Opuscula Mathematica • Vol. 31 • No. 4 • 2011

http://dx.doi.org/10.7494/OpMath.2011.31.4.669

NON SYMMETRIC RANDOM WALK ON INFINITE GRAPH

Marcin J. Zygmunt

Abstract. We investigate properties of a non symmetric Markov's chain on an infinite graph. We show the connection with matrix valued random walk polynomials which satisfy the orthogonality formula with respect to non a symmetric matrix valued measure.

Keywords: random walk on an infinite graph, block tridiagonal transition matrix, spectral measure matrix orthogonal polynomials.

Mathematics Subject Classification: Primary 60J10, Secondary 42C05, 47B36.

1. INTRODUCTION

In the Book of Genesis (cf. Gen 28,12) the biblical patriarch Jacob dreams about a ladder, set up on earth, and with its top reaching heaven. He sees also the angels of God ascending and descending on it.

Let us now assume that an angel standing on earth begins to climb the ladder in a very special way: he tosses a coin, then steps one step forward in the direction he actually is aiming in case of head, or reverses his direction in case of tail. The question is to investigate properties of his "random walk", i.e. what is the probability he eventually returns is, how much time his return takes or how high he climbs. The corresponding Markov's chain can be considered as a random walk on an infinite graph as in Figure 1.

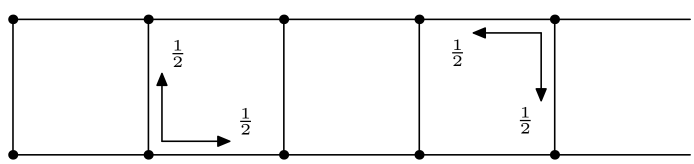

Fig. 1.

2. RANDOM WALK MATRIX POLYNOMIALS

We present a characterization of this specific random walk by properties of the blocks of the transition matrix

$$J = \begin{pmatrix} B_0 & A & 0 & 0 & \dots \\ C & B & A & 0 & \dots \\ 0 & C & B & A & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix}.$$

Hence we are going to investigate properties of matrix polynomials satisfying the following recurrence

$$xP_n(x) = AP_{n+1}(x) + BP_n(x) + CP_{n-1}, \tag{2.1}$$

with

$$AP_1(x) = \begin{pmatrix} x & -1/2 \\ -1 & x \end{pmatrix} \quad \& \quad P_0(x) = I.$$

The classical case, ie. random walk polynomials on the real line, has been studied extensively in the literature (see [5,7,8] among many others). But in our case the infinite Jacobi block matrix J instead is not self-adjoint or even symmetric as an operator on the Hilbert space $\ell_2(\mathbb{N})$. Moreover the matrix coefficient A is not invertible. Hence methods from the theory of Matrix Orthogonal Polynomials (cf. [2,4,6,9]) cannot be used directly. We need a new approach.

Note first that polynomials

$$P_n^o(x) = \begin{pmatrix} u_n(x) & -u_{n-1}(x) \\ u_{n-1}(x) & -u_{n-2}(x) \end{pmatrix},$$

where $u_n(x) = \cos(n\theta)$ are Tchebyshev polynomials of the first kind and $x = \cos\theta$, satisfy the recurrence (2.1). We recall that $xu_n(x) = \frac{1}{2}u_{n+1}(x) + \frac{1}{2}u_{n-1}(x)$ for $n \geq 1$, and $xu_0(x) = u_1(x)$ (it is assumed that $u_{-1} \equiv 0$).

