Opuscula Mathematica • Vol. 31 • No. 4 • 2011

http://dx.doi.org/10.7494/OpMath.2011.31.4.651

ON SOME CLASSES OF MEROMORPHIC FUNCTIONS DEFINED BY SUBORDINATION AND SUPERORDINATION

Alina Totoi

Abstract. Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and let Σ_p denote the class of meromorphic functions of the form $g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \ldots$, $z \in \dot{U}$, $a_{-p} \neq 0$.

We consider the integral operator $J_{p,\beta,\gamma} : K_{p,\beta,\gamma} \subset \Sigma_p \to \Sigma_p$ defined by

$$J_{p,\beta,\gamma}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma} \int_0^z g^\beta(t) t^{\gamma-1} dt\right]^{\frac{1}{\beta}}, \quad g \in K_{p,\beta,\gamma},\ z \in \dot{U}.$$

We introduce some new subclasses of the class Σ_p, associated with subordination and superordination, such that, in some particular cases, these new subclasses are the well-known classes of meromorphic starlike functions and we study the properties of these subclasses with respect to the operator $J_{p,\beta,\gamma}$.

Keywords: meromorphic functions, integral operators, subordination, superordination.

Mathematics Subject Classification: 30C45.

1. INTRODUCTION AND PRELIMINARIES

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane, $\dot{U} = U \setminus \{0\}$, $H(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic in } U\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

For $p \in \mathbb{N}^*$ let Σ_p denote the class of meromorphic functions of the form

$$g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \ldots, \quad z \in \dot{U},\ a_{-p} \neq 0.$$

We will also use the following notations:
$\Sigma_p^*(\alpha) = \left\{g \in \Sigma_p : \text{Re}\left[-\frac{zg'(z)}{g(z)}\right] > \alpha,\ z \in U\right\}$, where $\alpha < p$,

$\Sigma_p^*(\alpha,\delta) = \left\{ g \in \Sigma_p : \alpha < \text{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \delta,\ z \in U \right\}$, where $\alpha < p < \delta$,
$H[a,n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\}$ for $a \in \mathbb{C},\ n \in \mathbb{N}^*$,
$A_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots\}$, $n \in \mathbb{N}^*$, and for $n = 1$ we denote A_1 by A and this set is called *the class of analytic functions normalized at the origin*.

We remark that $\Sigma_1^*(\alpha)$ is the well-known class of meromorphic starlike functions of order α, when $0 \leq \alpha < 1$.

Definition 1.1 ([4, p. 4])**.** Let f and F be members of $H(U)$. The function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = F(w(z))$.

Definition 1.2 ([4, p. 16])**.** Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply, a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). (Note that the best dominant is unique up to a rotation of U).

If we require the more restrictive condition $p \in H[a,n]$, then p will be called an (a,n)-solution, q an (a,n)-dominant, and $\tilde{q}$ the best (a,n)-dominant.

Definition 1.3 ([**?**], [1, p. 98])**.** Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and satisfy the second order differential superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \quad z \in U, \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply, a subordinant, if $q \prec p$ for all p satisfying (1.2). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of U.

Definition 1.4 ([1, p. 99])**.** We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and they are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. The subclass of Q for which $f(0) = a$, is denoted by $Q(a)$.

Definition 1.5 ([4, p. 46])**.** Let c be a complex number such that $\operatorname{Re} c > 0$, let n be a positive integer, and let

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1+\frac{2\operatorname{Re} c}{n}} + \operatorname{Im} c\right]. \tag{1.3}$$

If $R(z)$ is the univalent function defined in U by $R(z) = \dfrac{2C_n z}{1-z^2}$, then the "Open Door" function is defined by

$$R_{c,n}(z) = R\left(\frac{z+b}{1+\bar{b}z}\right) = 2C_n\frac{(z+b)(1+\bar{b}z)}{(1+\bar{b}z)^2-(z+b)^2}, \tag{1.4}$$

where $b = R^{-1}(c)$.

Theorem 1.6 ([4, p. 83])**.** *Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U, with $h(0) = a$. Let n be a positive integer. Suppose that the differential equation*

$$q(z) + \frac{nzq'(z)}{\beta q(z)+\gamma} = h(z) \tag{1.5}$$

has a univalent solution q that satisfies $q(z) \prec h(z)$. If $p \in H[a,n]$ satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z), \tag{1.6}$$

then $p(z) \prec q(z)$, and q is the best (a,n)-dominant of (1.6)*.*

Corollary 1.7 ([4, p. 84])**.** *Let h be convex in U, with $h(0) = a$, and let m and n be positive integers. Let q_m and q_n be univalent solutions of the differential equation* (1.5) *for $n = m$ and n respectively, with $q_n \prec h$. If $m > n$, then $q_m \prec q_n$.*

Theorem 1.8 ([5], [1, p. 114])**.** *Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U with $h(0) = a$. Suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = h(z), \quad z \in U,$$

has the univalent solution q with $q(0) = a$, and $q(z) \prec h(z)$. If $p \in H[a,1] \cap Q$ and $p(z) + \dfrac{zp'(z)}{\beta p(z)+\gamma}$ is univalent in U, then

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \Rightarrow q(z) \prec p(z).$$

The function q is the best subordinant.

