Opuscula Mathematica • Vol. 31 • No. 4 • 2011

http://dx.doi.org/10.7494/OpMath.2011.31.4.549

FREE PROBABILITY INDUCED BY ELECTRIC RESISTANCE NETWORKS ON ENERGY HILBERT SPACES

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Abstract. We show that a class of countable weighted graphs arising in the study of electric resistance networks (ERNs) are naturally associated with groupoids. Starting with a fixed ERN, it is known that there is a canonical energy form and a derived energy Hilbert space $H_{\mathcal{E}}$. From $H_{\mathcal{E}}$, one then studies resistance metrics and boundaries of the ERNs. But in earlier research, there does not appear to be a natural algebra of bounded operators acting on $H_{\mathcal{E}}$. With the use of our ERN-groupoid, we show that $H_{\mathcal{E}}$ may be derived as a representation Hilbert space of a universal representation of a groupoid algebra $\mathfrak{A}_G$, and we display other representations. Among our applications, we identify a free structure of $\mathfrak{A}_G$ in terms of the energy form.

Keywords: directed graphs, graph groupoids, electric resistance networks, ERN-groupoids, energy Hilbert spaces, ERN-algebras, free moments, free cumulants.

Mathematics Subject Classification: 05C62, 05C90, 17A50, 18B40, 47A99.

1. INTRODUCTION

While the study of weighted infinite graphs G has a long history of interacting with a host of diverse areas of mathematics as well as applications (see the references cited below), its symbiotic relationship with the theory of operators in Hilbert space is of a more recent vintage. A case in point is the paper [11] dealing with sampling of signals with the sampling taking place on a prescribed but irregular point configuration. By a weighted graph we mean a pair of sets, vertices $V(G)$ and edges $E(G)$, and a positive function c defined on $E(G)$. The simplest model described very well by this is an *electrical resistance network*. In this case, the function c assigning weights to the edges is then representing a system of resistors assigned to the edges in G, the value on an edge of the function c is the reciprocal of the assigned resistance. For every pair (G, c), we introduce an associated difference operator Δ and a Hilbert space $H_{\mathcal{E}}$ such that Δ is a Hermitian operator, typically unbounded, with a natural and dense domain in $H_{\mathcal{E}}$. The operator Δ is defined intrinsically from a fixed (G, c), but in a special case, it is a discretized Laplace operator, so a discretization of a Laplace operator in classical analysis, in divergence form. Moreover, in this case, the function c is related to divergence. But we emphasize that not all discrete models with operator Δ are discretizations of classical PDE Laplacians. In the discrete models, the operator Δ plays a number of roles: It allows us to introduce such tools as random walk, spectral theory, and associated algebras of operators in $H_{\mathcal{E}}$. With this we get an associated mean value property: the Δ-harmonic functions h on $V(G)$ have the following discrete mean-value property: *For every x in $V(G)$, the value $h(x)$ is computed as a weighted average of the values $h(y)$, as the vertices y range over the neighbors to x, i.e., the points y such that (x, y) is in $E(G)$.* This idea in turn gives rise to a family of transfer operators, and radial operators (Definition 5.1). Our main results deal with the spectral theory of these operators. In Section 6, for a fixed (G, c), we compute a free probability model for an algebra of operators acting on $H_{\mathcal{E}}$.

Since we address several audiences, operator theory and operator algebras, as well as readers with interest in discrete analysis, before getting to our main theorems (Sections 5 and 6) we must first introduce a number of definitions and preliminaries.

Recently, the second named author and E. Pearse have studied *electric resistance networks* (ERNs) with the use of bounded and unbounded operators in Hilbert space (e.g., see [6] through [10]). Independently, the first named author initiated an approach to graph groupoids based on free probability theory and representations (e.g., see [1] through [5]).

In this paper, we combine these tools in a study of countable weighted graphs as they are used in the analysis of electric resistance networks. To do that, we re-establish electric resistance networks, but now as graph groupoids (intuitively a certain family of directed graphs). Further, we identify a canonical class of representations (of the groupoids) as a tool for analysis on electric resistance networks.

As an application, we offer a new energy calculus for groupoid-algebras of electric resistance networks. We begin with a definition of electric resistance networks as weighted directed graph G, specifically a system of resisters configured on the edges in a (typically) infinite graph.

We use this in computations of voltage configurations and current flows. In our analysis, the resisters in G are represented by weights on the edges of G. Directions in G in turn are prescribed by induced currents; a direction of an edge is determined by the sign of current. We then build corresponding graph groupoids from this, called the *ERN-groupoids*.

They in turn have groupoid actions, and we are interested in induced algebras of bounded operators on the energy Hilbert space $H_{\mathcal{E}}$, corresponding ERN-energy forms.

There are several early uses of tools from mathematics in the study of large electric networks, probability, random walks, harmonic analysis, spectral theory, to mention only a few (e.g., see [28]). We offer a sample of related problems, but caution the reader that the list is extensive, and we do not attempt anywhere a complete list.

While weighted graphs and random walks have a multitude of applications, let us mention one here which illuminates main ideas from our technical discussion. Consider a large network of resistors arranged on the vertices in a graph G. Here, we will have

G representing an electrical network. Of main interest to us is the case of models using an countable graph of vertices and specified weighted edges.

Below we list the precise mathematical axioms which are of relevance both for the physics (the laws of Ohm and of Kirchhoff) and for the mathematical models. The following intuitive notions may be helpful. We will be using functions on both of the set of vertices and of the set of edges: At the out set we have a fixed and given positive function c defined on the edges of G. This function c is taken to represent the reciprocal of fixed resistors arranged on each edge of G. One question which can be resolved with the use of our methods is that of determining induced currents. Consider for example two vertices, say x and y, in G. If one Amp is inserted at x, and then extracted at y, what is the induced current in G? What is the voltage drop from x to y? The latter number is a resistance metric, turning the set of vertices into a metric space.

For computations, it is useful to be able to evaluate two quadratic forms, one represents the energy of a voltage function on vertices, and the other the dissipation of a current, a function on edges.

An early pioneering paper using harmonic analysis in ERNs is [29] by Dole and Snell (see also [6]); and by now present authors with E. Pearse developed a global analysis of infinite weighted simplicial graphs (or network) based on a systematic study of different but related boundaries; we mention [7–9], and [10], and the papers cited there. However, the study of ERNs has not lent itself to natural algebras of bounded operators. Our aim here is to adapt tools from the theory of representations of groupoids to the solution of problems in the study of ERNs.

There is a variety of such problems in the literature, but studied with different methods, for example [12, 25] on geometry and telecommunications networks, [13] studying Jacobi matrices on trees, [14, 15] on random directed networks, [16, 24] on fractal networks and Sierpinski gaskets, [26] in potential theory, [17] on flow conductance in complex networks, [25] on equilibrium measures, [18] on impedance networks and resonances. See also [31, 32], and [33].

While groupoids have been used in representation theoretic problems (e.g., see [19]) on inequalities for measured groupoids and [20–23], their use in ERNs has so far gone unnoticed.

2. DEFINITIONS AND BACKGROUND

Starting with a graph G, for operator theory, we introduce a Hilbert space H_G naturally coming from G. Our approach is as follows: From G, pass to an enveloping groupoid $\mathbb{G}$ and an associated involutive algebra $\mathcal{A}_G$. We then introduce a conditional expectation E of $\mathcal{A}_G$ onto the subalgebra $\mathcal{D}_G$ of diagonal elements. To get a representation of $\mathcal{A}_G$ and an associated Hilbert space H_G, we then use the Stinespring construction on E (e.g., see [5]). In this section, we introduce the concepts and definitions we use.

2.1. GRAPH GROUPOIDS

Let G be a directed graph with its vertex set $V(G)$ and its edge set $E(G)$. Let $e \in E(G)$ be an edge connecting a vertex v_1 to a vertex v_2. Then we write $e = v_1 e v_2$, for emphasizing the initial vertex v_1 of e and the terminal vertex v_2 of e. For a fixed graph G, we define the oppositely directed graph G^{-1}, with $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{e^{-1} : e \in E(G)\}$, where each element e^{-1} of $E(G^{-1})$ satisfies that

$$e = v_1 e v_2 \quad \text{in} \quad E(G), \quad \text{with} \quad v_1, v_2 \in V(G),$$

if and only if

$$e^{-1} = v_2 e^{-1} v_1, \quad \text{in} \quad E(G^{-1}).$$

This opposite directed edge $e^{-1} \in E(G^{-1})$ of $e \in E(G)$ is called the *shadow of* e. Also, this new graph G^{-1} is said to be the *shadow of* G. Note that $(G^{-1})^{-1} = G$.

