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# TREES WHOSE 2-DOMINATION SUBDIVISION NUMBER IS 2

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**Abstract.** A set S of vertices in a graph $G = (V, E)$ is a 2-dominating set if every vertex of $V \setminus S$ is adjacent to at least two vertices of $S$. The 2-domination number of a graph $G$, denoted by $\gamma_2(G)$, is the minimum size of a 2-dominating set of G. The 2-domination subdivision number $\mathrm{sd}_{\gamma_2}(G)$ is the minimum number of edges that must be subdivided (each edge in $G$ can be subdivided at most once) in order to increase the 2-domination number. The authors have recently proved that for any tree $T$ of order at least 3, $1 \leq \mathrm{sd}_{\gamma_2}(T) \leq 2$. In this paper we provide a constructive characterization of the trees whose 2-domination subdivision number is 2.

**Keywords:** 2-dominating set, 2-domination number, 2-domination subdivision number.

**Mathematics Subject Classification:** 05C69.

## 1. INTRODUCTION

In this paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V$ and $E$). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of $v$ is the set $N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of $S$ is the set $N[S] = N(S) \cup S$. A *leaf* of a graph $G$ is a vertex of degree 1, while a *support vertex* of $G$ is a vertex adjacent to a leaf. A support vertex is *strong* if it is adjacent to at least two leaves. For a vertex $v$ in a rooted tree $T$, let $D(v)$ denote the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

A 2-*dominating set* of a graph $G = (V, E)$ is a subset $S$ of vertices where each vertex in $V \setminus S$ is adjacent to at least two vertices of $S$. The 2-*domination number* of a graph $G$, denoted by $\gamma_2(G)$, is the minimum size of a 2-dominating set of G. A $\gamma_2(G)$-*set* is a 2-dominating set of $G$ with size $\gamma_2(G)$. The 2-domination numbers have been studied by several authors (see for example [6, 7, 13, 15]).

The 2-*domination subdivision number* $\mathrm{sd}_{\gamma_2}(G)$ of a graph $G$ is the minimum number of edges that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to increase the 2-domination number of $G$. It is easy to see that [4] the 2-domination number of a graph cannot decrease when an edge of that graph is subdivided. For a more thorough treatment of domination parameters and for terminology not presented here see [12, 16].

Atapour *et al.* [4] showed the following result.

**Theorem 1.1.** *For any tree $T$ of order $n \geq 3$, $1 \leq \mathrm{sd}_{\gamma_2}(T) \leq 2$.*

Hence, trees can be classified as Class 1 or Class 2 depending on whether their 2-domination subdivision numbers are 1 or 2, respectively. In this paper we give a constructive characterization of trees in Class 2. For recent results on the topic "constructive characterization of graphs" the reader may consult [1–3, 9, 11, 14].

We make use of the following observations in this paper.

**Theorem 1.2** ([7])**.** *Every 2-dominating set of a graph $G$ contains every leaf.*

**Observation 1.3** ([7])**.** *Let $T$ be a tree obtained from a nontrivial tree $T'$ by adding a star $K_{1,p}$ with the center vertex $v$ attached by an edge $vw$ at a vertex $w$ of $T'$. Then $\gamma_2(T') + p \leq \gamma_2(T)$, with equality if $p \geq 2$ or $w$ is a leaf in $T'$.*

## 2. TREES WHOSE 2-DOMINATION SUBDIVISION NUMBER IS 2

In this section we provide a constructive characterization of all trees in Class 2. For this purpose, we describe a procedure to build a family $\mathcal{F}$ of labeled trees that are in Class 2 as follows. The label of a vertex is also called its *status* and denoted $sta(v)$. A labeled $P_4$ is a $P_4$ where the two leaves have status $A$ and the other two vertices have status $B$ and status $C$, respectively. Let $\mathcal{F}$ be the family of labeled trees that: A labeled $P_4$ is a tree in $\mathcal{F}$ and if $T$ is a tree in $\mathcal{F}$, then the tree $T'$ obtained from $T$ by the following five operations which extend the tree $T$ by attaching a tree to a vertex $y \in V(T)$, called an attacher, is also a tree in $\mathcal{F}$.

**Operation** $\mathfrak{T}_1$. If $sta(y) = B$ (respectively, $C$) and $y$ is a support vertex, then $\mathfrak{T}_1$ adds a vertex $x$ and an edge $xy$ to $T$ with $sta(x) = A$. Moreover, if $\deg(y) = 2$ and $y$ is adjacent to a vertex $z$ of status $C$ (respectively, $B$), then this operation changes the status of $z$ to $C'$ (respectively, $B'$).

**Operation** $\mathfrak{T}_2$. If $sta(y) = B$ (respectively, $C$) and $y$ is adjacent to a support vertex $z$ with $\deg(z) = 2$ and $sta(z) = C$ (respectively, $B$), then $\mathfrak{T}_2$ adds a vertex $x$ and an edge $xy$ to $T$ with $sta(x) = A$. Moreover, this operation changes the status of $z$ to $C'$ (respectively, $B'$).

**Operation** $\mathfrak{T}_3$. If $sta(y) = A, A', B'$ or $C'$, then $\mathfrak{T}_3$ adds a star $K_{1,2}$ with center $x$ and two leaves $x_1, x_2$ and an edge $xy$ to $T$ with $sta(x) = F$ and $sta(x_1) = sta(x_2) = A$. Moreover, this operation changes the status of $y$ from $A$ to $A'$.

**Operation** $\mathfrak{T}_4$. If $sta(y) = A$, then we have three cases:

*Case 1.* $y$ is adjacent to a vertex $z$ of status $B$ or $B'$. Then $\mathfrak{T}_4$ adds a path $yxu$ to $T$ with $sta(x) = B$, $sta(u) = A$ and changes the status of $y$ from $A$ to $C$.

*Case 2.* $y$ is adjacent to a vertex $z$ of status $C$ or $C'$. Then $\mathfrak{T}_4$ adds a path $yxu$ to $T$ with $sta(x) = C$, $sta(u) = A$ and changes the status of $y$ from $A$ to $B$.
*Case 3.* $y$ is adjacent to a vertex $z$ of status $F$. Then $\mathfrak{T}_4$ adds a path $yxu$ to $T$ with $sta(x) = C$, $sta(u) = A$ and changes the status of $y$ from $A$ to $B$.
**Operation** $\mathfrak{T}_5$. If $sta(y) = F$, then $\mathfrak{T}_5$ adds a vertex $x$ and an edge $xy$ to the tree $T$ with $sta(x) = A$.

The five operations are shown in Figure 1. Note that operation 3 adds two leaves and all the other operations add one leaf to tree T.