Theorem 1.9 ([4, p. 86]). *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, and let n be a positive integer. Let $R_{\beta a+\gamma,n}$ be as given in* (1.4)*, let h be analytic in U with $h(0) = a$, and let* $\operatorname{Re}[\beta a + \gamma] > 0$. *If*

$$\beta h(z) + \gamma \prec R_{\beta a+\gamma,n}(z),$$

then the solution q of

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z), \tag{1.7}$$

with $q(0) = a$, is analytic in U and satisfies $\operatorname{Re}[\beta q(z) + \gamma] > 0$.

If $a \neq 0$, then the solution for (1.7) *is given by*

$$q(z) = z^{\frac{\gamma}{n}} H^{\frac{\beta a}{n}}(z) \left[\frac{\beta}{n}\int_0^z H^{\frac{\beta a}{n}}(t) t^{\frac{\gamma}{n}-1} dt\right]^{-1} - \frac{\gamma}{\beta} = \tag{1.8}$$

$$= \left[\frac{\beta}{n}\int_0^1 \left[\frac{H(tz)}{H(z)}\right]^{\frac{\beta a}{n}} t^{\frac{\gamma}{n}-1} dt\right]^{-1} - \frac{\gamma}{\beta},$$

where

$$H(z) = z \exp \int_0^z \frac{h(t) - a}{at} dt.$$

If $a = 0$, then the solution is given by

$$q(z) = H^{\frac{\gamma}{n}}(z) \left[\frac{\beta}{n}\int_0^z H^{\frac{\gamma}{n}}(t) t^{-1} dt\right]^{-1} - \frac{\gamma}{\beta} =$$

$$= \left[\frac{\beta}{n}\int_0^1 \left[\frac{H(tz)}{H(z)}\right]^{\frac{\gamma}{n}} t^{-1} dt\right]^{-1} - \frac{\gamma}{\beta},$$

where

$$H(z) = z \exp \frac{\beta}{\gamma}\int_0^z \frac{h(t)}{t} dt.$$

Theorem 1.10 ([4, p. 97]). *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, and let n be a positive integer. Let $R_{\beta a+\gamma,n}$ be as given in* (1.4)*, let h be analytic in U, with $h(0) = a$,* $\operatorname{Re}[\beta a+\gamma] > 0$ *and*

$$(i) \quad \beta h(z) + \gamma \prec R_{\beta a+\gamma,n}(z).$$

If q is the analytic solution of the Briot-Bouquet differential equation

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z)$$

as given in (1.8), *and if*

$$(ii) \quad h \text{ is convex or } Q(z) = \frac{zq'(z)}{\beta q(z) + \gamma} \text{ is starlike,}$$

then q and h are univalent. Furthermore, if $p \in H[a,n]$ satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),$$

then $p \prec q$, and q is the best (a,n)-dominant.

Theorem 1.11 ([3])**.** *Let $\beta, \gamma \in \mathbb{C}$ and let h be a convex function in U, with*

$$\mathrm{Re}\,[\beta h(z) + \gamma] > 0, \quad z \in U.$$

Let q_m and q_k be the univalent solutions of the Briot-Bouquet differential equation

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z), \quad z \in U, \quad q(0) = h(0),$$

for $n = m$ and $n = k$ respectively. If m/k, then $q_k(z) \prec q_m(z) \prec h(z)$. So, $q_k(z) \prec q_1(z) \prec h(z)$.

Theorem 1.12 ([5], [1, p. 117])**.** *Let β, $\gamma \in \mathbb{C}$ and let the function $h \in H(U)$ with $h(0) = a$ and $\mathrm{Re}\, c > 0$, where $c = \beta a + \gamma$ and suppose that*

$$(i) \quad \beta h(z) + \gamma \prec R_{c,1}(z).$$

Let q be the analytic solution of the Briot-Bouquet differential equation

$$h(z) = q(z) + \frac{zq'(z)}{\beta q(z) + \gamma}$$

and suppose that

$$(ii) \quad \frac{zq'(z)}{\beta q(z) + \gamma} \quad \text{is starlike in } U.$$

If $p \in H[a,1] \cap Q$ and $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma}$ is univalent in U, then

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \Rightarrow q(z) \prec p(z)$$

and the function q is the best subordinant.

Corollary 1.13 ([8])**.** *Let $p \in \mathbb{N}^*$, $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. If $g \in \Sigma_p$ and*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),$$

then

$$G(z) = J_{p,\beta,\gamma}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma}\int_0^z g^\beta(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}} \in \Sigma_p,$$

with $z^pG(z) \neq 0$, $z \in U$, *and*

$$\mathrm{Re}\left[\beta\frac{zG'(z)}{G(z)} + \gamma\right] > 0, \quad z \in U.$$

All powers are chosen as principal ones.