Define the *shadowed graph* $\widehat{G}$ of G by a directed graph with its vertex set

$$V(\widehat{G}) = V(G) = V(G^{-1})$$

and its edge set

$$E(\widehat{G}) = E(G) \cup E(G^{-1}).$$

We say that two edges $e_1 = v_1 e_1 v_1'$ and $e_2 = v_2 e_2 v_2'$ are *admissible*, if $v_1' = v_2$, equivalently, the finite path $e_1 e_2$ is well-defined on $\widehat{G}$. Similarly, if w_1 and w_2 are finite paths on $\widehat{G}$, then we say w_1 and w_2 are *admissible*, if $w_1 w_2$ is a well-defined finite path on $\widehat{G}$, too. Similar to the edge case, if a finite path w has its initial vertex v and its terminal vertex v', then we write $w = v_1 w v_2$. Notice that every admissible finite path is a word in $E(\widehat{G})$. Denote the set of all finite path by $FP(\widehat{G})$. Then $FP(\widehat{G})$ is the subset of the set $E(\widehat{G})^*$, consisting of all finite words in $E(\widehat{G})$. Suppose we take a part

$$\begin{tikzcd} & \bullet \arrow[r, "e_3"] & \cdots \\ \cdots \arrow[r, "e_1"'] & \bullet \arrow[u, "e_2"'] & \end{tikzcd}$$

in a graph G or in the shadowed graph $\widehat{G}$, where e_1, e_2, e_3 are edges of G, respectively of $\widehat{G}$. Then the above admissibility shows that the edges e_1 and e_2 are admissible, since we obtain a finite path $e_1 e_2$, however, the edges e_1 and e_3 are not admissible, since a finite path $e_1 e_3$ is undefined.

Construct the *free semigroupoid* $\mathbb{F}^+(\widehat{G})$ of the shadowed graph $\widehat{G}$, as the union of all vertices in $V(\widehat{G}) = V(G) = V(G^{-1})$ and admissible words in $FP(\widehat{G})$, equipped with its binary operation, the *admissibility*. Naturally, we assume that $\mathbb{F}^+(\widehat{G})$ contains the *empty word* $\emptyset$, as the representative of all undefined (or non-admissible) finite words in $E(\widehat{G})$.

Remark that some free semigroupoid $\mathbb{F}^+(\widehat{G})$ of $\widehat{G}$ does not contain the empty word; for instance, if a graph G is a one-vertex-multi-edge graph, then the shadowed graph $\widehat{G}$ of G is also a one-vertex-multi-edge graph too, and hence its free semigroupoid

$\mathbb{F}^+(\widehat{G})$ does not have the empty word. However, in general, if $|V(G)| > 1$, then $\mathbb{F}^+(\widehat{G})$ always contain the empty word. Thus, if there is no confusion, we always assume the empty word $\emptyset$ is contained in the free semigroupoid $\mathbb{F}^+(\widehat{G})$ of $\widehat{G}$.

Definition 2.1. By defining the *reduction* (RR) on $\mathbb{F}^+(\widehat{G})$, we define the graph groupoid $\mathbb{G}$ of a given graph G, by the subset of $\mathbb{F}^+(\widehat{G})$, consisting of all "reduced" finite paths on $\widehat{G}$, with the inherited admissibility on $\mathbb{F}^+(\widehat{G})$ under (RR), where the *reduction* (RR) on $\mathbb{G}$ is as follows:

(RR) $\qquad ww^{-1} = v$ and $w^{-1}w = v'$,

for all $w = vwv' \in \mathbb{G}$, with $v, v' \in V(\widehat{G})$.

Such a graph groupoid $\mathbb{G}$ is indeed a categorial groupoid with its base $V(\widehat{G})$.

2.2. GRAPH-GROUPOID ALGEBRAS

Let $\mathcal{X}$ be a set and let $\mathcal{Y}$ be a subset of $\mathcal{X}$. Assume that each element x of $\mathcal{X}$ has its initial moment y_1 and its terminal moment y_2, where y_1, y_2 are contained in $\mathcal{Y}$. We denote this relation by $x = y_1 x y_2$. Clearly, every element y in $\mathcal{Y}$ is regarded as an element of $\mathcal{X}$, having its initial and terminal moments identified with itself. i.e., $y = yyy$. Thus, generally, we can conclude that

$$y^n = y, \quad \text{for all} \quad y \in \mathcal{Y}, \quad \text{and} \quad n \in \mathbb{N},$$

in $\mathcal{X}$. So, there exist functions

$$s, r : \mathcal{X} \to \mathcal{Y}$$

such that

$$s(x) = y_1, \quad \text{and} \quad r(x) = y_2,$$

whenever $x = y_1 x y_2$ in $\mathcal{X}$, with $y_1, y_2 \in \mathcal{Y}$. We call s and r, the *source map* and the *range map* on $\mathcal{X}$.

Definition 2.2. Let $\mathcal{X}$, $\mathcal{Y}$, s, and r be given as above. We say that the algebraic quadruple $\mathcal{X} = (\mathcal{X}, \mathcal{Y}, s, r)$ is a (*categorial*) *groupoid*, if it satisfies the followings conditions for binary operation $(\cdot)$:

(1) $x_1 x_2$ is well-defined only when $r(x_1) = s(x_2)$, for $x_1, x_2 \in \mathcal{X}$.
(2) $(x_1 x_2)x_3 = x_1(x_2 x_3)$, for $x_1, x_2, x_3 \in \mathcal{X}$.

The subset $\mathcal{Y}$ of $\mathcal{X}$ is called the *base of* $\mathcal{X}$.

In our case, we will define the empty element $\emptyset$ in $\mathcal{X}$, to represent the case where the products are undefined. So, if there is no confusion, then we always assume that the empty element $\emptyset$ is contained in groupoids. However, it is possible that certain groupoids have no empty words.

The merit of the empty element is that we can make the partially defined binary operation (satisfying (1)) be well-defined. i.e., the condition (1) is the above definition can be re-written by

$$x_1 x_2 = \begin{cases} x_1 x_2 & \text{if } r(x_1) = s(x_2), \\ \emptyset & \text{otherwise,} \end{cases}$$

for all $x_1, x_2 \in \mathcal{X}$. Trivially, all our graph groupoids are indeed groupoids with their bases, the vertex sets, in the above sense. Also, all groups are groupoids having their bases consisting only of group identities. Here, notice that all groups are the groupoids without empty word.

Definition 2.3. Let $\mathcal{X}$ be a groupoid. The (pure algebraic) algebra $A\lg(\mathcal{X})$, generated by $\mathcal{X}$, is called the groupoid algebra of $\mathcal{X}$. i.e., $A\lg(\mathcal{X}) = \mathbb{C}[\mathcal{X}]$, the algebra consisting of all polynomials in $\mathcal{X}$. However, in this paper, we also consider the pure algebraic algebra $\mathbb{C}[[\mathcal{X}]]$, consisting of all "formal" series in $\mathcal{X}$, too. And we denote $\mathbb{C}[[\mathcal{X}]]$ by $\mathfrak{A}_{\mathcal{X}}$.

Remember, by definition, the elements of $A\lg(\mathcal{X})$ are finite linear sums of $\mathcal{X}$, but the elements of $\mathfrak{A}_{\mathcal{X}}$ are either finite or infinite linear sums.

Similarly, for a graph groupoid $\mathbb{G}$, we can construct the corresponding groupoid algebra $\mathfrak{A}_{\mathbb{G}} = \mathbb{C}[[\mathbb{G}]]$. In such a case, we will denote it by $\mathfrak{A}_G$, to emphasize this groupoid algebra is induced by a graph G.

Definition 2.4. We call the groupoid algebra $\mathfrak{A}_G = \mathbb{C}[[\mathbb{G}]]$, generated by the graph groupoid $\mathbb{G}$ of a graph G, the *graph-groupoid algebra induced by* G.

