpha_1)$ and $\tilde{h} = h(fhf^{-1})\dots(f_i h f_i^{-1})$ on $N \times \overline{A^k}(R, \alpha_{i+1})$ for $i \geq 1$. Since $\mathrm{supp}(h) \subset N \times \mathbb{R}^k$, $\tilde{h}$ is well defined. Moreover, $f_j = (f_1)^j$ on $N \times \overline{A^k}(R, \infty)$, so $\tilde{h}$ is a homeomorphism.

From definition we have $\tilde{h} = hf\tilde{h}f^{-1}$, so $h = [\tilde{h}, f]$. Analogouly we obtain $\bar{h}$. Hence every g as above may be represented as two commutators. Using Theorem 1.1 we get that $\mathrm{Diff}^\infty_{[c]}(M^{(k)}, \mathcal{F}_k)$ is perfect. □

3. BOUNDEDNESS AND UNIFORM PERFECTNESS OF THE GROUP G^k

The notion of the conjugation-invariant norm is a basic tool in studies of the boundedness of groups. Let G be a group. A *conjugation-invariant norm* on G is a function $\nu : G \to [0, \infty)$ which satisfies the following conditions.

For any $g, h \in G$:

1. $\nu(g) > 0$ if and only if $g \neq e$,
2. $\nu(g^{-1}) = \nu(g)$,
3. $\nu(gh) \leq \nu(g) + \nu(h)$,
4. $\nu(hgh^{-1}) = \nu(g)$.

It is easy to see that if H is a subgroup of G and if ν is a conjugation-invariant norm on G then $\nu|_H$ is a conjugation-invariant norm on H. It is also easily seen that G is bounded if and only if any conjugation-invariant norm on G is bounded. Suppose that G is perfect. Then the commutator length cl_G is a conjugation-invariant norm on G.

In the sequel we will need some algebraic tools. A subgroup H of G is called *strongly m-displaceable* if there is $f \in G$ such that the subgroups H, $fHf^{-1}, \dots, f^m H f^{-m}$ pairwise commute. Then we say that *f m-displaces H*.

Fix a conjugation-invariant norm ν on G and assume that $H \subset G$ is strongly m-displaceable.

By Theorem 2.2 in [4] we have for any $h \in [H, H]$ that

$$\mathrm{cl}_G(h) \leq 2. \tag{3.1}$$

Moreover, if there exists $g \in G$ which m-displaces H for every $m \geq 1$ then for all $h \in [H, H]$

$$\nu(h) \leq 14\nu(g). \tag{3.2}$$

Proposition 3.1. *If U, W are open disjoint subsets of $M^{(k)}$ such that there is $f \in G^k$ with $\overline{f(U \cup W)} \subset W$ then f m-displaces G^k_U for all $m \geq 1$.*

Indeed, this follows from $f^m(U) \subset f^{m-1}(W) \setminus f^m(W)$ for every $m \geq 1$.

Proposition 3.2. *There exists disjoint open subsets $U, W \subset M^{(k)}$ and $f \in G^k_c$ with $\overline{f(U \cup W)} \subset W$ such that for every $g \in G^k_c$ there is $h \in G^k_c$ such that $h(\mathrm{supp}(g)) \subset U$.*

Proof. Indeed, let $U', W' \subset \mathbb{R}^k$ be open sets. From transitivity of $\mathrm{Diff}^\infty_c(\mathbb{R}^k)$ there exists $\varphi \in \mathrm{Diff}^\infty_c(\mathbb{R}^k)$ such that $\overline{\varphi(U' \cup W')} \subset W'$. Put $f = \mathrm{Id}_M \times \varphi$, $U = M \times U'$ and $W = M \times W'$.

Now for $g \in G^k_c$ we may assume that $\mathrm{supp}(g) \subset M \times B$ where B is a ball in $\mathbb{R}^k$. Then by taking $\varphi \in \mathrm{Diff}^\infty_c(\mathbb{R}^k)$ with $\varphi(B) \subset U$ we have $h = \mathrm{Id}_M \times \varphi \in G^k_c$ and $h(\mathrm{supp}(g)) \subset h(M \times B) \subset U$. □

Now we can prove the following theorem.

Theorem 3.3. *The group G^k_c is bounded.*

Proof. Fix U, W and f as in Proposition 3.2. Let ν be any conjugation invariant norm on G^k_c and $g \in G^k_c$. By taking $h \in G^k_c$ such that $h(\mathrm{supp}(g)) \subset U$ we get $\mathrm{supp}(hgh^{-1}) = h(\mathrm{supp}(g)) \subset U$. Then $hgh^{-1} \in G^k_U$.

From Proposition 3.1, f m-displaces G^k_U for every $m \geq 1$. From Theorem 1.1 we get that G^k_U is perfect and hence inequality (3.2) gives us

$$\nu(g) = \nu(hgh^{-1}) \leq 14\nu(f).$$

This completes the proof. □

We will use the following obvious fact.

Proposition 3.4. *The group G^k satisfies the following condition: For any sequence in (R, ∞) of the form*

$$R < a_1 < b_1 < a_2 < b_2 < \ldots < a_i < b_i < \ldots$$

tending to infinity there exists $f \in G^k_V$ such that

$$f\left(M \times (\overline{A^k}(a_{2i-1}, b_{2i-1}) \cup \overline{A^k}(a_{2i}, b_{2i}))\right) \subset M \times A^k(a_{2i}, b_{2i})$$

for $i = 1, 2, \ldots$. Moreover, for another sequence

$$R < \bar{a}_1 < \bar{b}_1 < \bar{a}_2 < \bar{b}_2 < \ldots < \bar{a}_i < \bar{b}_i < \ldots$$

there exists $\varphi \in G^k_V$ of the form $\varphi = \mathrm{Id}_M \times \tilde{\varphi}$ with $\tilde{\varphi}(\overline{A^k}(a_i, b_i)) = \overline{A^k}(\bar{a}_i, \bar{b}_i)$ for $i = 1, 2, \ldots$.