We remark that if $p \in \mathbb{N}^*$, $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, $\mathrm{Re}\,(\gamma - p\beta) > 0$ and $g \in \Sigma_p$ with

$$\beta\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z), \quad z \in U,$$

we have from Corollary 1.13 that $G = J_{p,\beta,\gamma}(g) \in \Sigma_p$ with $z^pG(z) \neq 0$, $z \in U$, so $P(z) = -\dfrac{zG'(z)}{G(z)} \in H[p,p]$. Having these conditions, it is easy to see that from

$$G(z) = \left[\frac{\gamma - p\beta}{z^\gamma}\int_0^z t^{\gamma-1}g^\beta(t)dt\right]^{\frac{1}{\beta}}, \quad z \in \dot{U},$$

we obtain

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad P(z) = -\frac{zG'(z)}{G(z)}. \tag{1.9}$$

2. MAIN RESULTS

In this section we present and prove five theorems and five corollaries concerning the integral operator $J_{p,\beta,\gamma}$. We consider some new subclasses of the class Σ_p, associated with superordination and subordination, and we establish the conditions such that when we apply the integral operator $J_{p,\beta,\gamma}$ to a function which belongs to one of these new subclasses, the result remains in a similar class.

The first result is a simple lemma and we will use it latter to present some examples for the results included in this paper. For this lemma we need the next criteria for convexity:

Theorem 2.1 ([6])**.** *If* $f \in A_n$ *and*

$$|f''(z)| \leq \frac{n}{n+1}, \quad z \in U,$$

then

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 1, \quad z \in U,$$

hence, f is convex. The result is sharp for the function

$$f(z) = z + \frac{z^{n+1}}{(n+1)^2}.$$

Lemma 2.2. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ with $\gamma \neq 0$, $\alpha + \gamma \neq 0$ and $|\beta| < |\gamma|$. Let h be the function*

$$h(z) = z + \frac{\alpha z}{\beta z + \gamma}, \quad z \in U.$$

If we have

$$4|\alpha\beta\gamma^2| \leq (|\gamma| - |\beta|)^3 |\alpha + \gamma|, \tag{2.1}$$

then h is convex in U.

Proof. Since $|\beta| < |\gamma|$ we have $\beta z + \gamma \neq 0$, $z \in U$, so, $h \in H(U)$. We also have $h'(0) = \frac{\alpha + \gamma}{\gamma} \neq 0$, hence $\frac{\gamma}{\alpha + \gamma} h \in A_1$.

It is easy to see that

$$h''(z) = -\frac{2\alpha\beta\gamma}{(\beta z + \gamma)^3}, \ z \in U,$$

hence

$$\left| \frac{\gamma}{\alpha + \gamma} h''(z) \right| = \frac{|\gamma|}{|\alpha + \gamma|} \cdot \frac{2|\alpha\beta\gamma|}{|\beta z + \gamma|^3} < \frac{2|\alpha\beta\gamma^2|}{(|\gamma| - |\beta|)^3 |\alpha + \gamma|} \leq \frac{1}{2}, \quad z \in U.$$

For the last inequality we used the fact that $4|\alpha\beta\gamma^2| \leq (|\gamma| - |\beta|)^3 |\alpha + \gamma|$.

Using Theorem 2.1, for $n = 1$, we obtain that h is convex in U. □

Remark 2.3. 1. It is obvious that if h is a convex function in U (with $h'(0) \neq 0$), then $\delta_1 + \delta_2 h(rz)$ is also a convex function, when $r \in (0, 1]$, $\delta_1, \delta_2 \in \mathbb{C}$, $\delta_2 \neq 0$.
2. If we consider $\alpha = |\beta| = 1$ in the above lemma, then the condition (2.1) becomes

$$4|\gamma|^2 \leq |\gamma + 1|(|\gamma| - 1)^3. \tag{2.2}$$

It is not difficult to verify that the condition (2.2) holds for each real number $\gamma \geq 3,2$.

In other words, the functions

$$z + \frac{z}{\gamma + z}, \ z + \frac{z}{\gamma - z}, \quad z \in U,$$

are convex functions when $\gamma \geq 3,2$.

We mention here that in [7] the authors proved that the function

$$h(z) = 1 + z + \frac{z}{z + 2}, \quad z \in U,$$

is convex in U, so the function $z + \frac{z}{2 + z}$ is also a convex function.

Next, we define some new subclasses of the class Σ_p, associated with superordination and subordination, such that, in some particular cases, these new subclasses are the well-known classes of meromorphic starlike functions.

Definition 2.4. Let $p \in \mathbb{N}^*$ and $h_1, h_2, h \in H(U)$ with $h_1(0) = h_2(0) = h(0) = p$ and $h_1(z) \prec h_2(z)$. We define:

$$\Sigma S_p(h_1, h_2) = \left\{ g \in \Sigma_p : h_1(z) \prec -\frac{zg'(z)}{g(z)} \prec h_2(z) \right\},$$

$$\Sigma S_p(h) = \left\{ g \in \Sigma_p : -\frac{zg'(z)}{g(z)} \prec h(z) \right\}.$$

We remark that if we consider $h(z) = h_{p,\alpha}(z) = \dfrac{p + (p - 2\alpha)z}{1 - z}$, $z \in U, 0 \le \alpha < p$, since $h_{p,\alpha}(U) = \{z \in \mathbb{C} : \operatorname{Re} z > \alpha\}$, we have $\Sigma S_p(h_{p,\alpha}) = \Sigma_p^*(\alpha)$.