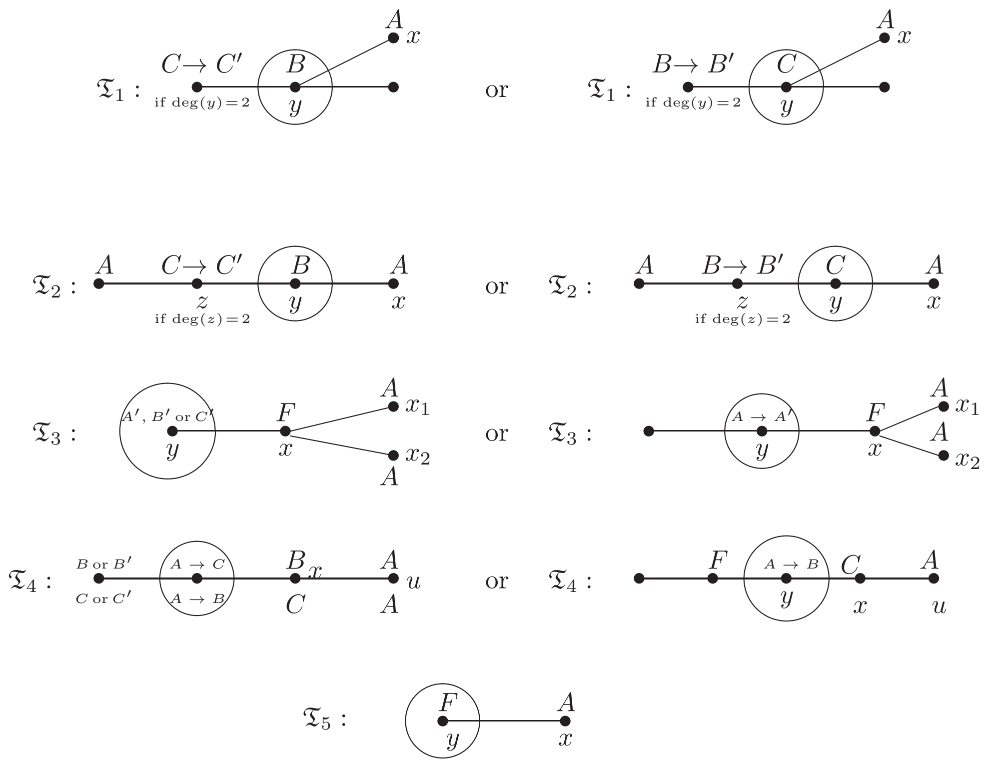



**Fig. 1.** The five operations

## The family $\mathcal{F}$

If $T \in \mathcal{F}$, we let $A(T)$, $B(T)$, $C(T)$, $F(T)$, $A'(T)$, $B'(T)$ and $C'(T)$ be the set of vertices of status $A, B, C, F, A', B'$, and $C'$, respectively, in $T$. The following observation comes from the way in which each tree in the family $\mathcal{F}$ is constructed.

**Observation 2.1.** *Let $T \in \mathcal{F}$ and $v \in V(T)$.*

1. *The set of vertices with status $A$ is the set of leaves of tree $T$.*

2. *If $v$ is a support vertex, then $sta(v) = B, C, F, B'$ or $C'$.*
3. *If $sta(v) = B$ or $B'$, then $v$ has at least one neighbor $y$ of status $C$ or $C'$ and $N(v) - \{y\} \subset A(T) \cup A'(T) \cup C(T) \cup C'(T) \cup F(T)$. Thus $A(T) \cup A'(T) \cup C(T) \cup C'(T) \cup F(T)$ is a 2-dominating set for $T$.*
4. *If $sta(v) = C$, $C'$ or $F$, then $v$ has at least two neighbors of status $A, A', B$* or *$B'$. Thus $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a 2-dominating set for $T$.*

We proceed with the following two propositions.

**Proposition 2.2.** 1. *Let $T'$ be a tree of order at least 3 and let $y$ be a leaf of $T'$. Let $T$ be a tree obtained from $T'$ by adding a path $yuv$ to $T'$. Then $\gamma_2(T) = \gamma_2(T') + 1$. Moreover, $\mathrm{sd}_{\gamma_2}(T) \le \mathrm{sd}_{\gamma_2}(T')$.*

2. *Let $T'$ be a tree of order at least 3 and let $y$ be a strong support vertex of $T'$. Let $T$ be a tree obtained from $T'$ by adding a pendant edge $yw$. Then $\gamma_2(T) = \gamma_2(T') + 1$. Moreover, $\mathrm{sd}_{\gamma_2}(T) \le \mathrm{sd}_{\gamma_2}(T')$.*
3. *Let $T'$ be a tree of order at least 3 and let $y$ be a leaf of $T'$. Let $T$ be a tree obtained from $T'$ by adding a path $yuv$ to $T'$ and $t(\ge 1)$ pendant edges at $y$. Then $\gamma_2(T) = \gamma_2(T') + t + 1$. Moreover, $\mathrm{sd}_{\gamma_2}(T) \le \mathrm{sd}_{\gamma_2}(T')$.*

*Proof.* (1) By Observation 1.3, $\gamma_2(T) = \gamma_2(T') + 1$. Let $F$ be a set of edges in $T'$ where subdividing the edges in $F$ increases the 2-domination number of $T'$. Let $T_1$ and $T_2$ be the trees obtained from $T'$ and $T$, respectively, by subdividing the edges in $F$. Then $y$ is a leaf in $T_1$ and $T_2$ is obtained from $T_1$ by adding a path $yuv$ to $T_1$. Thus

$$\gamma_2(T_2) = \gamma_2(T_1) + 1 > \gamma_2(T') + 1 = \gamma_2(T).$$

It follows that, $\mathrm{sd}_{\gamma_2}(T) \le \mathrm{sd}_{\gamma_2}(T')$.

(2) Let $u, v$ be the two leaves of $T'$ adjacent to $y$ in $T'$. Then $u, v, w$ are leaves in $T$. It is easy to see that for every $\gamma_2(T')$-set $S$, $S \cup \{w\}$ is a 2-dominating set of $T$. It follows that $\gamma_2(T) \le \gamma_2(T') + 1$. Now if $S'$ is a $\gamma_2(T)$-set, then $\{u, v, w\} \subseteq S'$. Hence, $S' - \{w\}$ is a 2-dominating set of $T'$. Thus $\gamma_2(T) = \gamma_2(T') + 1$.