Precisely, the graph-groupoid algebra $\mathfrak{A}_G$ have its zero element,

$$\emptyset = 0_{\mathfrak{A}_G},$$

and its identity element

$$1_{\mathfrak{A}_G} = \sum_{v \in V(G)} v.$$

Every element a of $\mathfrak{A}_G$ has its expression,

$$a = \sum_{w \in \mathbb{G}} t_w w \quad \text{with} \quad t_w \in \mathbb{C},$$

where $\sum_{w \in \mathbb{G}}$ means a finite or infinite sum.

Let $a = \sum_{w \in \mathbb{G}} t_w w$ be an element of the graph-groupoid algebra $\mathfrak{A}_G$. Define the subset $Supp(a)$ of $\mathbb{G}$ by

$$Supp(a) \overset{def}{=} \{w \in \mathbb{G} : t_w \neq 0 \quad \text{in} \quad \mathbb{C}\}.$$

This subset $Supp(a)$ is called the *support of* a. By definition,

$$|Supp(a)| \leq \infty.$$

By the admissibility on the graph groupoid $\mathbb{G}$ of a graph G, we obtain the following multiplication rule:

$$w_1 w_2 = \begin{cases} w_1 w_2 & \text{if } w_1 w_2 \neq \emptyset \quad \text{in} \quad \mathbb{G}, \\ 0_{\mathfrak{A}_G} & \text{if } w_1 w_2 = \emptyset \quad \text{in} \quad \mathbb{G}, \end{cases}$$

for all $w_1, w_2 \in \mathbb{G} \subset \mathfrak{A}_G$. Define now the unary operation $(*)$ on the graph-groupoid algebra $\mathfrak{A}_G$ by

$$* : w \in \mathbb{G} \subset \mathfrak{A}_G \longmapsto w^* \overset{def}{=} w^{-1} \in \mathfrak{A}_G,$$

with the linearity,

$$(t_1 w_1 + t_2 w_2)^* = \overline{t_1} w_1^{-1} + \overline{t_2} w_2^{-1},$$

for all $w_1, w_2 \in \mathbb{G} \subset \mathfrak{A}_G$, and $t_1, t_2 \in \mathbb{C}$, where $\overline{t}$ means the complex conjugate of t, for all $t \in \mathbb{C}$. By the uniqueness of the shadow w^{-1} for a fixed element w in $\mathbb{G}$, the element w^* for w in $\mathfrak{A}_G$ is uniquely determined. We will call $w^* = w^{-1}$ of w, the *adjoint* (or the *shadow*) of w, for $w \in \mathbb{G} \subset \mathfrak{A}_G$.

Proposition 2.5. *The graph-groupoid algebra $\mathfrak{A}_G$, generated by the graph groupoid $\mathbb{G}$ of a graph G, is a (pure algebraic) $*$-algebra.*

2.3. GROUPOID ACTIONS

Let $\mathcal{X} = (\mathcal{X}, \mathcal{Y}, s, r)$ be an arbitrary groupoid and let A be a set. We say that $\mathcal{X}$ *acts on* A, if (i) there exists a function g from $\mathcal{X}$ into $\mathfrak{F}(A)$, such that, for any fixed groupoid element $x \in \mathcal{X}, g(x) : A \to A$ is a well-defined function on A, where $\mathfrak{F}(A)$ means the collection of functions on A, and (ii) the images $g(x_1)$ and $g(x_2)$ satisfy that

$$g(x_1) \circ g(x_2) = g(x_1 x_2), \quad \text{on} \quad A,$$

for all $x_1, x_2 \in \mathcal{X}$, where $(\circ)$ means the usual functional composition. Sometimes, we call A, an $\mathcal{X}$- *set.* Also, we say that the function $g : \mathcal{X} \to \mathfrak{F}(A)$ is a *groupoid action* of $\mathcal{X}$ (acting) on A. As groupoids, our graph groupoids can have their groupoid actions. Canonical actions induced by graphs are introduced in [1, 2], and [3]. By considering groupoid elements as multiplication operators on certain Hilbert spaces, they become natural groupoid actions on Hilbert spaces. Such groupoid actions induce groupoid dynamical systems (e.g., [3, 4, 4, 6], and [5]).

2.4. UNIONS OF GRAPHS

Let G_1 and G_2 be countable directed graphs. Define a new graph G by a directed graph with its vertex set

$$V(G) = V(G_1) \cup V(G_2),$$

and its edge set

$$E(G) = E(G_1) \cup E(G_2).$$

Such a new graph G is called the *unioned graph of* G_1 *and* G_2. And we denote G by $G_1 \cup G_2$, to emphasize that the graph G is induced by G_1 and G_2. By definition, all "disjoint" unioned graphs are our unioned graph. Recall that the disjoint unioned graph U, denoted by $G_1 \sqcup G_2$, of G_1 and G_2 is the graph with

$$V(U) = V(G_1) \sqcup V(G_2),$$

and

$$E(U) = E(G_1) \sqcup E(G_2),$$

where $\sqcup$ on the above right-hand sides mean the disjoint union (set-theoretically).

However, if either

$$V(G_1) \cap V(G_2), \quad \text{or} \quad E(G_1) \cap E(G_2)$$

is nonempty, then our unioned graph $G_1 \cup G_2$ is completely different from the disjoint unioned graph $G_1 \sqcup G_2$. For instance, if

$$G_1 = \begin{array}{ccc} \bullet & & \\ \downarrow & & \\ \bullet & \rightarrow & \bullet \end{array}, \quad \text{and} \quad G_2 = \bullet \rightrightarrows \bullet,$$

then the disjoint unioned graph $G_1 \sqcup G_2$ is clearly,

$$\begin{array}{ccc} \bullet & & \\ \downarrow & & \\ \bullet & \rightarrow & \bullet \end{array} \qquad \bullet \rightrightarrows \bullet.$$

Assume now that the above graphs G_1 and G_2 satisfy

$$G_1 = \begin{array}{ccc} {}^{x_1}\bullet & & \\ {}^{e}\downarrow & & \\ {}^{x_2}\bullet & \rightarrow & \bullet \end{array}, \quad \text{and} \quad G_2 = {}_{x_1}\bullet \underset{e}{\rightrightarrows} \bullet_{x_2}.$$

Then the unioned graph $G_1 \cup G_2$ is determined by a "connected" graph

$$\begin{array}{ccc} \bullet & & \\ \downarrow\downarrow & & \\ \bullet & \rightarrow & \bullet \end{array}.$$

Therefore, we conclude that all disjoint unioned graphs are unioned graphs, but not all unioned graphs are disjoint unioned graphs. Another good examples for unioned graphs would be our shadowed graphs. Indeed, if G is a graph, then the shadowed graph $\widehat{G}$ of G is the unioned graph

$$\widehat{G} = G \cup G^{-1}$$

of G, and its shadow G^{-1}. It is shown that the groupoid sum $\mathbb{G}_1 + \mathbb{G}_2$ of two graph groupoids $\mathbb{G}_1$ and $\mathbb{G}_2$ is groupoid-isomorphic to the graph groupoid $\mathbb{G}$ of the unioned graph $G = G_1 \cup G_2$ in [5], where $\mathbb{G}_k$ are the graph groupoids of G_k, for $k = 1, 2$. Equivalently, the study of groupoid sums of graph groupoids is to investigate other graph groupoids determined by the unioned graphs. Clearly, the groupoid direct sum $\mathbb{G}_1 \oplus \mathbb{G}_2$ of $\mathbb{G}_1$ and $\mathbb{G}_2$ is groupoid-isomorphic to the graph groupoid $\mathbb{G}$ of the disjoint unioned graph $G = G_1 \sqcup G_2$.

3. ELECTRIC RESISTANCE NETWORKS (ERNS)

We refer to [7] for the study of electric resistance networks, and to [6] for the study of operator theory on electric resistance networks. In this section, we re-define electric resistance networks to apply our graph-groupoidal research (e.g., [1] through [5]). i.e., we construct graph groupoids induced by electric resistance networks and study certain operator algebras generated by electric resistance networks.

Remark first that the original electric resistance network theory assumes that electric resistance networks are weighted "undirected" graphs. However, graph groupoids are induced by "directed" graphs. We can solve this problem because we can regard our shadowed graphs of directed graphs as undirected versions of graphs.