Let's now set

$$A_\varepsilon = \begin{pmatrix} 1/2 & 0 \\ 0 & \varepsilon \end{pmatrix} \quad \text{and} \quad C_\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1/2 \end{pmatrix}.$$

In [10] it was shown that polynomials $P_n^{o,\varepsilon}$, which satisfy the recurrence formula

$$xP_n^{o,\varepsilon}(x) = A_\varepsilon P_{n+1}^{o,\varepsilon}(x) + BP_n^{o,\varepsilon}(x) + C_\varepsilon P_{n-1}^{o,\varepsilon},$$

fulfill also the following property:

$$\langle\!\langle P_n^{o,\varepsilon}, P \rangle\!\rangle = \int_{\mathbb{R}} P_n^{o,\varepsilon}(x) W_{o,\varepsilon}(x) P(x)^* \, dx \; = \; 0 \tag{2.2}$$

for any matrix polynomial P of degree lower than n. Matrix measure $W_{o,\varepsilon}(x)\,dx$ is given by the inverse Stieltjes-Perron formula

$$W_{o,\varepsilon}(x) \;=\; \lim_{\delta\to 0^+} \frac{F_{o,\varepsilon}(x-i\delta)-F_{o,\varepsilon}(x+i\delta)}{2\pi i}, \tag{2.3}$$

where $F_{o,\varepsilon}$ is a Stieltjes transform of $W_{o,\varepsilon}$ and can be obtained by equality (Lemma 2.4 in [10])

$$F_{o,\varepsilon}(z) \;=\; \lim_{n\to\infty} P_n^{o,\varepsilon}(z)^{-1} P_{n-1}^{o,\varepsilon}(z) A_\varepsilon^{-1}.$$

The corollary of Theorem 2.7 in [10] states that

$$F_{o,\varepsilon}(z) \;=\; \cfrac{1}{z-B_0-A_\varepsilon \cfrac{1}{z-B-A_\varepsilon \frac{1}{z-B-\ddots} C_\varepsilon} C_\varepsilon} \tag{2.4}$$

for $\operatorname{Im} z > 0$.

Polynomials P_n^o are the limit case of $P_n^{o,\varepsilon}$ as ε tends to 0. It is not difficult to see that then equation (2.2), (2.3) and (2.4) still hold, which leads to

$$\int_{\mathbb{R}} P_n^o(x) W_o(x) P(x)^*\,dx \;=\; 0,$$

where

$$W_o(x) = \lim_{\delta\to 0^+} \frac{1}{\pi} \operatorname{Im} F_o(x-i\delta) \tag{2.5}$$

and

$$F_o(z) \;=\; \cfrac{1}{z-B_0-A \cfrac{1}{z-B-A \frac{1}{z-B-\ddots} C} C}$$

for $\operatorname{Im} z > 0$. So the function F_o is given by equality

$$F_o(z)\bigl(z-B_0-AG_o(z)C\bigr) = I,$$

where

$$G_o(z)\bigl(z-B-AG_o(z)C\bigr) = I.$$

Thus the question is to solve

$$X(z)\bigl(z-B-AX(z)C\bigr) = I$$

with the additional condition $\lim_{z\to\infty} X(z) = 0$.

The exact solution is

$$X(z) = \begin{pmatrix} 2(z-\sqrt{z^2-1}\,) & 2(z-z\sqrt{z^2-1}\,)^2 \\ \frac{z-\sqrt{z^2-1}}{z} & 2(z-\sqrt{z^2-1}\,) \end{pmatrix},$$

so we get

$$F_o(z) = [z - B_0 - AX(z)C]^{-1} = \begin{pmatrix} \frac{1}{\sqrt{z^2-1}} & \frac{z}{\sqrt{z^2-1}} - 1 \\ \frac{1}{z\sqrt{z^2-1}} & \frac{1}{\sqrt{z^2-1}} \end{pmatrix}.$$

The matrix of functions $W_o(x)$ can be uniquely determined by the formula (2.5), hence

$$W_o(x) = \frac{1}{2\pi\sqrt{1-x^2}} \begin{pmatrix} 1 & x \\ \frac{1}{x} & 1 \end{pmatrix}.$$

Polynomials P_n^o are not MOP, but they satisfy

$$\int_{\mathbb{R}} P_n^o(x) W_o(x) P(x)^* \, dx \; = \; 0$$

for any polynomial P of degree lower then the degree of P_n^o. Thus P_n^o could be considered as orthogonal but with respect to a non positive definite matrix of measures (exactly non-symmetric).