Theorem 2.5. *Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\operatorname{Re}(\gamma - p\beta) > 0$. Let h_1 and h_2 be convex functions in U with $h_1(0) = h_2(0) = p$ and let $g \in \Sigma S_p(h_1, h_2)$ such that*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - \beta p, p}(z), \quad z \in U.$$

Suppose that the Briot-Bouquet differential equations

$$q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_1(z) \quad \text{and} \quad q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_2(z), \quad z \in U,$$

have the univalent solutions q_1^1 and, respectively, q_2^p, with $q_1^1(0) = q_2^p(0) = p$ and $q_1^1 \prec h_1$, $q_2^p \prec h_2$.

Let $G = J_{p,\beta,\gamma}(g)$. If $\dfrac{zg'(z)}{g(z)}$ is univalent in U and $\dfrac{zG'(z)}{G(z)} \in Q$, then

$$G \in \Sigma S_p(q_1^1, q_2^p).$$

The functions q_1^1 and q_2^p are the best subordinant and, respectively, the best (p, p)-dominant.

Proof. From $g \in \Sigma S_p(h_1, h_2)$ we have $\dfrac{zg'(z)}{g(z)} \in H(U)$ and

$$h_1(z) \prec -\frac{zg'(z)}{g(z)} \prec h_2(z), \tag{2.3}$$

with $h_1 \prec h_2$ and $h_1(0) = h_2(0) = p$.

Let $P(z) = -\dfrac{zG'(z)}{G(z)}$, $z \in U$. Since $\gamma + \beta \dfrac{zg'(z)}{g(z)} \prec R_{\gamma - \beta p, p}(z)$, $z \in U$, we have from Corollary 1.13 that $G = J_{p,\beta,\gamma}(g) \in \Sigma_p$ with $z^p G(z) \neq 0$, $z \in U$. Hence, $P \in H[p, p]$.

From (1.9) and (2.3), we obtain

$$h_1(z) \prec P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \prec h_2(z), \quad \text{where} \quad P(z) = -\frac{zG'(z)}{G(z)}, \quad z \in U.$$

If we apply Theorem 1.6 (for $a = n = p$, $h = h_2$ and with $-\beta$ instead of β) to the subordination

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \prec h_2(z), \quad z \in U,$$

we get

$$P(z) \prec q_2^p(z), \quad z \in U. \tag{2.4}$$

Because $P \in H[p,p] \cap Q$ and $P(z) + \dfrac{zP'(z)}{\gamma - \beta P(z)}$ is univalent in U, we may apply Theorem 1.8 (for $a = p$, $n = 1$, $h = h_1$ and with $-\beta$ instead of β) to

$$h_1(z) \prec P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}, \quad z \in U,$$

and we get

$$q_1^1(z) \prec P(z), \; z \in U. \tag{2.5}$$

From (2.4) and (2.5) we have

$$q_1^1(z) \prec P(z) \prec q_2^p(z), \quad z \in U,$$

which is equivalent to

$$q_1^1(z) \prec -\frac{zG'(z)}{G(z)} \prec q_2^p(z), \quad z \in U. \tag{2.6}$$

Since $G \in \Sigma_p$ we have from (2.6) that $G \in \Sigma S_p(q_1^1, q_2^p)$.

From Theorem 1.6 and Theorem 1.8 we also have that the functions q_1^1 and q_2^p are the best subordinant and, respectively, the best (p,p)-dominant. □

If we consider in the hypothesis of Theorem 2.5 the condition

$$\operatorname{Re}[\gamma - \beta h_2(z)] > 0, \quad z \in U,$$

instead of

$$\beta\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-\beta p,p}(z), \quad z \in U,$$

we get the next result.

Theorem 2.6. *Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\operatorname{Re}(\gamma - p\beta) > 0$. Let h_1 and h_2 be convex functions in U with $h_1(0) = h_2(0) = p$, $h_1 \prec h_2$ and*

$$\operatorname{Re}[\gamma - \beta h_2(z)] > 0, \quad z \in U.$$

Let $g \in \Sigma S_p(h_1, h_2)$ *and* $G = J_{p,\beta,\gamma}(g)$. *If* $\dfrac{zg'(z)}{g(z)}$ *is univalent in* U *and* $\dfrac{zG'(z)}{G(z)} \in Q$, *then*

$$G \in \Sigma S_p(q_1^1, q_2^p),$$

where q_1^1 *and* q_2^p *are the univalent solutions of the Briot-Bouquet differential equations*

$$q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_1(z), \quad z \in U, \tag{2.7}$$

and, respectively,

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_2(z), \quad z \in U, \tag{2.8}$$

with $q_1^1(0) = q_2^p(0) = p$.