Let $F$ be a set of edges in $T'$ where subdividing the edges in $F$ increases the 2-domination number of $T'$. Let $T_1$ and $T_2$ be the trees obtained from $T'$ and $T$, respectively, by subdividing the edges in $F$. Then $T_2$ is obtained from $T_1$ by adding the pendant edge $yw$. If $F \cap \{yu, yv\} = \emptyset$, then, as before, we have $\gamma_2(T_2) = \gamma_2(T_1) + 1$ and so

$$\gamma_2(T_2) = \gamma_2(T_1) + 1 > \gamma_2(T') + 1 = \gamma_2(T).$$

Now suppose that $|F \cap \{yu, yv\}| \ge 1$. We may assume the edge $yu$ is subdivided by inserting a vertex $x$. Obviously, for every $\gamma_2(T_1)$-set $S$, $S \cup \{w\}$ is a 2-dominating set of $T$ and so $\gamma_2(T_2) \le \gamma_2(T_1) + 1$. Now if $D$ is a $\gamma_2(T_2)$-set, then by Theorem 1.2, $w \in D$ and to dominate $x$ twice we must have $x \in D$ or $y \in D$. In each case $(D - \{x\}) \cup \{y\}$ is a 2-dominating set for $T_1$. It follows that $\gamma_2(T_2) = \gamma_2(T_1) + 1$. As before, we have

$$\gamma_2(T_2) = \gamma_2(T_1) + 1 > \gamma_2(T') + 1 = \gamma_2(T).$$

It follows that, $\mathrm{sd}_{\gamma_2}(T) \le \mathrm{sd}_{\gamma_2}(T')$.

(3) The proof is similar to (1) and (2) and therefore omitted. □

**Proposition 2.3.** *Let $T$ be a tree obtained from a tree $T'$ of order at least 3 by attaching a star $K_{1,t}$ $(t \geq 2)$ with center $x$ and joining $x$ to a vertex $y$ of $T'$. Then $\gamma_2(T) = \gamma_2(T') + t$. Moreover, $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$.*

*Proof.* By Observation 1.3, $\gamma_2(T) = \gamma_2(T') + t$. An argument similar to that described in Proposition 2.2 (Part 1) shows that $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$. □

### Reordering a set of operations with respect to a subset of $\{\mathfrak{T}_i\}_{i=1}^5$

Let $T$ be a tree obtained from a labeled $P_4$ by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ for $1 \leq i \leq m$. Let $J \subseteq \{1, 2, 3, 4, 5\}$ and $\mathfrak{T}_j \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ for $j \in J$. The following algorithm reorders operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$ with respect to $\mathfrak{T}_j$, $j \in J$. It is easy to see that if we apply operations $\mathfrak{T}^i$, $1 \leq i \leq m$ on a labeled $P_4$, according to the new ordering, we obtain $T$.

### Algorithm

1. Set $k = 0$.
2. Add one to $k$. If $k > m$, stop.
3. If $\mathfrak{T}^k \notin \{\mathfrak{T}_j \mid j \in J\}$, go to Step 2. If $\mathfrak{T}^k = \mathfrak{T}_j$ for some $j \in J$, proceed as follows. Find the smallest $\ell \in \{1, 2, \ldots, k-1\}$ such that applying $\mathfrak{T}_j$ before $\mathfrak{T}^\ell$ does not lead to a different tree from $T$. If such an $\ell$ does not exist, go to Step 2, otherwise apply $\mathfrak{T}_j$ before $\mathfrak{T}^\ell$.

Note that for given successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$ there exists a unique reordering with respect to a given subset of $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$.

**Example 2.4.** Let $T$ (Figure 2) be obtained by applying the sequence $\mathfrak{T}_3$, $\mathfrak{T}_5$, $\mathfrak{T}_1$, $\mathfrak{T}_4$, $\mathfrak{T}_1$, $\mathfrak{T}_4$, $\mathfrak{T}_3$, $\mathfrak{T}_5$, $\mathfrak{T}_4$ on the initial path $x_1x_2x_3x_4$. We see that $\mathfrak{T}_3$ adds the star with center $x_5$ and the leaves $x_6$ and $x_7$ to $x_4$ (Figure 3), $\mathfrak{T}_5$ adds $x_8$ to $x_5$ (Figure 4), $\mathfrak{T}_1$ adds $x_9$ to $x_2$, $\mathfrak{T}_4$ adds $x_{10}x_{11}$ to $x_8$, $\mathfrak{T}_1$ adds $x_{12}$ to $x_{10}$ (Figure 5), $\mathfrak{T}_4$ adds $x_{13}x_{14}$ to $x_{11}$, $\mathfrak{T}_3$ adds the star with center $x_{15}$ and the leaves $x_{16}$ and $x_{17}$ to $x_3$ (Figure 6), $\mathfrak{T}_5$ adds $x_{18}$ to $x_{15}$ and $\mathfrak{T}_4$ adds the path $x_{19}x_{20}$ to $x_{17}$ (Figure 7). Then $T$ is in Figure 2.

In what follows, we step by step show that how one can find the reordering of the operations $\mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_4$ with respect to $\{\mathfrak{T}_1, \mathfrak{T}_3, \mathfrak{T}_5\}$. The new ordering will be $\mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_1, \mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_4, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_4$.

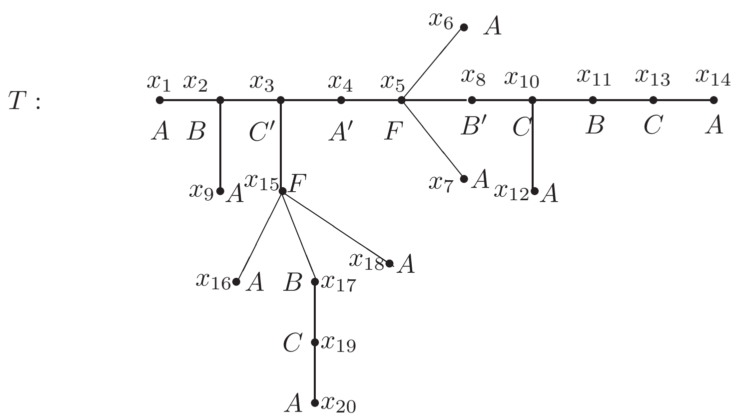



**Fig. 2**

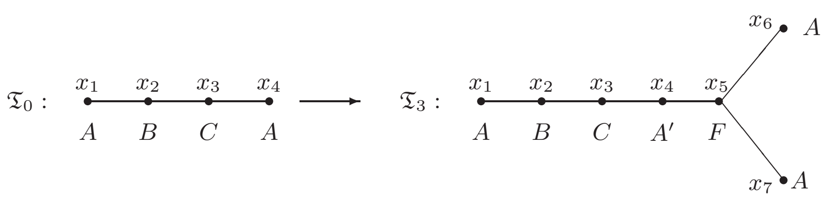



**Fig. 3**

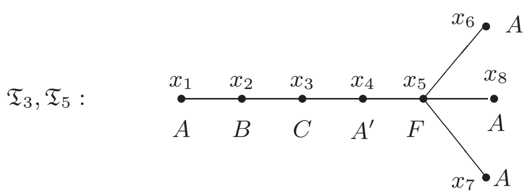
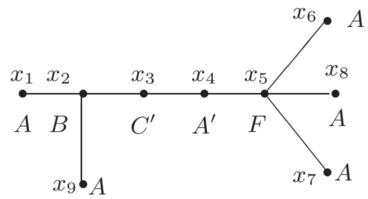



