MECHANICS AND CONTROL
Vol. 29 No. 4 2010

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THE INFLUENCE OF NUMERICAL ERRORS ON DETERMINING THE DISTRIBUTION OF VALUES OF STOCHASTIC IMPULSES FORCING AN OSCILLATOR

SUMMARY

The motion of an oscillator excited by a Poisson process is a stochastic process X_t. Knowing the trajectory of the motion we can find all the stochastic moments of X_t. for large t. This, in turn, allows us to find stochastic distribution of the forces exciting an oscillator. In this paper we evaluate the impact of errors in the computations of the moments on computed distribution of the forces exciting an oscillator.

Keywords: *stochastic impulses, stochastic moments, distributions of impulses, Poisson process*

*WPŁYW BŁĘDÓW NUMERYCZNYCH
NA WYZNACZANIE ROZKŁADU WIELKOŚCI STOCHASTYCZNYCH IMPULSÓW
DZIAŁAJĄCYCH NA OSCYLATOR*

Ruch oscylatora wymuszony przez proces stochastyczny Poissona jest również pewnym procesem stochastycznym X_t. Znając pewną trajektorię ruchu tego oscylatora, możemy znaleźć w przybliżeniu wszystkie momenty zmiennej losowej X_t dla dostatecznie dużych t. Momenty te pozwalają znaleźć rozkład stochastyczny sił działających na oscylator. W pracy badamy wpływ błędów w obliczeniach momentów na obliczanie prawdopodobieństw wielkości sił działających na oscylator.

Słowa kluczowe: *stochastyczne siły impulsowe, stochastyczne momenty, rozkłady impulsów, proces Poissona*

1. INTRODUCTION

The equation of vibrations of an oscillator with damping, with parameters $0 < b < a$ and excited by $f(t)$ assumes the form:

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + a^2x = f(t) \tag{1}$$

Due to the linear character of the differential equation, for our aim it is sufficient to consider the initial conditions of the form

$$x(0) = 0 \text{ i } \dot{x}(0) = 0 \tag{2}$$

Let us assume that function $f(t)$ has the form:

$$f(t) = \sum_{t_i<t} \eta_i \delta_{t_i} \tag{3}$$

where: t_i – time of action of an impulse of the value η_i, η_i for $i = 1, 2, 3, \ldots$, are independent and identically distributed random variables with finite mean value and $\tau_i = t_i - t_{i-1}$, $i = 1, 2, \ldots$, are independent and identically distributed random variables with exponential distribution

$$F(u) = \begin{cases} 1 - \exp(-\lambda u) & \text{for } u > 0 \\ 0 & \text{for } u \le 0 \end{cases} \tag{4}$$

for some $\lambda > 0$ and δ_{t_i} is the Dirac distribution at time t_i.

The motion of an oscillator excited by a Poisson process is a stochastic process X_t. The first partial mathematical results regarding vibration of oscillators forced by stochastic impulses and suggesting their possible technological applications can be found in the following works: (Campbell 1909a, 1909b; Hurwitz and Kac 1944; Roberts 1965a, b, 1972, 1973; Roberts and Spanes 1986, Rowland 1936, Khintchine 1938; Rice 1944; Takác 1994). Other works (Iwankiewicz and Nielsen 1996; Iwankiewicz 2002, 2003) include certain results concerning nonlinear systems subjected to stochastic forces that might not act in a continuous way, systems which are solved by stochastic equations with Ito integral. The methods of investigating the stochastic stability of systems are described in (Tylikowski 1991).

The deviation from the balanced position of the oscillator governed by (1) (2) (3) is a stochastic process and this process is given by the following formula (Jabłoński and Ozga 2006b, c, 2010)

$$X_{\mathbf{t}} = \frac{1}{c}\sum_{t_i<t} \eta_i \exp(-b(t-t_i))\sin\left(c(t-t_i)\right) \tag{5}$$

where $c = \sqrt{a^2 - b^2}$.

The developed model described below is based on the theorem proved in (Jabłoński and Ozga 2006a). This theorem allows for calculation of the statistical theoretical stochastic moments (Jabłoński and Ozga 2010) for assigned distributions of stochastic impulses.

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If assumes a finite number of values $\{\eta_1, \eta_2 \ldots \eta_k\}$ with probabilities $p_i = P(\eta = \eta_i)$ then for any $n > 0$.