Now we can return to the random walk on “Jacob's ladder”. The probability that an angel eventually returns to the ground is equal to

$$f_{00} = \lim_{z \to 1} \frac{p_{00}(z) - 1}{p_{00}(z)},$$

where

$$p_{00}(z) = (0,1) F_o(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2\pi} \frac{1}{z\sqrt{z^2-1}}.$$

This shows that the random walk considered in this section is recurrent ($f_{00} = 1$). The quantity $p_{00}(1)$ is equal to the average number of visits at the starting point (we refer the reader to [1] in case of a random walk on graphs, or to [7] in general case).

3. CASE OF AN UNFAIR COIN

What happens if the coin the angel tosses is unfair, ie. head and tail occur with probability p and $1-p$ respectively, with $0 < p < 1$ and $p \neq \frac{1}{2}$? In that case we should consider the following relation

$$xP_{p,n}(x) = A_p P_{p,n+1}(x) + B_p P_{p,n}(x) + C_p P_{p,n-1} \tag{3.1}$$

for $n \geq 1$ where

$$A_p = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \; C_p = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}, \; B_p = \begin{pmatrix} 0 & 1-p \\ 1-p & 0 \end{pmatrix}, \; B_{0,p} = \begin{pmatrix} 0 & 1-p \\ 1 & 0 \end{pmatrix}.$$

The corresponding function F_p satisfies

$$F_p(z)\big(z - B_{0,p} - A_p G_p(z) C_p\big) = I,$$

with

$$G_p(z)\big(z - B_p - A_p G_p(z) C_p\big) = I.$$

The solution is given then by

$$F_p(z) = \frac{1}{w_p(z)} \begin{pmatrix} 2(1-p)z & z^2 + (1-2p) - \sqrt{(z^2-1)(z^2-(1-2p)^2)} \\ 2(1-p) & 2(1-p)z \end{pmatrix},$$

where

$$w_p(z) = (z^2-1)(1-2p) + \sqrt{(z^2-1)(z^2-(1-2p)^2)}\,.$$

This shows that the corresponding random walk is still recurrent.

Acknowledgements

The author was supported by national grant N 201 4265 33 and partially by the Polish Ministry of Science and Higher Education.

REFERENCES

[1] R. Burioni, D. Cassi, *Random walks on graphs: ideas, techniques and results*, J. Phys. A: Math. Gen. **38** (2005), R45–R78.

[2] H. Dette, B. Reuther, W.J. Studden, M.J. Zygmunt, *Matrices measures and random walks with a block tridiagonal transition matrix*, SIAM J. Matrix Anal. Appl. **29** (2006) 1, 117–142.

[3] A.J. Duran, *A generalization of Favard's theorem for polynomials satisfying a recurrence relation*, J. Approx. Theory **74** (1993), 83–109.

[4] A.J. Duran, *On orthogonal polynomials with respect to a positive definite matrix of measures*, Canad. J. Math. **47** (1995), 88–112.

[5] S. Karlin, J. McGregor, *Random walks*, Illionis J. Math. **3** (1959), 66–81.

[6] D. Damanik, A. Pushnitski, B. Simon, *The Analytic Theory of Matrix Orthogonal Polynomials*, Surveys in Approx. Theory **4** (2008), 1–85.

[7] E.A. van Doorn, P. Schrijner, *Random walk polynomials and random walk measures*, J. Comput. Appl. Math. **49** (1993), 289–296.

[8] W. Woess, *Random walks and periodic continued fractions*, Adv. in Appl. Probab. **17** (1985), 67–84.

[9] M.J. Zygmunt, *Matrix orthogonal polynomials and continued fractions*, Linear Alg. Appl. **340** (2002) 1–3, 155–168.

[10] M.J. Zygmunt, *Matrix polynomials orthogonal with respect to a non-symmetric matrix of measures*, to appear.

Marcin J. Zygmunt
zygmunt@agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. A. Mickiewicza 30, 30-059 Krakow, Poland

Received: November 17, 2010.
Revised: March 7, 2011.
Accepted: March 15, 2011.