The functions q_1^1 *and* q_2^p *are the best subordinant and, respectively, the best* (p,p)*-dominant.*

Proof. From $g \in \Sigma S_p(h_1, h_2)$ we have

$$h_1(z) \prec -\frac{zg'(z)}{g(z)} \prec h_2(z), \quad z \in U,$$

hence

$$\gamma - \beta h_1(z) \prec \gamma + \beta\frac{zg'(z)}{g(z)} \prec \gamma - \beta h_2(z), \quad z \in U. \tag{2.9}$$

Since $\operatorname{Re}[\gamma - \beta h_2(z)] > 0$, $z \in U$, we get from (2.9) that

$$\operatorname{Re}[\gamma - \beta h_1(z)] > 0 \quad \text{and} \quad \operatorname{Re}\left[\gamma + \beta\frac{zg'(z)}{g(z)}\right] > 0, \quad z \in U.$$

Now, it is obvious that we have

$$\gamma + \beta\frac{zg'(z)}{g(z)} \prec R_{\gamma-\beta p,p}(z), \quad z \in U,$$

$$\gamma - \beta h_1(z) \prec R_{\gamma-p\beta,1}(z) \quad \text{and} \quad \gamma - \beta h_2(z) \prec R_{\gamma-p\beta,p}(z), \quad z \in U.$$

It is easy to see that the conditions from the hypothesis of Theorem 1.9 are fulfilled (for $h = h_1, a = p$ and $n = 1$) so, the solution q_1^1 of the equation (2.7) with $q_1^1(0) = p$ is analytic in U. Analogous we have that the solution q_2^p of the equation (2.8) with $q_2^p(0) = p$ is analytic in U.

Since h_1 and h_2 are convex functions, we have from Theorem 1.10 that the analytic functions q_1^1 and q_2^p are univalent in U, and from Theorem 1.11 (since $\operatorname{Re}[\gamma - \beta h_1(z)] > 0$ and $\operatorname{Re}[\gamma - \beta h_2(z)] > 0$, $z \in U$) we have the subordinations $q_1^1 \prec h_1$ and $q_2^p \prec h_2$.

Therefore, the conditions from the hypothesis of Theorem 2.5 are fulfilled and the result follows using this theorem. $\square$

Remark 2.7. Let the conditions from the hypothesis of Theorem 2.6 be fulfilled. If we consider, in addition, that q_1^p and q_2^1 are the univalent solutions of the Briot-Bouquet differential equations

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_1(z), \quad z \in U,$$

and, respectively,

$$q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_2(z), \quad z \in U,$$

with $q_1^p(0) = q_2^1(0) = p$, we have from the above theorem and Corollary 1.7, that

$$q_1^p(z) \prec q_1^1(z) \prec -\frac{zG'(z)}{G(z)} \prec q_2^p(z) \prec q_2^1(z), \quad z \in U.$$

Hence $G \in \Sigma S_p(q_1^1, q_2^p)$ is the best choice.

If we consider for Theorem 2.5 only the subordination, we obtain the next result.

Theorem 2.8. *Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\operatorname{Re}(\gamma - p\beta) > 0$. Let h be a convex function in U with $h(0) = p$ and $g \in \Sigma S_p(h)$ such that*

$$\beta\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-\beta p,p}(z), \quad z \in U.$$

Suppose that the Briot-Bouquet differential equation

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z), \quad z \in U,$$

has the univalent solution q with $q(0) = p$ and $q \prec h$. Then

$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(q).$$

The function q is the best (p,p)-dominant.

Proof. Let $P(z) = -\dfrac{zG'(z)}{G(z)}$, $z \in U$. We know from Corollary 1.13 that $G \in \Sigma_p$ with $z^pG(z) \neq 0$, $z \in U$, so $P \in H[p,p]$.

Since P is analytic in U, we have from (1.9) that

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad z \in U.$$

Because $g \in \Sigma S_p(h)$ we have $-\dfrac{zg'(z)}{g(z)} \prec h(z)$, $z \in U$, hence

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \prec h(z), \quad z \in U.$$

Using Theorem 1.6 (for $a = n = p$ and with $-\beta$ instead of β) we get $P \prec q$, so

$$-\frac{zG'(z)}{G(z)} \prec q(z), \quad z \in U. \tag{2.10}$$

Since $G \in \Sigma_p$ we obtain from (2.10) that

$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(q).$$

We also have from Theorem 1.6 that the function q is the best (p,p)-dominant. $\square$

Theorem 2.9. *Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\operatorname{Re}[\gamma - p\beta] > 0$. Let h_1 and h_2 be analytic functions in U with $h_1(0) = h_2(0) = p$, $h_1 \prec h_2$ and*

$$(i) \quad \gamma - \beta h_2(z) \prec R_{\gamma - p\beta, 1}(z), \quad z \in U.$$

If q_1 and q_2 are the analytic solutions of the Briot-Bouquet differential equations

$$q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_1(z), \quad z \in U, \tag{2.11}$$

and, respectively,

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_2(z), \quad z \in U, \tag{2.12}$$

with $q_1(0) = q_2(0) = p$ and if

$$(ii) \quad \frac{zq_1'(z)}{\gamma - \beta q_1(z)} \quad \text{is starlike in } U,$$

$$(iii) \quad h_2 \quad \text{is convex or} \quad \frac{zq_2'(z)}{\gamma - \beta q_2(z)} \quad \text{is starlike},$$

then q_1 and q_2 are univalent in U.