Firstly, we concluded that the solution of (6) is sensitive with respect to the deviation of m_i,

$$\sum_{i=1}^{k} p_i \left[(m_n m_1 - m_{n+1})\eta_i + \sum_{j=1}^{n} \binom{n}{j} m_{(n-j)} m_1 \eta_i^{(j+1)} \frac{C(j+1)}{C(1)c^j} \right] = 0 \quad (6)$$

where

$$m_n = \lim_{t\to\infty} E(X_t^n) \qquad (7)$$

and

$$C(j) = \lim_{t\to\infty} \int_0^{ct} \frac{\sin^j u}{\exp(jbu/c)} du \qquad (8)$$

By integration we can find that for an even j and $j > 0$

$$C(j) = \frac{j!}{\prod_{r=0}^{j/2-1}\left((jb/c)^2 + (2r)^2 \right)} \frac{c}{jb} \qquad (9)$$

and for an odd j and $j > 0$

$$C(j) = \frac{j!}{\prod_{r=0}^{(j-1)/2-1}\left((jb/c)^2 + (2r+1)^2 \right)} \qquad (10)$$

In particular

$$C(1) = \frac{1}{(b/c)^2 + 1}$$

$$C(2) = \frac{2!}{\left((2b/c)^2 + 2^2 \right)} \frac{c}{2b}$$

$$C(3) = \frac{3!}{\left((3b/c)^2 + 3^2 \right)} \frac{1}{(3b/c)^2 + 1}$$

To determine theoretical stochastic moments m_{n+1}, for $n \geq 0$we use the following equations

$$m_{n+1} = \sum_{j=0}^{n} \binom{n}{j} m_{(n-j)} \frac{\lambda E\left(\eta^{(j+1)}\right)}{c^{2+j}} C(j+1) \qquad (11)$$

In particular

$$m_0 = 1$$

$$m_1 = \frac{\lambda E(\eta) C(1)}{c^2}$$

$$m_2 = m_1^2 + E\left(\eta^2\right) C(2) \frac{\lambda}{c^3}$$

$$m_3 = m_2 m_1 + 2 m_1^2 \frac{E\left(\eta^2\right)}{cE(\eta)} \frac{C(2)}{C(1)} + m_1 \frac{E\left(\eta^3\right)}{c^2 E(\eta)} \frac{C(3)}{C(1)}$$

Since the process X_t, in the limit as $t \to \infty$, is ergodic (it is a consequence of independence of increases of X_t and damping), knowing the trajectory of the motion of an oscillator, we can calculate approximate values of m_k by the formula

$$E\left(X_t^k\right) \cong \frac{1}{k} \sum_{i=1}^{k} X_{ti/k}^k \qquad (12)$$

which is valid for large t and k. Here $X_{ti/k}^k$ is the random variable given by (4) at the time $t\frac{i}{k}$.

2. DESCRIPTION OF THE EXPERIMENT

RCL system consisting of capacity C = 2 nF and inductivity L = 0.5 H was subjected to examination. The forcing signal η is generated on the analogue output of the card NI USB-6251 at the sampling rate of 1 MHz, with simultaneous recording of the system's response on the analogue input. The application was built in Labview environment. The algorithm of the program takes into account that the distribution of probability of the random variable representing the distance between the impulses is exponential. It has also been taken into account that the distances between the impulses and the impulse values are probabilistically independent. The impulses were executed with the help of single samples of the shortest executable duration of 10^{-6} s, issuing from the sampling rate.

In order to check the possible applications of the above listed formulas in examination of physical phenomena, a physical experiment was conducted on an electric oscillator with parameters $c = \sqrt{a^2 - b^2} = 41\,960$ and b = 1970. Impulses were generated by an electronic generator on a pseudorandom variable where η_1 = 16 549 was almost 5 times as large as the second value η_2 = 8 661 and almost 10 times as large as the smallest η_3 = 1758 with probabilities $p_1 = p_2 = p_3$ = 1/3 and λ = 50 000. These impulses correspond with the values 10, 5 and 1 V produced by the generator. Every 10^{-6} second, tension was measured on an oscillator whose motion was forced by the impulses described above. Theoretical m_i for i = 1, 2, 3 was calculated from (11) where m_1 = 0.254928, m_2 = 0.488240, m_3 = 0.358256. Using formula (12) and the measurements, we got m_1 = 0.249375, m_2 = 0.498605, m_3 = 0.358362. Substituting them into (6) we calculated p_i, p_1 = 0.459, p_2 = 0.370, p_3 = 0.171 were obtained. This did not seem satisfactory, and a computer simulation with the same parameters was conducted using Matlab software package.