**Fig. 4**

$\mathfrak{T}_1, \mathfrak{T}_3, \mathfrak{T}_5 :$




$\mathfrak{T}_1, \mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_4, \mathfrak{T}_1 :$

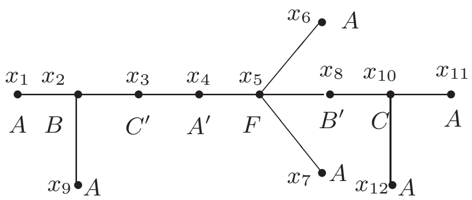



**Fig. 5**

$\mathfrak{T}_3, \mathfrak{T}_1, \mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_4, \mathfrak{T}_1, \mathfrak{T}_4 :$

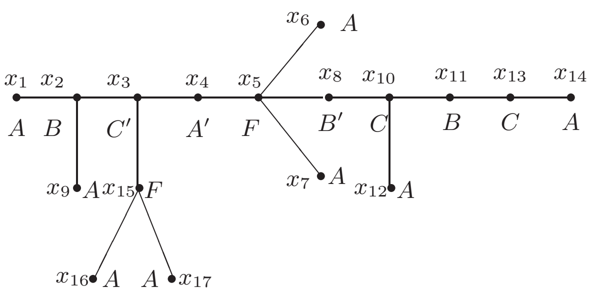



**Fig. 6**

$\mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_1, \mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_4, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_4 :$

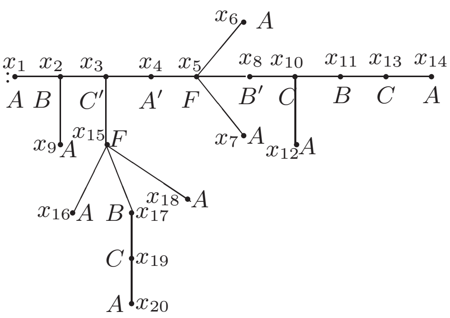



**Fig. 7**

In order to show that each tree in the family $\mathcal{F}$ is in Class 2, we first present three lemmas.

**Lemma 2.5.** *Let $T \in \mathcal{F}$ be obtained from a labeled $P_4$ by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ if $m \geq 1$ and $T = P_4$ if $m = 0$. Then $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a 2-dominating set of $T$ and $\gamma_2(T) = m + k + 3$, where $k$ is the number of operations of type $\mathfrak{T}_3$.*

*Proof.* By Part (4) of Observation 2.1, the set $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a 2-dominating set of $T$. This implies that $\gamma_2(T) \leq m + k + 3$. The proof of $\gamma_2(T) = m+k+3$ is by induction on $m$. If $m = 0$, then clearly the statement is true. Let $m \geq 1$ and that the statement holds for all trees which are obtained from $P_4$ by applying $m - 1$ operations $\mathfrak{T} \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$. Reorder the operations $\{\mathfrak{T}^1, \mathfrak{T}^2, \ldots, \mathfrak{T}^m\}$ with respect to $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_5\}$. Let $T_{m-1}$ be the tree obtained from $P_4$ by the first $m-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$. If $\mathfrak{T}^m = \mathfrak{T}_3$, then $T$ has been obtained from $T_{m-1}$ by adding a star $K_{1,2}$ with center $x$ and two leaves $x_1, x_2$ and an edge $xy$ to $T$. By the inductive hypothesis, $\gamma_2(T_{m-1}) = (m - 1) + (k - 1) + 3 = m + k + 1$ and the result follows by Proposition 2.3. If $\mathfrak{T}^m = \mathfrak{T}_5$, then $T$ has been obtained from $T_{m-1}$ by adding a vertex $x$ and an edge $xy$ to the tree $T_{m-1}$ where $sta_{T_{m-1}}(y) = F$. Then, by the choice of reordering, $y$ is a strong support vertex in $T_{m-1}$. By the inductive hypothesis, $\gamma_2(T_{m-1}) = (m - 1) + k + 3 = m + k + 2$ and the result follows by Proposition 2.2 (Part 2). If $\mathfrak{T}^m = \mathfrak{T}_4$, then the result follows by the inductive hypothesis and Proposition 2.2 (Part 1). Now consider the two remaining cases.

*Case 1.* $\mathfrak{T}^m = \mathfrak{T}_1$. Then $T$ has been obtained from $T_{m-1}$ by adding a vertex $x$ and an edge $xy$, where $y$ is a support vertex of $T_{m-1}$. Suppose that $w$ is a leaf adjacent to $y$ and $z$ is a vertex of status $B, C, C'$ or $B'$ adjacent to $y$ by Observation 2.1, Parts (2) and (3). First assume $y$ is in the original $P_4$. Then, by the choice of reordering, $\mathfrak{T}^1 = \mathfrak{T}^2 = \ldots = \mathfrak{T}^m = \mathfrak{T}_1$ and each operation adds a pendant edge at $y$. Therefore $\deg(z) = 2$. For any $\gamma_2(T_{m-1})$-set $S'$, $S' \cup \{x\}$ is a 2-dominating set of $T$ and so $\gamma_2(T) \leq \gamma_2(T_{m-1}) + 1$. On the other hand, if $S$ is a $\gamma_2(T)$-set, then clearly $x, w \in S$ and $|S \cap \{y, z\}| \geq 1$ since $\deg(z) = 2$. Then $S - \{x\}$ is a 2-dominating set of $T_{m-1}$. This implies that $\gamma_2(T_{m-1}) \leq \gamma_2(T) - 1$ and so $\gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Now the result follows by the inductive hypothesis.

Now assume $y$ is not in the original $P_4$. By the choice of reordering, we may assume for some positive integer $s$, $\mathfrak{T}^m = \ldots = \mathfrak{T}^{s+1} = \mathfrak{T}_1$ and each operation adds a pendant edge at $y$ and $\mathfrak{T}^s = \mathfrak{T}_4$ which adds the path $zyw$. Therefore, $z$ is a leaf in $T_{s-1}$ and so $sta_{T_{s-1}}(z) = A$ and $\deg_T(z) = 2$. By Proposition 2.3, $\gamma_2(T_{s-1}) + (m - s) + 1 = \gamma_2(T)$. Now the result follows by the inductive hypothesis.