Moreover, if $g \in \Sigma S_p(h_1, h_2)$ such that $\frac{zg'(z)}{g(z)}$ is univalent in U and $\frac{zG'(z)}{G(z)} \in Q$, where $G = J_{p,\beta,\gamma}(g)$, then

$$G \in \Sigma S_p(q_1, q_2).$$

The functions q_1 and q_2 are the best subordinant and, respectively, the best (p,p)-dominant.

Proof. From $h_1 \prec h_2$ and (i) we have

$$\gamma - \beta h_1(z) \prec \gamma - \beta h_2(z) \prec R_{\gamma - p\beta, 1}(z), \quad z \in U. \tag{2.13}$$

From (2.13), using also the fact that $R_{\gamma - p\beta, 1}(z) \prec R_{\gamma - p\beta, p}(z)$, $z \in U$, we have

$$\gamma - \beta h_1(z) \prec R_{\gamma - p\beta, 1}(z), \quad \gamma - \beta h_2(z) \prec R_{\gamma - p\beta, p}(z), \quad z \in U.$$

Therefore, from Theorem 1.9 (for $n = 1$ and $h = h_1$, respectively $n = p$ and $h = h_2$) we have the existence of the analytic solutions q_1 and q_2 of the equation (2.11), respectively (2.12).

Since we have conditions (*ii*) and (*iii*) in the hypothesis, we obtain from Theorem 1.10 the univalence of q_1 and q_2.

From $g \in \Sigma S_p(h_1, h_2)$ and (*i*) we have

$$\gamma - \beta h_1(z) \prec \gamma + \beta \frac{zg'(z)}{g(z)} \prec \gamma - \beta h_2(z) \prec R_{\gamma - p\beta, 1}(z), \quad z \in U. \tag{2.14}$$

Since $R_{\gamma-p\beta,1}(z) \prec R_{\gamma-p\beta,p}(z)$, $z \in U$, we have from (2.14)

$$\gamma + \beta \frac{zg'(z)}{g(z)} \prec R_{\gamma-p\beta,p}(z), \quad z \in U.$$

Using Corollary 1.13 we have $G = J_{p,\beta,\gamma}(g) \in \Sigma_p$ with $z^p G(z) \neq 0$, $z \in U$. Consequently,

$$P \in H[p,p], \quad \text{where} \quad P(z) = -\frac{zG'(z)}{G(z)}, \quad z \in U.$$

From (1.9) and $g \in \Sigma S_p(h_1, h_2)$ we obtain

$$h_1(z) \prec P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \prec h_2(z), \quad z \in U. \tag{2.15}$$

It is easy to see that we have $P \in H[p,p] \cap Q$ and $P(z) + \dfrac{zP'(z)}{\gamma - \beta P(z)}$ univalent in U.

We remark that the conditions from the hypotheses of Theorem 1.10 and Theorem 1.12 are met, so, using these two theorems we get from (2.15) that

$$q_1(z) \prec P(z) \prec q_2(z), \quad z \in U. \tag{2.16}$$

Since $P(z) = -\dfrac{zG'(z)}{G(z)}$, $z \in U$, and $G \in \Sigma_p$ we obtain from (2.16) that

$$G \in \Sigma S_p(q_1, q_2).$$

Of course, we also have from Theorem 1.10 and Theorem 1.12, that the functions q_1 and q_2 are the best subordinant and, respectively, the best (p,p)-dominant. □

From Theorem 1.9, since $p \neq 0$, we have that the solutions q_1 and q_2 (from the above theorem) are given by:

$$q_1(z) = z^{\gamma} H_1^{-p\beta}(z) \left[-\beta \int_0^z H_1^{-p\beta}(t) t^{\gamma-1} dt \right]^{-1} + \frac{\gamma}{\beta} =$$

$$= \left[-\beta \int_0^1 \left[\frac{H_1(tz)}{H_1(z)} \right]^{-p\beta} t^{\gamma-1} dt \right]^{-1} + \frac{\gamma}{\beta},$$

$$q_2(z) = z^{\frac{\gamma}{p}} H_2^{-\beta}(z) \left[\frac{-\beta}{p} \int_0^z H_2^{-\beta}(t) t^{\frac{\gamma}{p}-1} dt \right]^{-1} + \frac{\gamma}{\beta} =$$

$$= \left[\frac{-\beta}{p} \int_0^1 \left[\frac{H_2(tz)}{H_2(z)} \right]^{-\beta} t^{\frac{\gamma}{p}-1} dt \right]^{-1} + \frac{\gamma}{\beta},$$

where

$$H_k(z) = z \exp \int_0^z \frac{h_k(t) - p}{pt} dt, \quad k = 1, 2.$$

If we consider only the subordination for Theorem 2.9 we obtain the next result.

Theorem 2.10. *Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. Also let $h \in H(U)$ with $h(0) = p$ such that*

$$(i) \quad \gamma - \beta h(z) \prec R_{\gamma - \beta p, p}(z), \quad z \in U.$$

If q is the analytic solution of the Briot-Bouquet differential equation

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z), \quad z \in U,$$

with $q(0) = p$ and if

$$(ii) \quad h \quad \text{is convex or} \quad \frac{zq'(z)}{\gamma - \beta q(z)} \quad \text{is starlike,}$$

then q is univalent in U.

