

## RANK-ONE PERTURBATION OF TOEPLITZ OPERATORS AND REFLEXIVITY

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**Abstract.** It was shown that rank-one perturbation of the space of Toeplitz operators preserves 2-hyperreflexivity.

**Keywords:** Toeplitz operators, reflexivity, hyperreflexivity.

**Mathematics Subject Classification:** 47A15, 47L99.

### 1. INTRODUCTION

Let  $\mathcal{H}$  be a Hilbert space. By  $\mathcal{B}(\mathcal{H})$  we denote the algebra of all bounded linear operators on  $\mathcal{H}$ .

It is well known that the space of trace class operators  $\tau c$  is a predual to  $\mathcal{B}(\mathcal{H})$  with the dual action  $\langle A, f \rangle = \text{tr}(Af)$ , for  $A \in \mathcal{B}(\mathcal{H})$  and  $f \in \tau c$ . The trace norm in  $\tau c$  will be denoted by  $\|\cdot\|_1$ . Denote by  $F_k$  the set of operators of rank at most  $k$ . Every rank-one operator may be written as  $x \otimes y$ , for  $x, y \in \mathcal{H}$ , and  $(x \otimes y)z = \langle z, y \rangle x$  for  $z \in \mathcal{H}$ . Moreover,  $\text{tr}(T(x \otimes y)) = \langle Tx, y \rangle$ .

Let  $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$  be a subspace (when we write subspace we mean a norm closed linear manifold). By  $d(T, \mathcal{M})$  we will denote the standard distance from an operator  $T$  to a subspace  $\mathcal{M}$ , i.e.,  $d(T, \mathcal{M}) = \inf\{\|T - M\| : M \in \mathcal{M}\}$ . It is known that when  $\mathcal{M}$  is weak\* closed  $d(T, \mathcal{M}) = \sup\{|\text{tr}(Tf)| : f \in \mathcal{M}_\perp, \|f\|_1 \leq 1\}$ , where  $\mathcal{M}_\perp$  denotes the preannihilator of  $\mathcal{M}$ .

Recall that the *reflexive closure* of a subspace  $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$  is given by

$$\text{ref } \mathcal{M} = \{T \in \mathcal{B}(\mathcal{H}) : Tx \in [\mathcal{M}x] \text{ for all } x \in \mathcal{H}\},$$

where  $[\cdot]$  denotes the norm-closure. A subspace  $\mathcal{M}$  is called *reflexive* if  $\mathcal{M} = \text{ref } \mathcal{M}$ . Due to Longstaff [14] we know that when  $\mathcal{M}$  is a weak\* closed subspace of  $\mathcal{B}(\mathcal{H})$ , then  $\mathcal{M}$  is reflexive if and only if  $\mathcal{M}_\perp$  is a closed linear span of the set of all operators of rank one contained in  $\mathcal{M}_\perp$  (i.e.,  $\mathcal{M}_\perp = [\mathcal{M}_\perp \cap F_1]$ ). A subspace  $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$  is called *k-reflexive* if  $\mathcal{M}^{(k)} = \{M^{(k)} : M \in \mathcal{M}\}$  is reflexive in  $B(\mathcal{H}^{(k)})$ , where

$M^{(k)} = M \oplus \dots \oplus M$  and  $\mathcal{H}^{(k)} = \mathcal{H} \oplus \dots \oplus \mathcal{H}$ . Kraus and Larson [12, Theorem 2.1] proved that a weak\* closed subspace  $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$  is  $k$ -reflexive if and only if  $\mathcal{M}_\perp$  is a closed linear span of rank- $k$  operators contained in  $\mathcal{M}_\perp$  (i.e.,  $\mathcal{M}_\perp = [\mathcal{M}_\perp \cap F_k]$ ).

In [2] Arveson defines an algebra  $\mathcal{A}$  as *hypercentral* if there is a constant  $a$  such that  $d(T, \mathcal{A}) \leq a \sup\{\|P^\perp T P\| : P \in \text{Lat } \mathcal{A}\}$  for all  $T \in \mathcal{B}(\mathcal{H})$ . In [11] this definition was generalized to subspaces of operators. A subspace  $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$  is called *hypercentral* if there is a constant  $a$  such that

$$d(T, \mathcal{M}) \leq a \sup\{\|Q^\perp T P\| : P, Q \text{ are projections and } Q^\perp \mathcal{M} P = 0\}$$

for all  $T \in \mathcal{B}(\mathcal{H})$ . As it was shown in [12] the supremum on the right hand side is equal to  $\sup\{|\langle T, g \otimes h \rangle| : g \otimes h \in \mathcal{M}_\perp, \|g \otimes h\|_1 \leq 1\}$ .