*Case 2.* $\mathfrak{T}^m = \mathfrak{T}_2$. Then $T$ has been obtained from $T_{m-1}$ by adding a vertex $x$ and an edge $xy$, where $y$ is adjacent to a support vertex $z$ of $T_{m-1}$ with $\deg(z) = 2$. For any $\gamma_2(T_{m-1})$-set $S'$, $S' \cup \{x\}$ is a 2-dominating set of $T$ and so $\gamma_2(T) \leq \gamma_2(T_{m-1}) + 1$. On the other hand, if $S$ is a $\gamma_2(T)$-set, then clearly $x, w \in S$ and $|S \cap \{y, z\}| \geq 1$ since $\deg(z) = 2$. Then $S - \{x\}$ is a 2-dominating set of $T_{m-1}$. This implies that $\gamma_2(T_{m-1}) \leq \gamma_2(T) - 1$ and so $\gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Now the result follows by the inductive hypothesis. $\square$

**Lemma 2.6.** *Let $T \in \mathcal{F}$ be obtained from a labeled $P_4$ by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ if $m \geq 1$ and $T = P_4$ if $m = 0$. Then:*

1. *for every $v \in V(T)$, there exists a $\gamma_2(T)$-set containing $v$,*
2. *if $v \in A(T)$, then there is a $\gamma_2(T)$-set $S$ containing $v$ and its support vertex. Therefore, $S - \{v\}$ is a 2-dominating set of $T - \{v\}$.*

*Proof.* The proof is by induction on $m$. If $m = 0$, then clearly the statements are true. Let $m \geq 1$ and the statements hold for all trees which are obtained from a labeled $P_4$ by applying at most $m - 1$ operations $\mathfrak{T} \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$. Let $T_{m-1}$ be the tree obtained from $P_4$ by the first $m - 1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$. Reorder the operations $\{\mathfrak{T}^1, \mathfrak{T}^2, \ldots, \mathfrak{T}^m\}$ with respect to $\{\mathfrak{T}_3\}$.

(1) Since by Lemma 2.5, $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a $\gamma_2(T)$-set, we assume that $v \in C(T) \cup C'(T) \cup F(T)$. We consider three cases.

*Case 1.* $\mathfrak{T}^m = \mathfrak{T}_1, \mathfrak{T}_2$ or $\mathfrak{T}_5$. Then $T$ is obtained from $T_{m-1}$ by adding a vertex $x$ and an edge $xy$, where $y \in B(T_{m-1}) \cup C(T_{m-1}) \cup F(T_{m-1})$. Since $C(T) \cup C'(T) \cup F(T) = C(T_{m-1}) \cup C'(T_{m-1}) \cup F(T_{m-1})$, by the inductive hypothesis $v$ is contained in some $\gamma_2(T_{m-1})$-set $S$. Now $S \cup \{x\}$ is a $\gamma_2(T)$-set containing $v$ by Lemma 2.5.

*Case 2.* $\mathfrak{T}^m = \mathfrak{T}_3$. Then $T$ is obtained from $T_{m-1}$ by adding a star $K_{1,2}$ with center $x$ and two leaves $x_1, x_2$ and an edge $xy$, where $y \in A(T_{m-1}) \cup A'(T_{m-1}) \cup B'(T_{m-1}) \cup C'(T_{m-1})$. We have $C(T) \cup C'(T) \cup F(T) = (C(T_{m-1}) \cup C'(T_{m-1}) \cup F(T_{m-1})) \cup \{x\}$. If $v \in V(T_{m-1})$, then by the inductive hypothesis there is a $\gamma_2(T_{m-1})$-set $S$ containing $v$ and $S \cup \{x_1, x_2\}$ is a $\gamma_2(T)$-set by Lemma 2.5. Let $v = x$. By the choice of reordering, for some integer $0 \leq s \leq m - 1$, each of $\mathfrak{T}^m, \mathfrak{T}^{m-1}, \cdots, \mathfrak{T}^{s+1}$ adds a star $K_{1,2}$ and joins its center to $y$ but $\mathfrak{T}^s$ does not add a star $K_{1,2}$ to $y$. If $s < m - 1$, then we may assume $\mathfrak{T}^{m-1}$ adds a star $K_{1,2}$ with center $x'$ and leaves $x'_1, x'_2$. Obviously, we can rearrange the order of the operations to have $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-2}$, $\mathfrak{T}^m, \mathfrak{T}^{m-1}$. By the inductive hypothesis, the tree $T'$ obtained from $P_4$ by the operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-2}$, $\mathfrak{T}^m$ has a $\gamma_2(T')$-set $S$ containing $v$. Then $S \cup \{x'_1, x'_2\}$ is a $\gamma_2(T)$-set containing $v$ by Lemma 2.5. Now we assume $s = m - 1$. Let first $\mathrm{sta}(y) = B'$ or $C'$. Then, by the choice of reordering, we may assume $\mathfrak{T}^s \in \{\mathfrak{T}_1, \mathfrak{T}_2\}$. We consider two subcases.

*Subcase 2.1.* $\mathfrak{T}^{m-1} = \mathfrak{T}_1$. This forces that $y$ is adjacent to a strong support vertex $z$ with status $B$ or $C$ and $\deg(z) = 3$. By Lemma 2.5 and the inductive hypothesis, $z$ is contained in a $\gamma_2(T_{m-1})$-set $S$. Now obviously $(S \setminus \{z\}) \cup \{x, x_1, x_2\}$ is a $\gamma_2(T)$-set containing $v$.

*Subcase 2.2.* $\mathfrak{T}^{m-1} = \mathfrak{T}_2$. Then $T_{m-1}$ is obtained from $T_{m-2}$ by adding a vertex $u$ and an edge $uz$, where $z$ is a vertex of status $B$ or $C$ adjacent to the support vertex $y$ of status $C$ or $B$ and degree 2 in $T_{m-2}$. Thus we have $\deg_{T_{m-1}}(z) \geq 3$, $\mathrm{sta}_{T_{m-1}}(z) = B$ or $C$ and $\mathrm{sta}_{T_{m-1}}(y) = C'$ or $B'$. Let $z'$ be a vertex adjacent to $z$ other than $y$ and $u$. By the inductive hypothesis, $z'$ is contained in a $\gamma_2(T_{m-1})$-set say $S$. Then we have $z \in S$ or $y \in S$. By Lemma 2.5, $(S \setminus \{z, y\}) \cup \{x, x_1, x_2\}$ is a $\gamma_2(T)$-set containing $v$.

Now let $sta(y) = A$. Then $y$ is a leaf in $T_{m-1}$, and by the inductive hypotheses there is a $\gamma_2(T_{m-1})$-set $S$ containing $y$ and its support vertex and so $(S \setminus \{y\}) \cup \{x, x_1, x_2\}$ is a $\gamma_2(T)$-set containing $v$.