3.1. NETWORKS AND NETWORK-GROUPOIDS

Let X be an arbitrary countable discrete set. Define a set $\mathcal{V}$ of positive-real-valued functions on X,

$$\mathcal{V} \overset{def}{=} \{v : X \to \mathbb{R}^+ : v \quad \text{is a function}\} \tag{3.1}$$

satisfying the additional properties;

$$v_1, v_2 \in \mathcal{V} \Longrightarrow v_1 + v_2 \in \mathcal{V} \tag{3.2}$$

$$\alpha \in \mathbb{R}^+, \quad \text{and} \quad v \in \mathcal{V} \Longrightarrow \alpha v \in \mathcal{V}, \quad \text{where} \quad \mathbb{R}^+ \overset{def}{=} \{r \in \mathbb{R} : r > 0\}. \tag{3.3}$$

Definition 3.1. We call a set $\mathcal{V}$, satisfying (3.1), (3.2), and (3.3), a *voltage set on* X.

For instance, let $v_0 : X \to \mathbb{R}^+$ be an arbitrary function. Then the set,

$$\mathcal{V}_0 = \{\alpha v_0 : \alpha \in \mathbb{R}^+\} \tag{3.4}$$

is a voltage set on X, too, since

$$n v_0 = \underbrace{v_0 + \ldots + v_0}_{n\text{-ties}} \in \mathcal{V}_0,$$

for all $n \in \mathbb{N}$, and

$$\alpha v_0 \in \mathcal{V}_0, \quad \text{for all} \quad \alpha \in \mathbb{R}^+.$$

Definition 3.2. The voltage sets $\mathcal{V}_0$ on X, determined only by fixed functions v_0 (like (3.4)), are called *single-voltage sets* (or the v_0*-voltage sets*) *on* X.

Let X be a countable discrete set and $\mathcal{V}$, a fixed voltage set on X. Define the collection $\mathcal{I}$ by a certain $\mathbb{R}$-valued set of functions on $X \times X$,

$$\mathcal{I} \overset{def}{=} \left\{ i_v : X \times X \to \mathbb{R} \;\middle|\; \begin{array}{c} i_v\left((x,y)\right) = v(x) - v(y) \\ \forall v \in \mathcal{V},\ \forall (x,y) \in X \times X \end{array} \right\}. \tag{3.5}$$

Definition 3.3. The set $\mathcal{I}$ of (3.5), induced by a set X and a fixed voltage set $\mathcal{V}$ on X, is called the *current set on X induced by* $\mathcal{V}$. If $\mathcal{V}$ is the v_0-voltage set on X, where $v_0 : X \to \mathbb{R}^+$ is a fixed function, then the corresponding current set $\mathcal{I}$ is called the v_0 -current set on X. Also, in this case, we denote $\mathcal{I}$ by $\mathcal{I}_{v_0}$.

By definition, we obtain the following fundamental facts.

Proposition 3.4. *Let $\mathcal{I}$ be the current set on X induced by a voltage set $\mathcal{V}$.*

(1) *Each element $i_v \in \mathcal{I}$ is skew-symmetric, in the sense that*

$$i_v((x, y)) = -i_v((y, x)),$$

for all $(x, y) \in X \times X$.

(2) *If $i_{v_1}, i_{v_2} \in \mathcal{I}$, then $i_{v_1} + i_{v_2} = i_{v_1+v_2}$ in $\mathcal{I}$.*

(3) *If $\alpha \in \mathbb{R} \setminus \{0\}$, and $i_v \in \mathcal{I}$, then*

$$\alpha i_v = \begin{cases} i_{\alpha v} & \text{if } \alpha > 0, \\ -i_{|\alpha| v} & \text{if } \alpha < 0, \end{cases}$$

in $\mathcal{I}$.

Proof. (1) By definition, it is clear. Since

$$i_v((x,\ y)) \overset{def}{=} v(x) - v(y),$$

we have

$$i_v((x,\ y)) = v(x) - v(y) = -(v(y) - v(x)) = -i_v((y,\ x)),$$

for all $(x, y) \in X \times X$. Thus, each element i_v in $\mathcal{I}$ is skew-symmetric.

(2) Let $i_{v_1}, i_{v_2} \in \mathcal{I}$. Then, for any $(x, y) \in X \times X$,

$$\begin{aligned}(i_{v_1} + i_{v_2})((x,\ y)) &= i_{v_1}((x,\ y)) + i_{v_2}((x,\ y)) = \\ &= v_1(x) - v_1(y) + v_2(x) - v_2(y) = \\ &= (v_1 + v_2)(x) - (v_1 + v_2)(y) = \\ &= i_{v_1+v_2}((x,\ y)).\end{aligned}$$

Notice here that, by definition, if $v_1, v_2 \in \mathcal{V}$, then $v_1 + v_2 \in \mathcal{V}$. And hence $i_{v_1+v_2}$ is well-defined in $\mathcal{I}$. (3) Let $\alpha > 0$ in $\mathbb{R}$. Then

$$\begin{aligned}\alpha i_v((x,\ y)) &= \alpha(v(x) - v(y)) = \\ &= \alpha v(x) - \alpha v(y) = i_{\alpha v}((x,\ y)), \quad \text{for all} \quad (x, y) \in X \times X.\end{aligned}$$

Therefore, if $\alpha > 0$, then

$$\alpha i_v = i_{\alpha v}, \quad \text{for all} \quad v \in \mathcal{V}.$$

Recall that, if $\alpha \in \mathbb{R}^+$, and $v \in \mathcal{V}$, then $\alpha v \in \mathcal{V}$, too.

Assume that $\alpha < 0$ in $\mathbb{R}$. Then

$$\begin{aligned}\alpha i_v((x,\ y)) &= \alpha(v(x) - v(y)) = \\ &= -\alpha(v(y) - v(x)) = (-\alpha v)(y) - (-\alpha v)(x) = \\ &= i_{-\alpha v}((y,\ x)) = -i_{-\alpha v}((x,\ y))\end{aligned}$$

by (1)

$$= -i_{|\alpha| v}((x,\ y)).$$

$\square$

By the statements (2) and (3) of the above proposition, the current set $\mathcal{I}$ on X induced by a fixed voltage set $\mathcal{V}$ is a real-vector-space-like set.

Now, we construct graphs determined by a countable discrete set X, a fixed voltage set $\mathcal{V}$ on X, and the current on X. First, fix v in $\mathcal{V}$. Construct a graph G_v as a directed graph with its vertex set

$$V(G_v) = X,$$

and its edge set

$$E(G_v) = \{(x, y) \in X \times X : i_v\left((x,\ y)\right) > 0\}.$$

We call G_v, the *v-graph on X*. The edges of the v-graph G_v represent the current (or the flow of current) on X when we put the voltage v on X. Recall that we say two graphs G_1 and G_2 are *graph-isomorphic*, if there exists a bijection

$$g : V(G_1) \cup E(G_1) \to V(G_2) \cup E(G_2)$$

such that

(i) g is bijective from $V(G_1)$ onto $V(G_2)$,
(ii) g is bijective from $E(G_1)$ onto $E(G_2)$,
(iii) $g(e) = g(x_1 e x_2) = g(x_1)g(e)g(x_2)$ in $E(G_2)$, for all $e = x_1 e x_2 \in E(G_1)$, with $x_1, x_2 \in V(G_1)$.

In particular, the bijection g is called a *graph-isomorphism.* In [1] and [2], we showed that if two graphs G_1 and G_2 have graph-isomorphic shadowed graphs $\widehat{G_1}$ and $\widehat{G_2}$, then the corresponding graph groupoids $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomorphic.