3. ERROR ANALYSIS IN CALCULATION OF p_i

Firstly, we concluded that the solution of (6) is sensitive with respect to the deviation of m_i (Fig. 1, Tab. 1). Small deviations of m_i from the accuracy of m_i give large differences in calculating of p_i with regard to the real p_i. To get satisfactory p_i from formula (6) we need m_i computed with precision 10^{-4}. Secondly, the pseudo random variable implemented in Matlab is not perfect. For example, the value of $\frac{1}{n}\sum_{i=1}^{n} \chi_{\left\langle 0,\frac{1}{3} \right)}(Y_i)$, where Y_i is the i-th rand function call is close to 1/3 with the precision 10^{-4} if n is larger than 10^9. This means that in the simulation as well as in the physical experiment we need at least 10^9 measurements to get satisfactory results.

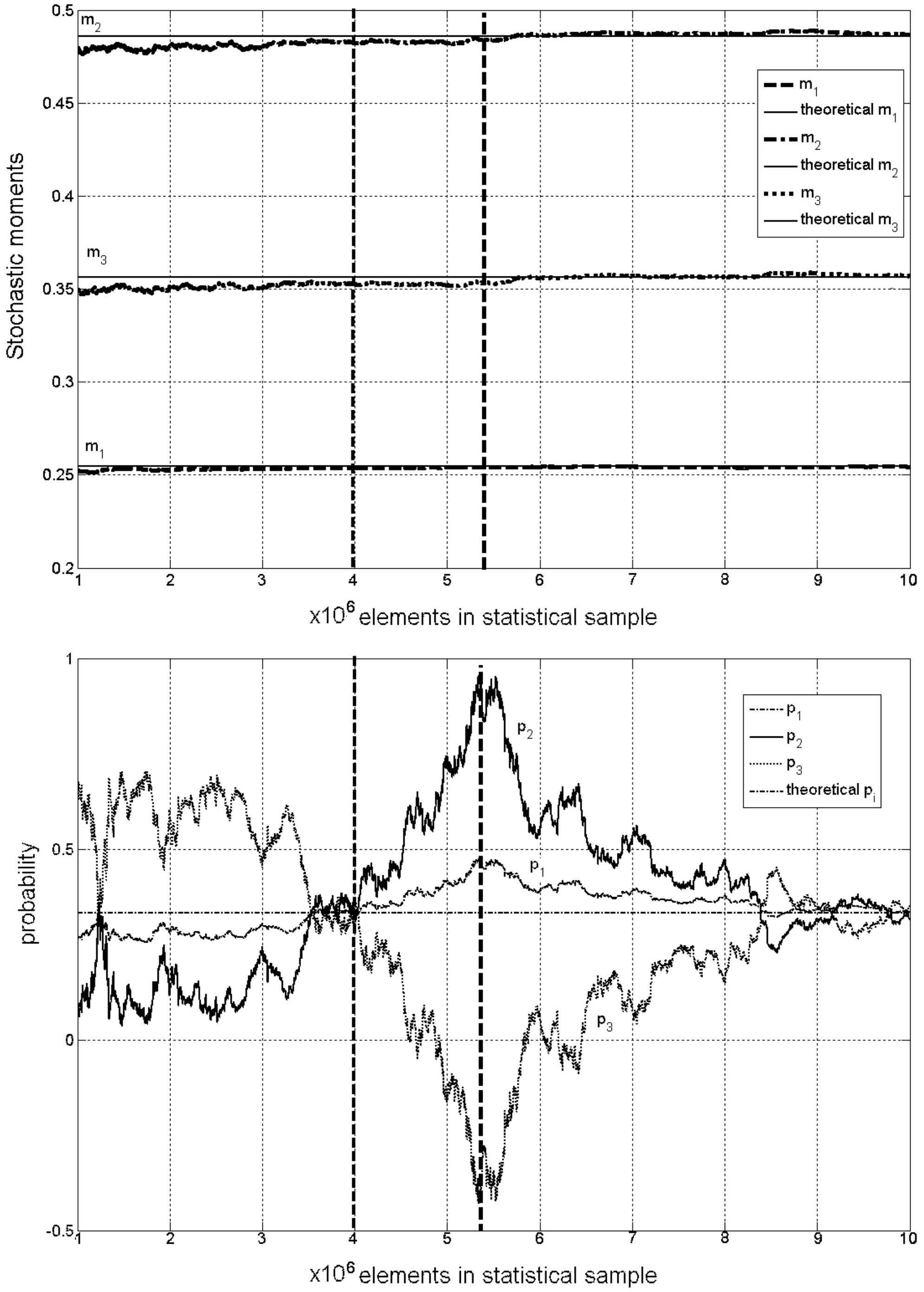

Fig. 1. Estimation of stochastic moments and probabilities p_i as a function of the number of measurements