Moreover, if $g \in \Sigma S_p(h)$ and $G = J_{p,\beta,\gamma}(g)$, then $G \in \Sigma S_p(q)$.

The function q is the best (p, p)-dominant.

Proof. The fact that the function q is univalent in U results from Theorem 1.10. Since $g \in \Sigma S_p(h)$ we have

$$-\frac{zg'(z)}{g(z)} \prec h(z), \quad z \in U, \tag{2.17}$$

and using (i) we obtain

$$\gamma + \beta \frac{zg'(z)}{g(z)} \prec R_{\gamma - p\beta, p}(z), \quad z \in U.$$

Using now Corollary 1.13 we get that $G = J_{p,\beta,\gamma}(g) \in \Sigma_p$ with $z^p G(z) \neq 0$, $z \in U$. Hence, $P \in H[p, p]$, where $P(z) = -\dfrac{zG'(z)}{G(z)}$, $z \in U$. We know that

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad z \in U,$$

and using (2.17) we get

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \prec h(z), \quad z \in U.$$

Using now Theorem 1.10 for $a = n = p$ and with $-\beta$ instead of β, we obtain that $P(z) \prec q(z)$, so

$$-\frac{zG'(z)}{G(z)} \prec q(z), \quad z \in U. \tag{2.18}$$

Since $G \in \Sigma_p$ we have from (2.18) that $G \in \Sigma S_p(q)$.

It is obvious that the function q is the best (p,p)-dominant. □

If we consider, in the above theorem, that the function h is convex we obtain the corollary:

Corollary 2.11. *Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. Also let $g \in \Sigma S_p(h)$ with h convex in U, $h(0) = p$. If the function h satisfies the condition*

$$\gamma - \beta h(z) \prec R_{\gamma - \beta p, p}(z), \quad z \in U,$$

then

$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z), \quad z \in U,$$

with $q(0) = p$.

The function q is the best (p,p)-dominant.

Next, we present an application for the above corollary, when $\beta = 1, \gamma \in \mathbb{R}$, for a particular function h. We will use the notation $J_{p,\gamma}$ instead of $J_{p,1,\gamma}$.

Corollary 2.12. *Let $p \in \mathbb{N}^*$ and $\gamma \geq p + 3$ such that $4p(\gamma - p)^2 \leq \gamma(\gamma - p - 1)^3$.*

If $g \in \Sigma S_p(h)$ with $h(z) = p + z + \dfrac{pz}{\gamma - p - z}$, then

$$G = J_{p,\gamma}(g) \in \Sigma S_p(p + z),$$

which is equivalent to $\left|\dfrac{zG'(z)}{G(z)} + p\right| < 1$, $z \in U$. Therefore,

$$p - 1 < \mathrm{Re}\left[-\frac{zG'(z)}{G(z)}\right] < p + 1, \quad z \in U,$$

this meaning that $G \in \Sigma_p^*(p - 1, p + 1)$.

Proof. Considering $\alpha = p$, $\beta = -1$, $\gamma \to \gamma - p$ in Lemma 2.2, we remark that the conditions from this lemma are met in the hypothesis of this corollary, so, the function $h(z) = p + z + \dfrac{pz}{\gamma - p - z}$ is convex in U.

It is easy to see that the function $q(z) = p + z$ is the univalent solution for the differential equation

$$q(z) + \frac{pzq'(z)}{\gamma - q(z)} = h(z), \quad z \in U, \quad \text{with} \quad q(0) = p.$$

Next we verify that $|\mathrm{Im}\, h(z)| < C_p(\gamma - p)$, $z \in U$, which is equivalent to

$$\left|\mathrm{Im}\left(z + \frac{pz}{\gamma - p - z}\right)\right| < \sqrt{p^2 + 2p(\gamma - p)}, \quad z \in U.$$

We have

$$\left|\mathrm{Im}\left(z + \frac{pz}{\gamma - p - z}\right)\right| = \left|\mathrm{Im}\left[z - p - \frac{p(\gamma - p)}{z - \gamma + p}\right]\right| \le |\mathrm{Im}\, z| + p(\gamma - p)\left|\mathrm{Im}\, \frac{1}{z - \gamma + p}\right|.$$

If we denote $\gamma - p$ with a we have from the hypothesis $a \ge 3$ and

$$\left|\mathrm{Im}\, \frac{1}{z - a}\right| = \frac{|\mathrm{Im}\, z|}{|z - a|^2} < \frac{1}{|z - a|^2} \le \frac{1}{(a - \mathrm{Re}\, z)^2} \le \frac{1}{a}, \quad z \in U, \quad a \ge 3,$$

so

$$\left|\mathrm{Im}\, \frac{1}{z - \gamma + p}\right| < \frac{1}{\gamma - p}, \quad z \in U.$$

Therefore, we get $\left|\mathrm{Im}\left(z + \dfrac{pz}{\gamma - p - z}\right)\right| < p + 1$, $z \in U$, so $|\mathrm{Im}\, h(z)| < p + 1$, $z \in U$.