Proposition 3.5. *Fix $v \in \mathcal{V}$, and let G_v be the v-graph on X, and $G_{\alpha v}$, the αv-graph on G, for all $\alpha \in \mathbb{R}^+$. Then they are graph-isomorphic from each other, for all $\alpha \in \mathbb{R}^+$.*

Proof. Let $v \in \mathcal{V}$ be fixed and let $\alpha \in \mathbb{R}^+$ be arbitrary. For the fixed voltage v, the v-graph G_v is well-defined, and since $\alpha v \in \mathcal{V}$, we can have the αv-graph $G_{\alpha v}$, too. By definition, the vertex set $V(G_{\alpha v})$ satisfies

$$V(G_{\alpha v}) = X = V(G_v), \tag{3.6}$$

and the edge set $E(G_{\alpha v})$ is defined by the subset

$$E(G_{\alpha v}) = \{(x, y) \in X \times X : i_{\alpha v}\left((x,\ y)\right) > 0\}$$

in $X \times X$. Since $i_{\alpha v} = \alpha i_v$, for $\alpha \in \mathbb{R}^+$, we conclude that

$$i_{\alpha v}\left((x,\ y)\right) > 0 \Longleftrightarrow i_v\left((x,\ y)\right) > 0$$

for $(x, y) \in X \times X$. Therefore,

$$E(G_{\alpha v}) = E(G_v). \tag{3.7}$$

Thus, by (3.6) and (3.7), we can define a bijection

$$g : V(G_v) \cup E(G_v) \to V(G_{\alpha v}) \cup E(G_{\alpha v}) \tag{3.8}$$

by

$$g(y) = y, \quad \text{for all} \quad y \in V(G_v) \cup E(G_v).$$

It is easy to check that the map g of (3.8) is a graph-isomorphism. So, the graphs G_v and $G_{\alpha v}$ are graph-isomorphic. Since $\alpha \in \mathbb{R}^+$ is arbitrary, the graphs G_v and $\{G_{\alpha v}\}_{\alpha \in \mathbb{R}^+}$ are graph-isomorphic from each other. □

The above proposition says that the v-graph G_v is kind of a representative of all αv-graphs $G_{\alpha v}$, for all $\alpha \in \mathbb{R}^+$. So, from now on, if we simply mention about v-graphs G_v, then they are also regarded as αv-graphs $G_{\alpha v}$, for all $\alpha \in \mathbb{R}^+$. Also, we obtain the following proposition.

Proposition 3.6. *Let $v = v_1 + v_2$ in the voltage set $\mathcal{V}$ on X, for $v_1, v_2 \in \mathcal{V}$. Then the $(v_1 + v_2)$-graph $G_{v_1+v_2}$ is graph-isomorphic to the unioned graph $G_{v_1} \cup G_{v_2}$.*

Proof. Let $v_1, v_2 \in \mathcal{V}$. Then the corresponding voltage graphs G_{v_1} and G_{v_2} are well-defined. Since $v_1 + v_2$ is also an element in $\mathcal{V}$, we have the $(v_1 + v_2)$-graph $G_{v_1+v_2}$. By definition,

$$\begin{aligned} V(G_{v_1+v_2}) \quad &= X = V(G_{v_1}) = V(G_{v_2}) = \\ &= V(G_{v_1}) \cup V(G_{v_2}), \end{aligned} \tag{3.9}$$

and

$$E(G_{v_1+v_2}) = \{(x, y) \in X \times X : i_{v_1+v_2}((x,\ y)) > 0\}.$$

Consider the edge set $E(G_{v_1+v_2})$. Since $i_{v_1+v_2} = i_{v_1} + i_{v_2}$, we can get that $(x, y) \in E(G_{v_1+v_2})$, if and only if

$$i_{v_1+v_2}((x,\ y)) > 0,$$

if and only if one of the followings holds

$$i_{v_1}((x,\ y)) > 0, \quad \text{and} \quad i_{v_2}((x,\ y)) > 0, \quad \text{or} \tag{3.10}$$

$$i_{v_1}((x,\ y)) > i_{v_2}((x,\ y)), \quad \text{or} \tag{3.11}$$

$$i_{v_1}((x,\ y)) < i_{v_2}((x,\ y)). \tag{3.12}$$

The condition (3.10) holds, if and only if

$$(x, y) \in E(G_{v_1}) \cap E(G_{v_2}),$$

and the condition (3.11) (resp., the condition (3.12)) holds, if and only if

$$(x, y) \in E(G_{v_1}) \quad (\text{resp., } (x, y) \in E(G_{v_2})).$$

Therefore, we get that

$$(x, y) \in E(G_{v_1+v_2}) \Longleftrightarrow (x, y) \in E(G_{v_1}) \cup E(G_{v_2}).$$

Equivalently,

$$E(G_{v_1+v_2}) = E(G_{v_1}) \cup E(G_{v_2}). \tag{3.13}$$

So, by (3.9) and (3.13), the $(v_1 + v_2)$-graph $G_{v_1+v_2}$ is graph-isomorphic to the unioned graph $G_{v_1} \cup G_{v_2}$, via a graph-isomorphism, defined like (3.8). □

The above proposition is one of our motivation to re-define electric resistance networks. Now, let

$$\mathcal{G} \overset{def}{=} \{G_v : v \in \mathcal{V}\}$$

be the collection of voltage graphs on X induced by a voltage set $\mathcal{V}$. The iterated unioned graph

$$G = \cup\mathcal{G} = \underset{v\in\mathcal{V}}{\cup} G_v \tag{3.14}$$

is well-defined. This new graph G is called the *directed network on X induced by voltage $\mathcal{V}$*.

Definition 3.7. Let X be a countable set and $\mathcal{V}$, a voltage set on X, and let $\mathcal{G} = \{G_v\}_{v\in\mathcal{V}}$ be the collection of all voltage-graphs. The iterated unioned graph $G = \cup\mathcal{G}$ of (3.14) is called the direct network on X induced by voltage $\mathcal{V}$. The shadow G^{-1} of G is called the negative network, and the corresponding graph groupoid $\mathbb{G}$ of G is called the network groupoid.

Observe the detailed property of the directed network G.

Recall now that we say a graph G is *simplicial*, if (i) G has no loop-edges, and (ii) G does not allow multi-edges. Here, *loop-edges* are the edges e satisfying $e = xex$, for some $x \in V(G)$. And *multi-edges* mean more than one edge, connecting same initial vertex to same terminal vertex, for example, if a graph G contains a pair (x_1, x_2) of vertices satisfying

$$_{x_1}\bullet \rightrightarrows \bullet_{x_2},$$

then G allows multi-edges (in fact, 2-edges) connecting x_1 to x_2. Note here that, if a graph G is either

$$\bullet \leftrightarrows \bullet, \quad \text{or} \quad \bullet \rightleftarrows \bullet,$$

then it has no multi-edges, because there is no multi-edges connecting same initial vertex to same terminal vertex.

Proposition 3.8. *Let G be the direct network on a set X induced by voltage $\mathcal{V}$. Then G is simplicial.*

Proof. Let G be the direct network. To show G is simplicial, we need to prove this graph G contains neither loop-edges nor multi-edges. Assume now that G contains a loop-edge $l = xlx$ with $x \in X = V(G)$. This means that the pair $(x, x) \in X \times X$ is contained in the edge set $E(G)$. Since G is the unioned graph $\underset{v\in\mathcal{V}}{\cup} G_v$ of voltage-graphs G_v, there exists at least one $v \in \mathcal{V}$, such that (x, x) is contained in $E(G_v)$. Equivalently,

$$i_v\left((x,\ x)\right) = v(x) - v(x) > 0.$$

This contradicts the definition of G_v's, for all $v \in \mathcal{V}$ (and hence the definition of G). Therefore, the directed network G does not have loop-edges.

Assume now that there exist two distinct edges e_1 and e_2 in $E(G)$, such that

$$e_k = x_1 e_k x_2, \quad \text{with} \quad x_1, x_2 \in X = V(G),$$

for all $k = 1, 2$. Then there exist(s) $v_1, v_2 \in \mathcal{V}$, such that

$$e_1, e_2 \in E(G_{v_1}), \quad (\text{or, } e_1, e_2 \in E(G_{v_2})) \quad \text{or} \tag{3.15}$$

$$e_1 \in E(G_{v_1}) \quad \text{and } e_2 \in E(G_{v_2}) \quad (\text{or}, e_1 \in E(G_{v_2}) \quad \text{and } e_2 \in E(G_{v_1})). \tag{3.16}$$

Suppose first that (3.16) holds. By definition of voltage graphs and unioned graph, if e_1 and e_2 satisfies (3.16), then they are identically same edges, i.e., $e_1 = e_2$ in $E(G)$. So, it contradicts our assumption.