Proof of Theorem 1.4. Let ν be a conjugation-invariant norm on G^k and let $g \in G^k$. From Proposition 2.2 (1) we have a decomposition $g = fh$, $f \in G^k_c$, $h \in G^k_V$. In view of Theorem 3.3, $\nu|_{G^k_c}$ is bounded and we may assume that $\nu(f)$ is bounded for every g. It suffices to show that $\nu(h)$ is also bounded.

Let V be a neighbourhood of infinity such that $h \in G^k_V$. From Proposition 2.2 (2) we get a sequence

$$R < a_1 < \bar{a}_1 < \bar{b}_1 < b_1 < \ldots$$

and diffeomorphisms $h_1, h_2 \in G^k_V$ such that $h_1 = h$ on a neighbourhood of $\bigcup_{i=1}^\infty M \times \overline{A^k}(\bar{a}_{2i-1}, \bar{b}_{2i-1})$ with $\mathrm{supp}(h_1) \subseteq U_1 := \bigcup_{i=1}^\infty M \times A^k(a_{2i-1}, b_{2i-1})$ and

$h_2 = h$ on a neighbourhood of $\bigcup_{i=1}^{\infty} M \times \overline{A^k}(\bar{a}_{2i}, \bar{b}_{2i})$ with $\mathrm{supp}(h_2) \subseteq U_2 := \bigcup_{i=1}^{\infty} M \times A^k(a_{2i}, b_{2i})$.

Now we may decompose h into $h = h_1 h_2 h_3 h_4$, where $h_3 = h$ on a neighbourhood of $\bigcup_{i=1}^{\infty} M \times \overline{A^k}(b_{2i-1}, a_{2i})$, $\mathrm{supp}(h_3) \subseteq U_3 := \bigcup_{i=1}^{\infty} M \times A^k(\bar{b}_{2i-1}, \bar{a}_{2i})$, $h_4 = h$ on a neighbourhood of $\bigcup_{i=1}^{\infty} M \times \overline{A^k}(b_{2i}, a_{2i+1})$ and $\mathrm{supp}(h_4) \subseteq U_4 := \bigcup_{i=1}^{\infty} M \times A^k(\bar{b}_{2i}, \bar{a}_{2i+1})$.

On the other hand, for a fixed sequence

$$R < c_1 < d_1 < c_2 < d_2 < \dots$$

tending to infinity, from Proposition 3.4 there exists $f \in G_V^k$ such that

$$f\left(M \times (\overline{A^k}(c_{2i-1}, d_{2i-1}) \cup \overline{A^k}(c_{2i}, d_{2i}))\right) \subset M \times A^k(c_{2i}, d_{2i})$$

and we get $\varphi_1 \in G_V^k$ with $\overline{\varphi_1 f \varphi_1^{-1}(U_1 \cup U_2)} \subset U_2$. From Proposition 3.1 diffeomorphism $\varphi_1 f \varphi_1^{-1}$ m-displaces $G_{U_1}^k$ for every $m \geq 1$. Since $G_{U_1}^k \cong G^k$ is perfect, (3.2) gives us

$$\nu(h_1) \leq 14\nu(\varphi_1 f \varphi_1^{-1}) = 14\nu(f).$$

We may obtain analogous estimations for h_2, h_3, h_4 with some $\varphi_2, \varphi_3, \varphi_4 \in G_V^k$ and then

$$\nu(h) \leq \nu(h_1) + \nu(h_2) + \nu(h_3) + \nu(h_4) \leq 56\nu(f).$$

Hence $\nu(g) \leq \nu(f) + \nu(h)$ is bounded for every $g \in G^k$. □

As a consequence we have the proof of Corollary 1.5.

Proof of Corollary 1.5. Obviously, as the groups G_c^k and G^k are both perfect and bounded, they are uniformly perfect as well.

We will show estimations of commutator length diameters of G_c^k and G^k.

Let U, W and $f \in G_c^k$ be defined as in Proposition 3.2 and let $\tilde{g} \in G_c^k$. There exists $h \in G_c^k$ with $h(\mathrm{supp}(\tilde{g})) \subset U$. By Proposition 3.1, f m-displaces G_U^k for any $m \geq 1$.

For $h\tilde{g}h^{-1}$ we have $\mathrm{supp}(h\tilde{g}h^{-1}) = h(\mathrm{supp}(\tilde{g})) \subset U$. Since G_U^k is perfect and it is m-displaceable in G_c^k, then by (3.1) we get $\mathrm{cl}_{G_c^k}(\tilde{g}) = \mathrm{cl}_{G_c^k}(h\tilde{g}h^{-1}) \leq 2$.

Now let $g \in G^k$. According to (1) in Proposition 2.2 there are $\tilde{g} \in G_c^k$ and $h \in G_V^k$ such that $g = \tilde{g}h$, where V is a neighbourhood of infinity. Since $\mathrm{cl}_{G_c^k}(\tilde{g}) \leq 2$ and $\mathrm{cl}_{G^k}(h) = 2$ by Lemma 2.4 then, $\mathrm{cl}_{G^k}(g) \leq 4$. □

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