Table 1

Examples of values of m_i and p_i received from graph 1

THEORETICAL STOCHASTIC MOMENTS AND ASSIGNED DISTRIBUTIONS OF STOCHASTIC IMPULSES	Received from graph 1:	
	Results of measurements for 5 358 000 elements in statistical sample	Results of measurements for 3 992 000 elements in statistical sample
$m_1 = 0.2547365$ $m_2 = 0.4859736$ $m_3 = 0.3565185$	$m_1 = 0.254301$ $m_2 = 0.484279$ $m_3 = 0.353593$	$m_1 = 0.253669$ $m_2 = 0.482310$ $m_3 = 0.352424$
$p_1 = 1/3$ $p_2 = 1/3$ $p_3 = 1/3$	$p_1 = 0.4694$ $p_2 = 0.9344$ $p_3 = -0.4038$	$p_1 = 0.3287$ $p_2 = 0.3339$ $p_3 = 0.3374$

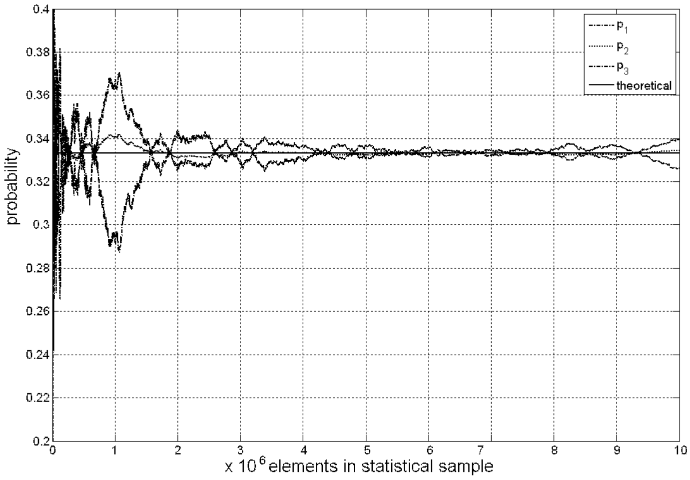

Fig. 2. Estimation of probabilities p_i as a function of number of measurements in the mean value of 1000 samples for 10 million elements in a sample