Let's assume that (3.15) holds. Then both e_1 and e_2 are represented as (x_1, x_2) in $X \times X$ (in $E(G_{v_1})$ or $E(G_{v_2})$). This means that they are identical element in $E(G)$. It contradicts our assumption. Thus the directed network G does not allow multi-edges.

So, the graph G has neither loop-edges nor multi-edges, and hence it is simplicial.

□

The above proposition shows that our directed network G on a countable set X induced by voltage $\mathcal{V}$ is simplicial. So, without loss of generality, we can write the length-k reduced finite paths by the $(k+1)$-tuples of vertices, for all $k \in \mathbb{N}$. i.e., if w is a length-k reduced finite path in the network groupoid $\mathbb{G}$, then there exists $(k+1)$-vertices $x_1, \ldots, x_{k+1} \in X = V(\widehat{G})$, such that

$$w = (x_1, \ldots, x_{k+1}),$$

for $k \in \mathbb{N}$. Again, by the simpliciality of G, the above tuple-notation is uniquely determined for each reduced finite path in $\mathbb{G}$.

Now, consider the currents i_v on $E(\widehat{G})$ more in detail, for $v \in \mathcal{V}$. We can extend the current i on $E(\widehat{G})$ to the current, also denoted by i, on $\mathbb{G}$. i.e., we define

$$i_v : \mathbb{G} \to \mathbb{R}$$

by a function

$$i_v(w) \overset{def}{=} \begin{cases} \sum_{j=1}^{k} i_v\left((x_j, x_{j+1})\right) & \text{if } w = (x_1, \ \ldots, \ x_{k+1}) \in FP_r(\widehat{G}), \\ \sum_{w \sim x} i_v\left((w, x)\right) & \text{if } w \in V(\widehat{G}), \\ 0 & \text{if } w = \emptyset, \end{cases} \tag{3.17}$$

for all $w \in \mathbb{G}$, and for all $v \in \mathcal{V}$, where

$$x_1 \sim x_2 \overset{def}{\iff} \exists (x_1, x_2), (x_2, x_1) \in E(\widehat{G}), \tag{3.18}$$

for all $x_1, x_2 \in X = V(\widehat{G})$.

Lemma 3.9. *Let $(x_1, \ldots, x_{k+1})$ be a reduced finite path in the network groupoid $\mathbb{G}$. Then*

$$i_v\left((x_1, \ldots, x_{k+1})\right) = v(x_1) - v(x_{k+1}), \tag{3.19}$$

for all $v \in \mathcal{V}$, for all $k \in \mathbb{N}$.

Proof. The proof is straightforward. Indeed,

$$i_v\left((x_1,\ \ldots,\ x_{k+1})\right) = \sum_{j=1}^{k} i_v\left((x_j,\ x_{j+1})\right) =$$
$$= \sum_{j=1}^{k} \left(v(x_j) - v(x_{j+1})\right) = v(x_1) - v(x_{j+1}), \quad \text{for} \quad k \in \mathbb{N}. \qquad \square$$

3.2. OHM'S LAW AND ERNS

Let X be a countable set and $\mathcal{V}$, a voltage set on X, and let $G = \underset{v \in \mathcal{V}}{\cup} G_v$ be the directed network on X induced by $\mathcal{V}$, with its network groupoid $\mathbb{G}$. Like in basic physics, consider *Ohm's law*,

$$V = RI, \quad \text{equivalently,} \quad I = cV,$$

where V, R, I, $c = \frac{1}{R}$ mean the voltage, the *resistance*, the current, and the *conductance*, respectively.

Since the network groupoid $\mathbb{G}$ of the direct network G is determined by the voltage and current, the resistance (or the conductance) on $\mathbb{G}$ would be defined naturally. Remark that, as a graph groupoid, the network groupoid $\mathbb{G}$ is generated by the edge set $E(\widehat{G})$ of the network $\widehat{G}$, which is the shadowed graph of the directed network G. So, we can define the conductance c on $E(\widehat{G})$, and then we may extend it to that on $\mathbb{G}$. Define the *conductance c on $E(\widehat{G})$* by a function,

$$c : E(\widehat{G}) \to \mathbb{R}^{+} \tag{3.20}$$

satisfying

$$c\left((x,\ y)\right) = c\left((y,\ x)\right),$$

for all $(x, y) \in E(\widehat{G})$. i.e., the conductance is the symmetric positive-real-valued function on $E(\widehat{G})$. For convenience, we denote $c\left((x,\ y)\right)$ simply by c_{xy}, for all $(x, y) \in E(\widehat{G})$.

Remark 3.10. If a conductance c is determined on $E(\widehat{G})$, then the resistance R is also well-defined on $E(\widehat{G})$. By physics,

$$R \overset{def}{=} \frac{1}{c} : E(\widehat{G}) \to \mathbb{R}^{+}.$$

Since c is nonzero, the rational function R is well-defined on $E(\widehat{G})$.

By the Ohm's law, we define *the currents I_v with conductance c* by a function

$$I_v : E(\widehat{G}) \to \mathbb{R}$$

such that

$$I_v\left((x,\ y)\right) \overset{def}{=} c_{xy} i_v\left((x,\ y)\right) = c_{xy}\left(v(x) - v(y)\right),$$

for all $(x, y) \in E(\widehat{G})$, and for all $v \in \mathcal{V}$.

Proposition 3.11. *Let I_v be the currents with conductance c, for $v \in \mathcal{V}$.*

(1) *I_v is skew-symmetric on $E(\widehat{G})$, for all $v \in \mathcal{V}$.*
(2) *$I_{v_1+v_2} = I_{v_1} + I_{v_2}$, on $E(\widehat{G})$, for all $v_1, v_2 \in \mathcal{V}$.*
(3) $\alpha I_v = \begin{cases} I_{\alpha v} & \text{if } \alpha > 0, \\ -I_{|\alpha| v} & \text{if } \alpha < 0, \end{cases}$ *on $E(\widehat{G})$, for all $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof. (1) For any $v \in \mathcal{V}$, the current I_v with conductance c is defined by

$$I_v\left((x,\ y)\right) = c_{xy} i_v\left((x,\ y)\right).$$

Thus, we have

$$\begin{aligned} I_v\left((x,\ y)\right) = c_{xy}\left(v(x) - v(y)\right) = \\ = -c_{xy}\left(v(y) - v(x)\right) = -c_{yx} i_v\left((y, x)\right) = \\ = -I_v\left((y,\ x)\right), \quad \text{for all} \quad (x, y) \in E(\widehat{G}). \end{aligned}$$

Therefore, each current I_v with conductance c is skew-symmetric.

(2) Take $v_1, v_2 \in \mathcal{V}$, and $(x, y) \in E(\widehat{G})$. Then

$$\begin{aligned} I_{v_1+v_2}\left((x,\ y)\right) = c_{xy} i_{v_1+v_2}\left((x,\ y)\right) = \\ = c_{xy}\left((i_{v_1} + i_{v_2})((x,\ y))\right) \end{aligned}$$

since $i_{v_1+v_2} = i_{v_1} + i_{v_2}$, for all $v_1, v_2 \in \mathcal{V}$

$$\begin{aligned} &= c_{xy} i_{v_1}\left((x,\ y)\right) + c_{xy} i_{v_2}\left((x,\ y)\right) = \\ &= I_{v_1}\left((x,\ y)\right) + I_{v_2}\left((x,\ y)\right) = \\ &= \left(I_{v_1} + I_{v_2}\right)\left((x,\ y)\right). \end{aligned}$$

Therefore, $I_{v_1+v_2} = I_{v_1} + I_{v_2}$ on $E(\widehat{G})$.