Recall after [10] the definition of  $k$ -hypercentrality. Let  $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$  be a subspace. For any  $T \in \mathcal{B}(\mathcal{H})$  denote

$$\alpha_k(T, \mathcal{M}) = \sup\{|\text{tr}(Tf)| : f \in \mathcal{M}_\perp \cap F_k, \|f\|_1 \leq 1\}.$$

A subspace  $\mathcal{M}$  is called  *$k$ -hypercentral* if there is  $a > 0$  such that for any  $T \in \mathcal{B}(\mathcal{H})$  the following inequality holds:

$$d(T, \mathcal{M}) \leq a \alpha_k(T, \mathcal{M}). \quad (1.1)$$

Let  $\kappa_k(\mathcal{M})$  be the infimum of the collection of all constants  $a$  such that inequality (1.1) holds, then  $\kappa_k(\mathcal{M})$  is a constant of  $k$ -hypercentrality. Operator  $T$  is  *$k$ -hypercentral* if the WOT closed algebra generated by  $T$  and identity is  $k$ -hypercentral.

When  $k = 1$  the definition above coincides with the definition of hypercentral and the letter  $k$  will be omitted.

## 2. REFLEXIVITY OF PERTURBATED TOEPLITZ OPERATORS

Let  $\mathbb{T}$  be the unit circle on the complex plane  $\mathbb{C}$ . Denote  $L^2 = L^2(\mathbb{T}, m)$  and  $L^\infty = L^\infty(\mathbb{T}, m)$ , where  $m$  is the normalized Lebesgue measure on  $\mathbb{T}$ . Let  $H^2$  be the Hardy space corresponding to  $L^2$  and  $P_{H^2}$  be a projection from  $L^2$  onto  $H^2$ . For each  $\phi \in L^\infty$  we define  $T_\phi : H^2 \rightarrow H^2$  by  $T_\phi f = P_{H^2}(\phi f)$  for  $f \in H^2$ . Operator  $T_\phi$  is called a *Toeplitz operator* and  $\mathcal{T}$  will denote the space of all Toeplitz operators.

The unilateral shift  $S$  can be realized as the multiplication operator by independent variable  $T_z$ . Moreover,  $\mathcal{T} = \{T_\phi : \phi \in L^\infty\} = \{A : T_z^* A T_z = A\}$  ([9, Corollary 1 to Problem 194]). Hence  $\mathcal{T}$  is weak\* closed.

Let  $\{e_j\}_{j \in \mathbb{N}}$  be the usual basis in  $H^2$ . Denote by  $\mathcal{M}_{lm}$  the subspace  $\mathcal{T} + \mathbb{C}(e_l \otimes e_m)$ . In [4, Theorem 3.1] the authors proved that the space of all Toeplitz operators is not reflexive but it is 2-reflexive. We will show that the subspace  $\mathcal{M}_{lm}$  has the same properties.

**Proposition 2.1.** *The subspace  $\mathcal{M}_{lm}$  is not reflexive but it is 2-reflexive.*

*Proof.* Notice that  $(\mathcal{M}_{lm})_\perp = \mathcal{T}_\perp \cap (e_l \otimes e_m)_\perp$ . Since  $\mathcal{T}_\perp$  contains no nonzero rank-one operators, then  $\mathcal{M}_{lm}$  is not reflexive.

Notice that

$$\mathcal{T}_\perp = \text{span}\{e_i \otimes e_j - Se_i \otimes Se_j : i, j = 1, 2, \dots\},$$

where  $S$  is the unilateral shift. Therefore,

$$(\mathcal{M}_{lm})_\perp = \text{span}\{e_i \otimes e_j - Se_i \otimes Se_j : i, j = 1, 2, \dots, (i, j) \neq (l, m) \text{ and } (i + 1, j + 1) \neq (l, m)\}.$$

Hence  $\mathcal{M}_{lm}$  is 2-reflexive. □

Recall after [5] the following definition.

**Definition 2.2.** Subspace  $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$  has *property  $\mathbb{A}_{1/k}$*  if  $\mathcal{M}$  is weak\* closed and for any weak\* continuous functional  $\phi$  on  $\mathcal{M}$  there is  $g \in F_k$  such that  $\phi(M) = \text{tr}(Mg)$  for  $M \in \mathcal{M}$ .