Finally, let $sta(y) = A'$. Then $\mathfrak{T}^{m-1}$ adds a star $K_{1,2}$ with center $x'$ and leaves $x'_1, x'_2$ and changes the status of $y$ from $A$ to $A'$. Thus $y$ is a leaf in $T_{m-2}$, and by the inductive hypothesis there is a $\gamma_2(T_{m-2})$-set $S$ containing $y$ and its support vertex $w$. Now obviously $(S \setminus \{y\}) \cup \{x'_1, x'_2, x, x_1, x_2\}$ is a $\gamma_2(T)$-set containing $v$.

*Case 3.* $\mathfrak{T}^m = \mathfrak{T}_4$. Then $T$ is obtained from $T_{m-1}$ by adding a path $yxu$ to $T_{m-1}$, where $y \in A(T_{m-1})$. Thus $y$ is a leaf in $T_{m-1}$. Suppose that $z$ is the support vertex of $y$ in $T_{m-1}$. If $v \in T_{m-1}$, then by the inductive hypothesis $v$ is contained in some $\gamma_2(T_{m-2})$-set $S$ and $S \cup \{u\}$ is a $\gamma_2(T)$-set by Lemma 2.5. Now let $v = x$. By the inductive hypothesis, there is a $\gamma_2(T_{m-1})$-set $S$ containing $y$ and its support vertex and obviously $(S - \{y\}) \cup \{x, u\}$ is a $\gamma_2(T)$-set containing $v$.

(2) Let $u$ be the support vertex of $v$. Then by Part (2) of Observation 2.1, $sta(u) = B, C, B', C'$, or $F$. Now the result follows by Lemma 2.5, Part (1) of this theorem and the fact that each $\gamma_2(T)$-set contains all leaves □

**Lemma 2.7.** *Let $T \in \mathcal{F}$ and let $T^*$ be a tree obtained from $T$ by subdividing an edge of $T$. Then $\gamma_2(T^*) = \gamma_2(T)$.*

*Proof.* Let $T \in \mathcal{F}$. First note that $\gamma_2(T^*) \geq \gamma_2(T)$ and that any 2-dominating set of $T^*$ of size $\gamma_2(T)$ is a $\gamma_2(T^*)$-set. Let $e \in E(T)$ and let $T^*$ be obtained from $T$ by subdividing the edge $e$ with vertex $u$. Let $T$ be obtained from a labeled $P_4$ by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, respectively, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ for $1 \leq i \leq m$ if $m \geq 1$ and $T = P_4$ if $m = 0$. The proof is by induction on $m$. If $m = 0$, then clearly the statement is true. Assume $m \geq 1$ and that the statement holds for all trees which are obtained from a labeled $P_4$ by applying at most $m-1$ operations. Suppose $T_{m-1}$ is a tree obtained by applying the first $m-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$. When $e \in E(T_{m-1})$, let $T^*_{m-1}$ be obtained from $T_{m-1}$ by subdividing the edge $e$ with vertex $u$. We consider three cases.

*Case 1.* $\mathfrak{T}^m = \mathfrak{T}_1, \mathfrak{T}_2$ or $\mathfrak{T}_5$. Then $T$ is obtained from $T_{m-1}$ by attaching the path $yx$ to $y \in B(T_{m-1}) \cup C(T_{m-1}) \cup F(T_{m-1})$. If $e \in E(T_{m-1})$, then by the inductive hypothesis we have

$$\gamma_2(T^*) \leq \gamma_2(T^*_{m-1}) + 1 = \gamma_2(T_{m-1}) + 1 = \gamma_2(T).$$

Let $e = xy$. By Lemmas 2.5 and 2.6, there exists a $\gamma_2(T_{m-1})$-set $S$ containing $y$. Now $S \cup \{x\}$ is a 2-dominating set of $T^*$ of size $\gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Hence, $\gamma_2(T^*) = \gamma_2(T)$.

*Case 2.* $\mathfrak{T}^m = \mathfrak{T}_3$. Then $T$ is obtained from $T_{m-1}$ by attaching a star $K_{1,2}$ with center $x$ and two leaves $x_1, x_2$ to the attacher $y \in A(T_{m-1}) \cup A'(T_{m-1}) \cup C'(T_{m-1}) \cup B'(T_{m-1})$. If $e \in E(T_{m-1})$, then by Proposition 2.3 and the inductive hypothesis we have

$$\gamma_2(T^*) = \gamma_2(T^*_{m-1}) + 2 = \gamma_2(T_{m-1}) + 2 = \gamma_2(T).$$

Let $e \in E(T) \setminus E(T_{m-1})$. By Lemma 2.6, there is a $\gamma_2(T)$-set $S$ containing $x$. Now $S$ is a 2-dominating set of $T^*$ of size $\gamma_2(T)$ if $e = xx_1$ or $xx_2$ and $(S - \{x\}) \cup \{u\}$ is a 2-dominating set for $T^*$ of size $\gamma_2(T)$ if $e = xy$. Recall that $u$ is the subdividing vertex.

*Case 3.* $\mathfrak{T}^m = \mathfrak{T}_4$. Then $T$ is obtained from $T_{m-1}$ by attaching the path $yxw$ to the attacher $y \in A(T_{m-1})$. If $e \in E(T_{m-1})$, then by Proposition 2.2 and the inductive hypothesis $\gamma_2(T^*) = \gamma_2(T^*_{m-1}) + 1 = \gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Let $e \notin E(T_{m-1})$. Without loss of generality, we may subdivide $e = yx$ with $u$. By Lemma 2.6, $T_{m-1}$ has a $\gamma_2(T_{m-1})$-set $S$ containing $y$ and its support vertex. Now $(S - \{y\}) \cup \{u, w\}$ is a $\gamma_2(T^*)$-set of size $\gamma_2(T)$. This completes the proof. □

An immediate consequence of Theorem 1.1 and Lemma 2.7 now follows.

**Theorem 2.8.** *Each tree in Family $\mathcal{F}$ is in Class 2.*

In order to prove that any tree in Class 2 is indeed in $\mathcal{F}$ we need the following lemma.