(3) Let $\alpha \in \mathbb{R} \setminus \{0\}$. First, assume that $\alpha > 0$. Take arbitrary v in $\mathcal{V}$. Then

$$\begin{aligned} I_v\left((x,\ y)\right) = \alpha\left(c_{xy} i_v(x,\ y)\right) = \\ = c_{xy}\left(\alpha i_v\left((x,\ y)\right)\right) = c_{xy} i_{\alpha v}\left((x,\ y)\right) = \\ = I_{\alpha v}\left((x,\ y)\right) \end{aligned}$$

for all $(x, y) \in E(\widehat{G})$, because if $\alpha > 0$, then $\alpha i_v = i_{\alpha v}$. Thus, if $\alpha > 0$, then $\alpha I_v = I_{\alpha v}$. Assume now that $\alpha < 0$ in $\mathbb{R}$. Then

$$\begin{aligned} \alpha I_v\left((x,\ y)\right) = c_{xy}\left(\alpha i_v\left((x,\ y)\right)\right) = c_{xy}\left(-i_{-\alpha v}\left((x,\ y)\right)\right) = -c_{xy} i_{|\alpha| v}\left((x,\ y)\right) = \\ = -I_{|\alpha| v}\left((x,\ y)\right) \end{aligned}$$

for all $(x, y) \in E(\widehat{G})$, because if $\alpha < 0$, then $\alpha i_v = -i_{|\alpha| v}$. Thus, if $\alpha < 0$, then $\alpha I_v = -I_{|\alpha| v}$. $\square$

Now, we can define *the current set $\mathcal{I}_c$ with conductance c* by

$$\mathcal{I}_c \overset{def}{=} \left\{ I_v : E(\widehat{G}) \to \mathbb{R} \;\middle|\; \begin{array}{c} I_v \text{ are the current} \\ \text{with conductance } c, \\ \forall v \in \mathcal{V} \end{array} \right\}.$$

Now, extend the conductance c of (3.20) on $E(\widehat{G})$ to the *conductance, also denoted by c, on* $\mathbb{G}$. Define the $\mathbb{R}_0^+$-valued function

$$c : \mathbb{G} \to \mathbb{R}_0^+$$

by

$$c(w) = c_w \overset{def}{=} \begin{cases} \sum_{j=1}^{k} c_{x_j x_{j+1}} & \text{if } w = (x_1, \ \ldots, \ x_{k+1}) \in FP_r(\widehat{G}), \\ \sum\limits_{w \sim x} c_{wx} & \text{if } w \in V(\widehat{G}), \\ 0 & \text{if } w = \emptyset, \end{cases} \tag{3.21}$$

for all $w \in \mathbb{G}$, where

$$\mathbb{R}_0^+ \overset{def}{=} \mathbb{R}^+ \cup \{0\}.$$

Similarly, extend the current I on $E(\widehat{G})$ with conductance c to the current, also denoted by I, on $\mathbb{G}$ with conductance c. Define

$$I_v\,(w) \overset{def}{=} c_w i_v\,(w)\,, \tag{3.22}$$

for all $w \in \mathbb{G}$, and for all $v \in \mathcal{V}$. Recall that $i_v(w)$ is defined in (3.17), and $c_w = c(w)$ is defined in (3.19).

So, if $w = (x_1, \ldots, x_{k+1})$ is a reduced finite path in $\mathbb{G}$, then

$$I_v(w) = c_w i_v(w) = \left(\sum_{j=1}^{k} c_{x_j x_{j+1}} \right) (v(x_1) - v(x_{k+1}))\,. \tag{3.23}$$

And, if w is a vertex in $\mathbb{G}$, then

$$I_v(w) = c_w i_v(w) = \left(\sum_{w \sim x} c_{wx} \right) \left(\sum_{w \sim x} i_v\,((w, \ x)) \right), \tag{3.24}$$

where the relation $w \sim x$ is defined in (3.18).

Finally, if $w = \emptyset$, then $I_v(w) = 0$.

Definition 3.12. Let $\widehat{G}$ be the network on a countable set X induced by the direct network G and by voltage $\mathcal{V}$, and let $\mathbb{G}$ be the network groupoid of G. Assume that we define the conductance c on $\mathbb{G}$ as in (3.21) (extended by (3.20) on $E(\widehat{G})$). Then the weighted graph $\widehat{G} = (\widehat{G}, c)$ is called the electric resistance network (in short, ERN). The weighted groupoid $\mathbb{G} = (\mathbb{G}, c)$ is called the ERN-groupoid.

4. REPRESENTATIONS OF ERNS

Throughout this section, let X be a countable set, and $\mathcal{V}$, a voltage set on X, and let G be the direct network on X induced by $\mathcal{V}$. Also, let $\widehat{G} = (\widehat{G}, c)$ be the ERN, and $\mathbb{G} = (\mathbb{G}, c)$, the ERN-groupoid, where c is a conductance in the sense of (3.21).

Since the ERN-groupoid $\mathbb{G}$ is a groupoid in the sense of Section 2.3, we may consider suitable groupoid actions of $\mathbb{G}$ acting on certain Hilbert spaces. We will consider two Hilbert spaces where $\mathbb{G}$ acts.

4.1. ENERGY HILBERT SPACE

Recall that a set C is *convex*, if, for any $t \in [0,1]$,

$$tx_1 + (1-t)x_2 \in C,$$

for all $x_1, x_2 \in C$, where $[0,1]$ is the closed interval in $\mathbb{R}$. Let X be an arbitrary set. Then the *convex hull* $conH(X)$ *of* X is defined to be a set of all covex combinations in X. i.e.,

$$conH(X) \overset{def}{=} \cap \left\{ C \,\middle|\, \begin{array}{c} C \text{ is convex, and} \\ X \subseteq C \end{array} \right\}.$$

By definition, the set X, itself, is convex, if and only if $conH(X) = X$. Similarly, we say that a set D is a *convex cone*, if, for any $t \in \mathbb{R}_0^+$, if (i) D is convex, and (ii) for any $t \in \mathbb{R}^+, tx \in D$, too, where $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$. For an arbitrary set X, the *convex cone* $conC(X)$ *of* X is defined to be a set,

$$conC(X) \overset{def}{=} \cap \left\{ C \,\middle|\, \begin{array}{c} C \text{ is a convex cone, and} \\ X \subseteq C \end{array} \right\}.$$

Fix an arbitrary countable set X, from now on. Take a subset V of the set of all positive-real-valued functions on X. i.e.,

$$V = \{f : X \to \mathbb{R}^+ : f \text{ is a function}\}.$$

Construct the convex cone $conC(V)$ of the set V. We will denote this convex cone $conC(V)$ of V simply by $\mathcal{V}$, and we call $\mathcal{V}$, a *voltage set on* X. Let $\mathcal{V}$ be a voltage set on X. Construct a vector space $\mathfrak{V}$ generated by $\mathcal{V}$. i.e., $\mathfrak{V}$ is the set of all linear combinations of voltages in $\mathcal{V}$, equipped with the vector addition $(+)$, defined by the usual functional addition, and the usual $\mathbb{C}$-scalar product.

Remark that, by definition, a voltage set $\mathcal{V}$ satisfies

$$v_1, v_2 \in \mathcal{V} \Longrightarrow v_1 + v_2 \in \mathcal{V},$$

and

$$\alpha \in \mathbb{R}^+, v \in \mathcal{V} \Longrightarrow \alpha v \in \mathcal{V},$$

since $\mathcal{V}$ is a convex cone. So, the vector space $\mathfrak{V}$ generated by $\mathcal{V}$ is naturally determined, by defining the natural $\mathbb{C}$-scalar product.

On $\mathfrak{V}$, define an inner product $\langle , \rangle_c$ on $\mathfrak{V}$ by the sesqui-linear form satisfying that

$$\langle v_1, v_2 \rangle_c \overset{def}{=} \frac{1}{2} \sum_{(x,y) \in E(G)} c_{xy} \left(i_{v_1} \left((x,\ y) \right) \right) \left(i_{v_2} \left((x,\ y) \right) \right), \tag{4.1}$$

having its norm $\|\cdot\|_c$ on $\mathfrak{V}$, satisfying

$$\|v\|_c^2 = \frac{1}{2} \sum_{(x,y)} c_{xy} \left| i_v \left((x,\ y) \right) \right|^2. \tag{4.2}$$

We call the inner product (4.1) and (4.2), the *energy form* and the *energy norm* on $\mathfrak{V}$, respectively.