Now it is obvious that we have $|\mathrm{Im}\, h(z)| < \sqrt{p^2 + 2p(\gamma - p)} = C_p(\gamma - p)$, hence $|\mathrm{Im}\,[\gamma - h(z)]| < C_p(\gamma - p)$, $z \in U$, this means that

$$\gamma - h(z) \prec R_{\gamma - p, p}(z), \quad z \in U.$$

Therefore, from Corollary 2.11, we obtain

$$G = J_{p,\gamma}(g) \in \Sigma S_p(p + z),$$

which is equivalent to $\left|\dfrac{zG'(z)}{G(z)} + p\right| < 1$, $z \in U$. □

If we consider for Corollary 2.11 the condition $\mathrm{Re}\,[\gamma - \beta h(z)] > 0$, $z \in U$, instead of $\gamma - \beta h(z) \prec R_{\gamma - \beta p, p}(z)$, $z \in U$, we get:

Corollary 2.13. *Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. Also let $g \in \Sigma S_p(h)$ with h convex in U and $h(0) = p$. If*

$$\mathrm{Re}\,[\gamma - \beta h(z)] > 0, \quad z \in U,$$

then

$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z), \quad z \in U,\ q(0) = p.$$

The function q is the best (p,p)-dominant.

Proof. The result follows from Corollary 2.11. □

Since for Corollary 2.13 we have $q \prec h$ (see Theorem 1.11), we get the next corollary:

Corollary 2.14. *Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. Also let $g \in \Sigma S_p(h)$ with h convex in U and $h(0) = p$. If*

$$\mathrm{Re}\,[\gamma - \beta h(z)] > 0, \quad z \in U,$$

then

$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(h).$$

Furthermore, using Corollary 2.14 for a particular function h, we present a result which was also obtained in [8] but using a different method.

We consider $h(z) = h_{p,\alpha}(z) = \dfrac{p + (p - 2\alpha)z}{1 - z}$, $z \in U$, where $p \in \mathbb{N}^*$ and $0 \leq \alpha < p$. It is not difficult to see that $h_{p,\alpha}(U) = \{z \in \mathbb{C}/\mathrm{Re}\, z > \alpha\}$ and $h_{p,\alpha}(0) = p$.

Using the notations given at the beginning of this paper we have

$$g \in \Sigma S_p(h_{p,\alpha}) \Leftrightarrow g \in \Sigma_p^*(\alpha).$$

We now get the next result:

Corollary 2.15. [8] *Let $p \in \mathbb{N}^*$, $\beta < 0$, $\gamma \in \mathbb{C}$ and $\dfrac{\mathrm{Re}\,\gamma}{\beta} \leq \alpha < p$. Then we have*

$$g \in \Sigma_p^*(\alpha) \Rightarrow G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha).$$

Proof. From $\dfrac{\mathrm{Re}\,\gamma}{\beta} \leq \alpha < p$ and $\beta < 0$ we have $\mathrm{Re}\,\gamma - \beta\alpha \geq 0$ and $\mathrm{Re}\,\gamma - p\beta > 0$.

It is easy to see that

$$\mathrm{Re}\,\gamma - \beta \mathrm{Re}\, h_{p,\alpha}(z) > \mathrm{Re}\,\gamma - \alpha\beta \geq 0, \quad z \in U,$$

hence $\mathrm{Re}\,[\gamma - \beta h_{p,\alpha}(z)] > 0$, $z \in U$.

We know that $g \in \Sigma_p^*(\alpha) \Leftrightarrow g \in \Sigma S_p(h_{p,\alpha})$.

Since the conditions from Corollary 2.14 holds, we get $G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(h_{p,\alpha})$ which is equivalent to $G \in \Sigma_p^*(\alpha)$. □

REFERENCES

[1] T. Bulboacă, *Differential subordinations and superordinations. Recent results*, Casa Cărţii de Ştiinţa, Cluj-Napoca, 2005.

[2] S.S. Miller, P.T. Mocanu, *Briot-Bouquet differential equations and differential subordinations*, Complex Variables **33** (1997), 217–237.

[3] S.S. Miller, P.T. Mocanu, *Differential subordinations. Theory and applications*, Marcel Dekker Inc., New York, Basel, 2000.

[4] S.S. Miller, P.T. Mocanu, *Subordinants of differential superordinations*, Complex Variables **48** (2003) 10, 815–826.

[5] S.S. Miller, P.T. Mocanu, *Briot-Boquet differential superordinations and sandwich theorems*, J. Math. Anal. Appl. **329** (2007), 327–335.

[6] P.T. Mocanu, *Some simple criteria for starlikeness and convexity*, Libertas Mathematica **13** (1993), 27–40.

[7] Gh. Oros, G.I. Oros, *An application of Briot-Bouquet differential subordinations*, Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica **50** (2006), 101–104.

[8] A. Totoi, *On integral operators for meromorphic functions*, General Mathematics **18** (2010) 3, 91–108.

Alina Totoi
totoialina@yahoo.com

"Lucian Blaga" University of Sibiu
Faculty of Science
Department of Mathematics
Str. Dr. I. Ratiu, no. 5, Sibiu, Romania

Received: July 7, 2010.
Revised: November 12, 2010.
Accepted: November 15, 2010.