**Lemma 2.9.** *Let $T \in \mathcal{F}$, $v \in B(T) \cup C(T) \cup F(T)$ and let $T^*$ be obtained from $T$ by adding a star $K_{1,2}$ and an edge joining the center of the star to $v$. Then $\mathrm{sd}_{\gamma_2}(T^*) = 1$.*

*Proof.* Let $T \in \mathcal{F}$ be obtained from a labeled $P_4$ by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ if $m \geq 1$ and $T = P_4$ if $m = 0$. The proof is by induction on $m$. If $m = 0$, then clearly the statement is true. Assume $m \geq 1$ and that the statement holds for all trees which are obtained from a labeled $P_4$ by applying at most $m - 1$ operations. Suppose $T_{m-1}$ is the tree obtained by applying the first $m - 1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$. When $v \in V(T_{m-1})$, let $T^*_{m-1}$ be obtained from $T_{m-1}$ by adding a star $K_{1,2}$ and an edge joining the center of the star to $v$. Reorder the operations $\{\mathfrak{T}^1, \mathfrak{T}^2, \ldots, \mathfrak{T}^m\}$ with respect to $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_5\}$. Let $z, z_1$ and $z_2$ be the center and leaves of the added star to $T$, respectively. We consider five cases.

*Case 1.* $\mathfrak{T}^m = \mathfrak{T}_3$. Then $T$ is obtained from $T_{m-1}$ by adding a star $K_{1,2}$ and an edge joining the center $x$ of the star to $y \in A(T_{m-1}) \cup A'(T_{m-1}) \cup B'(T_{m-1}) \cup C'(T_{m-1})$. If $v \in V(T_{m-1})$, then by the inductive hypothesis $\mathrm{sd}_{\gamma_2}(T^*_{m-1}) = 1$. Since $T^*$ is formed from $T^*_{m-1}$ by adding a star $K_{1,2}$, by Proposition 2.3 we have $\mathrm{sd}_{\gamma_2}(T^*) \leq \mathrm{sd}_{\gamma_2}(T^*_{m-1}) = 1$. Thus by Theorem 1.1, $\mathrm{sd}_{\gamma_2}(T^*) = 1$. If $v = x$, then let $T'$ be obtained from $T^*$ by subdividing the edge $xz$ by inserting a vertex $t$. Since $y$ and $z$ are strong support vertices, for each $\gamma_2(T^*)$-set $S$ we have $z \notin S$, for otherwise $S - \{z\}$ is a 2-dominating set for $T^*$, a contradiction. Let $D$ be a $\gamma_2(T')$-set. Then $u \in D$ or $y, z \in D$ and hence $D - \{u\}$ or $D - \{z\}$ is a 2-dominating set for $T^*$. Therefore $\mathrm{sd}_{\gamma_2}(T^*) \leq 1$ and the result follows by Theorem 1.1.

*Case 2.* $\mathfrak{T}^m = \mathfrak{T}_4$. Then $T$ is obtained from $T_{m-1}$ by adding a path $xw$ and an edge joining $x$ to $y \in A(T_{m-1})$. First let $v \in V(T_{m-1}) - \{y\}$. Then by the inductive hypothesis $\mathrm{sd}_{\gamma_2}(T^*_{m-1}) = 1$. Assume $e$ is an edge of $T^*_{m-1}$ such that subdividing $e$ increases the 2-domination number. Let $T'_{m-1}$ and $T'$ be obtained from $T^*_{m-1}$ and

$T^*$ by subdividing the edge $e$, respectively. By Proposition 2.2 (Part (1)), $\gamma_2(T^*) = \gamma_2(T^*_{m-1}) + 1$ and $\gamma_2(T') = \gamma_2(T'_{m-1}) + 1$. Now

$$\gamma_2(T') = \gamma_2(T'_{m-1}) + 1 \geq \gamma_2(T^*_{m-1}) + 2 = \gamma_2(T^*) + 1.$$

Therefore, $\mathrm{sd}_{\gamma_2}(T^*) = 1$ by Theorem 1.1.

Let $v = y$. Obviously, $\deg(x) = 2$. Let $T'$ be obtained from $T^*$ by subdividing the edge $yz$ by inserting a vertex $t$. Suppose that $S$ is a $\gamma_2(T')$-set. Since $\deg(x) = 2$, $y \in S$ or $x \in S$. We may assume $y \in S$, otherwise $(S - \{x\}) \cup \{y\}$ is a $\gamma_2(T')$-set. Since $t$ is a subdividing vertex, $\deg(t) = 2$. To dominate $t$ we must have $S \cap \{t, z\} \neq \emptyset$. Now obviously $S - \{t, z\}$ is a 2-dominating set for $T^*$ and so $\mathrm{sd}_{\gamma_2}(T^*) = 1$ by Theorem 1.1.

Now let $v = x$. Then $\deg_{T^*}(y) = 2$. Suppose that $w \neq x$ is adjacent to $y$. Let $T'$ be obtained from $T^*$ by subdividing the edge $xz$ by inserting a vertex $t$. Suppose that $S$ is a $\gamma_2(T')$-set. Since $\deg_{T'}(y) = 2$, $y \in S$ or $\{x, w\} \subseteq S$. To dominate $t$ we must have $S \cap \{t, z\} \neq \emptyset$. Now obviously $S - \{t, z\}$ is a 2-dominating set for $T^*$ and so $\mathrm{sd}_{\gamma_2}(T^*) = 1$ by Theorem 1.1.

*Case 3.* $\mathfrak{T}^m = \mathfrak{T}_5$. Then $T$ is obtained from $T_{m-1}$ by adding a vertex $x$ and an edge joining $x$ to $y \in F(T_{m-1})$. By the choice of reordering, we may assume $\mathfrak{T}^m = \ldots = \mathfrak{T}^{k+1} = \mathfrak{T}_5$ and $\mathfrak{T}^k = \mathfrak{T}_3$ which adds a star $K_{1,2}$ with center $y$. Suppose $T_{k-1}$ is the tree obtained by applying the first $k-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{k-1}$. If $v \in V(T_{k-1})$, then Proposition 3 and an argument similar to that described in Case 2 show that the statement is true. If $v = y$, then let $T'$ be obtained from $T^*$ by subdividing the edge $vz$ by inserting a vertex $t$. Since $y$ and $z$ are strong support vertices, for each $\gamma_2(T^*)$-set $S$ we have $z \notin S$, for otherwise $S - \{z\}$ is a 2-dominating set for $T^*$, a contradiction. Let $D$ be a $\gamma_2(T')$-set. Then $u \in D$ or $y, z \in D$ and hence $D - \{u\}$ or $D - \{z\}$ is a 2-dominating set for $T^*$. Therefore $\mathrm{sd}_{\gamma_2}(T^*) \leq 1$ and the result follows by Theorem 1.1.