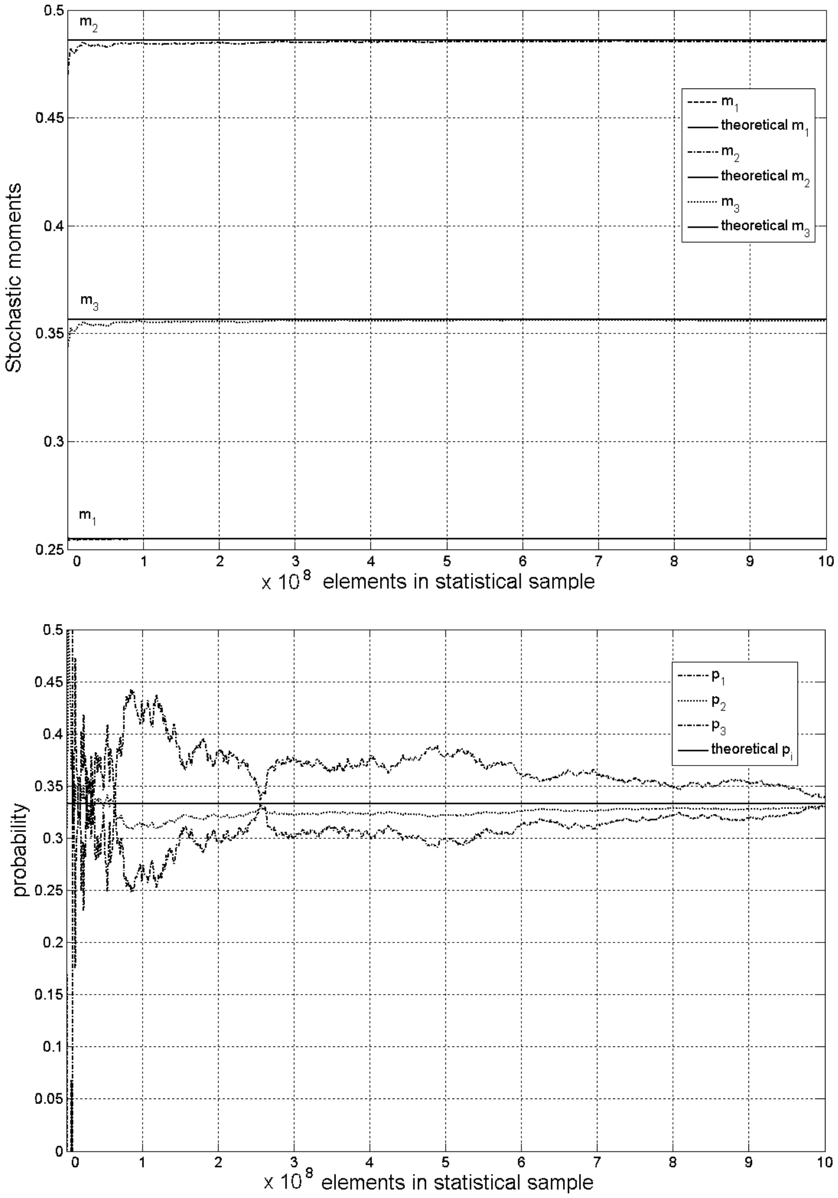

Fig. 3. Estimation of stochastic moments and probabilities p_i as a function of the number of measurements for one milliard elements in a sample

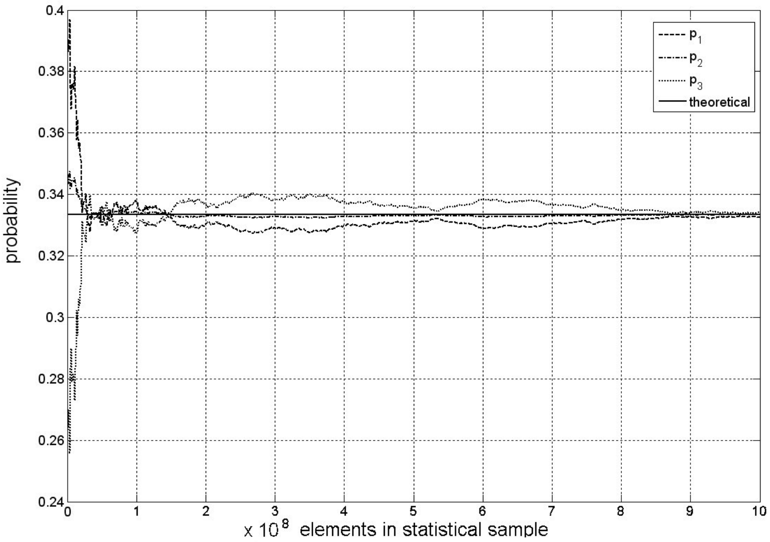

Fig. 4. Stochastic moments determined from the trajectory of the oscillator's motion for a milliard elements in a sample and probabilities computed for them, for the impulse distributions $p_i = 1/3$

Table 2

Stochastic moments

	m_1	m_2	m_3
THEORETICAL STOCHASTIC MOMENTS	0.254737	0.485974	0.356519
Stochastic moments determined from the movement trajectory of the oscillator after computing the average of 1000 samples consisting of 10 milion elements each	0.254709	0.485815	0.356341
Stochastic moments determined from the movement trajectory of the oscillator after computing the average of 86 samples consisting of 1 miliard elements each	0.254716	0.485962	0.356491

It should be noticed that if mi is given with the precision 10^{-2} the determined p_i assumes negative values (Tab. 1), which is impossible.

We can obtain more precise results computing the mean value of a large number of statistical samples. We present a graph (Fig. 2) showing an average of 1000 samples consisting of 10 million elements each.

Figure 3 shows the convergence of estimations of m_i and p_i to the correct values as the number of measurements is close to 10^9.

Figure 4 shows the exactness of the computed p_i if we take the mean value of 86 samples consisting of 10^9 elements. It seems that the value obtained in this case is satisfactory.

The precision of the computed p_i shown in Figures 3 and 4 is presented in Table 2.

4. CONCLUSIONS

The physical experiment as well as the simulation imply that to get suffcently exact values of p_i we need samples consisting of at least 10^9 elements. Since measurements were taken after every 10^{-6} s this means that we need to observe our oscillator for 10^3 s (about 17 minutes). Moreover, the real systems need some correction in theoretical formulas taking into account their specific properties. For example, we have to take into account the errors in measurements. It is worth noticing that the solution of the practical problem imposes **high requirements** on the measuring instruments.

Acknowledgment

This work has been financed by the Polish Ministry of Science and Higher Education, research project No. N N501 180636.

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