**Proposition 2.3.** *The subspace  $\mathcal{M}_{lm} = \mathcal{T} + \mathbb{C}(e_l \otimes e_m)$  has property  $\mathbb{A}_{1/4}$ .*

*Proof.* Let  $t \in \tau c$ . Since  $\mathcal{T}$  has property  $\mathbb{A}_{1/2}$  ([10, Proposition 4.1]), there is  $f \in F_2$  such that  $(t - f) \in \mathcal{T}_\perp$ . If  $(t - f) \in (\mathbb{C}e_l \otimes e_m)_\perp$ , then  $(t - f) \in (\mathcal{M}_{lm})_\perp$ . If  $(t - f) \notin (\mathbb{C}e_l \otimes e_m)_\perp$ , then  $(t - f - \lambda e_l \otimes e_m + \lambda e_{l+1} \otimes e_{m+1}) \in (\mathcal{M}_{lm})_\perp$ , where  $\lambda = P_{\mathbb{C}e_l}(t - f)P_{\mathbb{C}e_m}$  and  $P_{\mathbb{C}e_i}$  denotes the orthogonal projection on  $\mathbb{C}e_i$ . So  $\mathcal{M}_{lm}$  has property  $\mathbb{A}_{1/4}$ . □

In [13] Larson proved that if  $\mathcal{M}$  is  $k$ -reflexive, then any weak\* closed subspace  $\mathcal{L} \subset \mathcal{M}$  is  $k$ -reflexive if and only if  $\mathcal{M}$  has property  $\mathbb{A}_{1/k}$ . It follows immediately from Proposition 2.1 and Proposition 2.3 that:

**Corollary 2.4.** *Every weak\*-closed subspace of  $\mathcal{M}_{lm} = \mathcal{T} + \mathbb{C}(e_l \otimes e_m)$  is 4-reflexive.*

On the other hand, due to [8] we know that the algebra of analytic Toeplitz operators is hyperreflexive. Moreover, the space of all Toeplitz operators  $\mathcal{T}$  is 2-hyperreflexive and  $\kappa_2(\mathcal{T}) \leq 2$  (see [10, 15]). We will show that the subspace  $\mathcal{M}_{lm}$  is 2-hyperreflexive. In the proof we will use the projection  $\pi : \mathcal{B}(H^2) \rightarrow \mathcal{T}$  constructed by Arveson in [1, Proposition 5.2], which has the property that for any  $A \in \mathcal{B}(H^2)$  the operator  $\pi(A)$  belongs to the weak\* closed convex hull of the set  $\{T_{z^n}^* A T_{z^n} : n \in \mathbb{N}\}$ .

**Proposition 2.5.** *Subspace  $\mathcal{M}_{lm} = \mathcal{T} + \mathbb{C}(e_l \otimes e_m)$  is 2-hyperreflexive with constant  $\kappa_2(\mathcal{M}_{lm}) \leq 2$ .*

*Proof.* Let  $A \in \mathcal{B}(H^2)$ . For  $\lambda \in \mathbb{C}$  define  $A_\lambda = A - \lambda e_l \otimes e_m$ . Notice that for any  $\lambda \in \mathbb{C}$

$$d(A, \mathcal{M}_{lm}) \leq \|A - \pi(A) - \lambda e_l \otimes e_m\| = \|A_\lambda - \pi(A_\lambda)\|.$$

Since the space of Toeplitz operators  $\mathcal{T}$  is 2-hyperreflexive with constant at most 2, we have that

$$d(A_\lambda, \mathcal{T}) \leq \|A_\lambda - \pi(A_\lambda)\| \leq 2\alpha_2(A_\lambda, \mathcal{T}) \text{ (for details see [10]).}$$

To complete the proof it is enough to show that for any  $A \in \mathcal{B}(H^2)$  there is  $\lambda \in \mathbb{C}$  such that

$$\alpha_2(A_\lambda, \mathcal{T}) = \alpha_2(A, \mathcal{M}_{lm}). \quad (2.1)$$

Note that

$$\alpha_2(A_\lambda, \mathcal{T}) = \sup\{|tr(A_\lambda t)| : 2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k}, k \geq 1, i, j = 0, 1, 2, \dots\}.$$

If this supremum is realized by  $2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k}$  for  $(i, j) \neq (l, m)$  and  $(i+k, j+k) \neq (l, m)$ , then equality (2.1) holds. So, it is enough to consider the case when

$$\begin{aligned} \alpha_2(A_\lambda, \mathcal{T}) &= \sup\{|tr(A_\lambda t)| : 2t = e_l \otimes e_m - e_{l+k} \otimes e_{m+k}, k \geq \min\{-l, -m\}\} = \\ &= \sup\{\frac{1}{2}|a_{lm} - \lambda - a_{l+k, m+k}| : k \geq \min\{-l, -m\}\}. \end{aligned}$$

Suppose that  $\alpha_2(A, \mathcal{M}_{lm}) = \beta > 0$ . Note that for any  $\lambda$  we have  $\beta \leq \alpha_2(A_\lambda, \mathcal{T})$ . If we choose  $\lambda = a_{lm} - a_{l+1, m+1}$ , then

$$\alpha_2(A_\lambda, \mathcal{T}) = \sup\{\frac{1}{2}|a_{l+1, m+1} - a_{l+k, m+k}| : k \geq \min\{-l, -m\}\} \leq \beta.$$

Hence  $\alpha_2(A_\lambda, \mathcal{T}) = \alpha_2(A, \mathcal{M}_{lm})$ , which completes the proof.  $\square$

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