*Case 4.* $\mathfrak{T}^m = \mathfrak{T}_1$. Then $T$ is obtained from $T_{m-1}$ by adding a vertex $x$ and an edge joining $x$ to a support vertex $y \in B(T_{m-1}) \cup C(T_{m-1})$. If $y$ belongs to the original $P_4$, then obviously $\mathfrak{T}^1 = \ldots = \mathfrak{T}^m = \mathfrak{T}_1$ and each operation adds a pendant edge at $y$. This forces $v = y$ and as Case 3, it is easy to see that subdividing the edge $yz$ increases the 2-domination number. Suppose $y$ is not contained in the original $P_4$. By the choice of reordering, we may assume $\mathfrak{T}^m = \ldots = \mathfrak{T}^{s+1} = \mathfrak{T}_1$ where each operation adds a pendant edge at $y$ and $\mathfrak{T}^s = \mathfrak{T}_4$ which adds a path $yw$ and an edge joining $y$ to some vertex in $T_{s-1}$. Suppose $T_{s-1}$ is the tree obtained by applying the first $s-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{s-1}$. If $v \in V(T_{s-1})$, then Proposition 2.3 and an argument similar to that described in Case 2 show that the statement is true. If $v = y$ then, as before, we can see that subdividing the edge $yz$ increases the 2-domination number.

*Case 5.* $\mathfrak{T}^m = \mathfrak{T}_2$. Then $T$ is obtained from $T_{m-1}$ by adding a vertex $x$ and an edge that joins $x$ to a vertex $y \in B(T_{m-1}) \cup C(T_{m-1})$, where $y$ is adjacent to a support vertex $z$ of degree 2 in $T_{m-1}$. If $y$ belongs to the original $P_4$, then the result follows as Case 4. Suppose $y$ is not contained in the original $P_4$. By the choice of reordering, we may assume $\mathfrak{T}^m = \ldots = \mathfrak{T}^{s+1} = \mathfrak{T}_2$ where each operation adds a pendant edge at $y$ and $\mathfrak{T}^s = \mathfrak{T}_4$ which adds the path $zw$ and an edge joining $z$ to $y$ in $T_{s-1}$.

Suppose $T_{s-1}$ is the tree obtained by applying the first $s-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{s-1}$. If $v \in V(T_{s-1})$, then the result follows by Proposition 2.2 (Part (3)) and the inductive hypothesis. Let $v = y$. We show that subdividing the edge $zz_1$, where $z_1$ is a leaf at $z$, increases the 2-domination number. Let $T'$ be obtained from $T^*$ by subdividing the edge $zz_1$ by inserting a vertex $u$. Let $S$ be a $\gamma_2(T)$-set. Since $\deg_T(z) = 2$, we may assume $y \in S$. Now to dominate $u$ we must have $u \in S$ or $z \in S$. Then clearly $S - \{u, z\}$ is a 2-dominating set for $T^*$. It follows that $\mathrm{sd}_{\gamma_2}(T) = 1$. This completes the proof. □

**Theorem 2.10.** *A tree $T$ of order $n \geq 3$ is in Class 2 if and only if $T \in \mathcal{F}$.*

*Proof.* By Theorem 2.8, we only need to prove that every tree in Class 2 is in $\mathcal{F}$. We prove this by induction on $n$. Since $\mathrm{sd}_{\gamma_2}(T) = 2$, we have $n \geq 4$. If $n = 4$, then the only tree $T$ of order 4 and $\mathrm{sd}_{\gamma_2}(T) = 2$ is $P_4 \in \mathcal{F}$. Let $n \geq 5$ and assume the statement holds for every tree in Class 2 of order less than $n$. Let $T$ be a tree of order $n$ and $\mathrm{sd}_{\gamma_2}(T) = 2$. Assume $P = v_1 v_2 \ldots v_r$ is the longest path in $T$. Obviously, $\deg(v_1) = \deg(v_r) = 1$ and $r \geq 4$. Suppose $T$ is rooted at $v_r$.

First let $\deg(v_2) \geq 3$. Then $v_2$ is a strong support vertex. Let $v_1 = u_1, u_2, \ldots, u_{\deg(v_2)-1}$ be the leaves adjacent to $v_2$ and $T_1 = T - T_{v_2}$. By Proposition 2.3, $\mathrm{sd}_{\gamma_2}(T_1) = 2$ and by the inductive hypothesis, $T_1 \in \mathcal{F}$. Since $\mathrm{sd}_{\gamma_2}(T) = 2$, by Lemma 2.9, $sta_{T_1}(v_3) = A, A', B'$, or $C'$, and hence $T$ can be obtained from $T_1$ by applying operation $\mathfrak{T}_3$ once and operation $\mathfrak{T}_5$, $\deg(v_2) - 3$ times.

Now let $\deg(v_2) = 2$. First let $\deg(v_3) = 2$. Then by Proposition 2.2 (Part (1)), $\gamma_2(T) = \gamma_2(T - T_{v_2}) + 1$ and $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T - T_{v_2})$. Therefore $\mathrm{sd}_{\gamma_2}(T - T_{v_2}) = 2$ and by the inductive hypothesis, $T - T_{v_2} \in \mathcal{F}$. Now $T$ can be obtained from $T - T_{v_2}$ by operation $\mathfrak{T}_4$. Now let $\deg(v_3) \geq 3$. First assume that $v_3$ is adjacent to a support vertex $u$ such that $u \neq v_2$. Let $w$ be a leaf adjacent to $u$. As before, we may assume that $\deg(u) = 2$. Let $T'$ be obtained from $T$ by subdividing the edge $v_3 u$ by inserting a vertex $s$. For any $\gamma_2(T)$-set $S$ of $T$, $|S \cap \{v_1, v_2, v_3\}| \geq 2$ and $|S \cap \{s, u, w\}| \geq 2$. Obviously, $(S - \{v_1, v_2, v_3, s, u, w\}) \cup \{v_1, v_3, w\}$ is a 2-dominating set for $T$ with cardinality less than $|S|$. Therefore, $\mathrm{sd}_{\gamma_2}(T) = 1$, a contradiction. Thus $v_3$ is adjacent to $\deg(v_3) - 2$ leaves. Let $u_1, \ldots, u_{\deg(v_3)-2}$ be the leaves adjacent to $v_3$. Assume $T' = T - \{u_1, \ldots, u_{\deg(v_3)-2}, v_1, v_2\}$. By Proposition 2.2 (Part 3) $\gamma_2(T) = \gamma_2(T') + \deg(v_3) - 1$ and $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$. Since $\mathrm{sd}_{\gamma_2}(T) = 2$, by Theorem 1.1 , $\mathrm{sd}_{\gamma_2}(T') = 2$. Hence, by the inductive hypothesis, $T' \in \mathcal{F}$. Since $v_3$ is a leaf in $T'$, $sta_{T'}(v_3) = A$ and $T$ can be obtained from $T'$ by applying operation $\mathfrak{T}_4$ once and operations $\mathfrak{T}_1$ or $\mathfrak{T}_2$, $\deg(v_3) - 2$ times. Thus $T \in \mathcal{F}$ and the proof is complete. □

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