Remark 4.1. (1) In (4.1), we need to keep in mind that the sum $\sum\limits_{(x,y) \in E(G)}$ is over the edge set $E(G)$ of the "directed" network, not the edge set $E(\widehat{G})$ of the network! If we take the sum over $E(\widehat{G})$, then the form (4.1) becomes 0, for any $v_1, v_2 \in \mathcal{V}$. Indeed, the edge $(x, y) \in E(\widehat{G})$, if and only if its shadow $(y, x) \in E(\widehat{G})$. So, for a summand

$$c_{xy} \left(v_1(x) - v_1(y) \right) \left(v_2(x) - v_2(y) \right),$$

there always exists its pair,

$$c_{yx} \left(v_1(y) - v_1(x) \right) \left(v_2(y) - v_2(x) \right).$$

Since $c_{xy} = c_{yx}$, for all $(x, y) \in E(\widehat{G})$, we always obtain a factor of 2 from

$$c_{xy} \left(v_1(x) - v_1(y) \right) \left(v_2(x) - v_2(y) \right) + c_{yx} \left(v_1(y) - v_1(x) \right) \left(v_2(y) - v_2(x) \right)$$

in the sum $\sum_{(x,y) \in E(\widehat{G})}$ over $E(\widehat{G})$. And hence, if we take the sum over $E(\widehat{G})$, then the form (4.1) goes to 0.

(2) Moreover, if we take a sum $\sum_{(x,y) \in E(G)}$ over $E(G)$, then it is equivalent to the original definition of the energy form in the sense of Jorgensen and Pearse (See [6]).

(3) Under the inner product (4.1), our energy Hilbert spaces, defined below, are equivalent to the original energy Hilbert spaces in the sense of Jorgensen and Pearse (Also, see [6]).

Definition 4.2. The Hilbert space, the norm closure of the normed space $(\mathfrak{V}, \|.\|_c)$, is called the energy Hilbert space, equipped with its energy form (or its energy inner product) $\langle , \rangle_c$. We denote the energy Hilbert space by $H_{\mathcal{E}}$.

We consider the ERN-groupoid $\mathbb{G}$ acts on the energy Hilbert space $H_{\mathcal{E}}$. Indeed, there exists a groupoid action λ,

$$\lambda : \mathbb{G} \to B(H_{\mathcal{E}})$$

such that

$$\lambda(w) = \lambda_w \quad \text{for all} \quad w \in \mathbb{G},$$

where

$$\lambda_w(v)(x) \overset{def}{=} v\left(wxw^{-1}\right) \tag{4.3}$$

for all $v \in \mathcal{V} \subset \mathfrak{V}$, and for all $x \in X = V(\widehat{G})$, with identity

$$v(\emptyset) = 0, \quad \text{as an image of the zero element of} \quad H_{\mathcal{E}}.$$

Theorem 4.3. *The groupoid action λ of $\mathbb{G}$ acting on the energy Hilbert space $H_{\mathcal{E}}$ is well-defined.*

Proof. To check λ is a well-defined groupoid action, it is sufficient to show that

$$\lambda_{w_1}\lambda_{w_2} = \lambda_{w_1 w_2}, \quad \text{for} \quad w_1, w_2 \in \mathbb{G},$$

and λ_w are linear on $H_{\mathcal{E}}$, for $w \in \mathbb{G}$. By definition, λ_w are linear on $H_{\mathcal{E}}$. Let $w_1, w_2 \in \mathbb{G}$. Then

$$\begin{aligned}\left(\lambda_{w_1}\lambda_{w_2}\right)(v)(x) &= \lambda_{w_1}\left(v(w_2 x w_2^{-1})\right) = \\ &= v\left(w_1 w_2 x w_2^{-1} w_1^{-1}\right) = v\left((w_1 w_2) x (w_1 w_2)^{-1}\right) = \\ &= \lambda_{w_1 w_2}(v)(x), \quad \text{for all} \quad v \in \mathcal{V} \subset \mathfrak{V}, \text{ and } x \in X.\end{aligned}$$

Therefore,

$$\lambda_{w_1}\lambda_{w_2} = \lambda_{w_1 w_2}. \tag{4.4}$$

□

The above theorem guarantees that our ERN-groupoid $\mathbb{G}$ acts on the energy Hilbert space $H_{\mathcal{E}}$.

Definition 4.4. The pair $(H_{\mathcal{E}}, \lambda)$ of the energy Hilbert space $H_{\mathcal{E}}$ and the ERN-groupoid action λ of (4.3) is called the energy representation of $\mathbb{G}$ (or, by abusing of notation, the energy representation of ERN).

In the rest of this section, we will consider some fundamental relations between the ERN-groupoid action λ of the ERN $\mathbb{G}$, and the energy form $\langle , \rangle_c$. The following computation would be the basic tool to establish our next sections.

Theorem 4.5. *Let λ be the ERN-groupoid action of the ERN $\mathbb{G}$, acting on the energy Hilbert space $H_{\mathcal{E}}$. Then*

$$\langle \lambda_w v_1, v_2 \rangle_c = \frac{1}{2} v_1\left(s(w)\right) \sum_{r(w) \sim y} c_{r(w)y}\left(v_2\left(r(w)\right) - v_2(y)\right). \tag{4.5}$$

for all $w \in \mathbb{G}$, and for all $v_1, v_2 \in \mathcal{V} \subset \mathfrak{V} \subseteq H_{\mathcal{E}}$.

Proof. Take $w \in \mathbb{G}$, and $v_1, v_2 \in \mathcal{V}$. Then in $\mathcal{L}(H_{\mathcal{E}})$

$$\begin{aligned}
\langle \lambda_w v_1, v_2 \rangle_c &= \frac{1}{2} \sum_{(x,\ y) \in E(G)} c_{xy} \left(\lambda_w v_1(x) - \lambda_w v_1(y) \right) \left(v_2(x) - v_2(y) \right) = \\
&= \frac{1}{2} \sum_{(x,y) \in E(G)} c_{xy} \left(v_1(wxw^{-1}) - v_1(wyw^{-1}) \right) \left(v_2(x) - v_2(y) \right) = \\
&= \frac{1}{2} \sum_{(x,y) \in E(G)} c_{xy} \left(\delta_{r(w),x} v_1 \left(s(w) \right) - \delta_{r(w),y} v_1 \left(s(w) \right) \right) \left(v_2(x) - v_2(y) \right) = \\
&= \frac{1}{2} \sum_{(x,y) \in E(G)} c_{xy} \left(\delta_{r(w),x} - \delta_{r(w),y} \right) v_1 \left(s(w) \right) \left(v_2(x) - v_2(y) \right) = \\
&= \frac{1}{2} \sum_{(r(w),y) \in E(G)} c_{r(w)y} v_1 \left(s(w) \right) \left(v_2 \left(r(w) \right) - v_2(y) \right) - \\
&\quad - \frac{1}{2} \sum_{(x,\ r(w)) \in E(G)} c_{xr(w)} v_1 \left(s(w) \right) \left(v_2(x) - v_2 \left(r(w) \right) \right) =
\end{aligned}$$

since we have

$$\delta_{r(w),x} = 1 \Longleftrightarrow \delta_{r(w),y} = 0,$$

and

$$\delta_{r(w),x} = 0 \Longleftrightarrow \delta_{r(w),\ y} = 1$$

(Remark that the directed network G is simplicial, and hence if $(x, y) \in E(G)$, then $x \neq y$ in $X = V(G)$.)

$$\begin{aligned}
&= \frac{1}{2} \sum_{(r(w),y) \in E(G)} c_{r(w)y} v_1 \left(s(w) \right) \left(v_2 \left(r(w) \right) - v_2(y) \right) + \\
&\quad + \frac{1}{2} \sum_{(x,\ r(w)) \in E(G)} c_{xr(w)} v_1 \left(s(w) \right) \left(v_2 \left(r(w) \right) - v_2(x) \right) = \\
&= \frac{1}{2} \sum_{r(w) \sim y} c_{r(w)y} v_1 \left(s(w) \right) \left(v_2 \left(r(w) \right) - v_2(y) \right) =
\end{aligned}$$

since $c_{xy} = c_{yx}$, where the relation $\sim$ is defined in (3.18),

$$= \frac{1}{2} v_1 \left(s(w) \right) \sum_{r(w) \sim y} c_{r(w)y} \left(v_2 \left(r(w) \right) - v_2(y) \right). \qquad \square$$

We will use the formula (4.5) in Section 5.

4.2. DISSIPATION HILBERT SPACE

Let $X, \mathcal{V}, G, \widehat{G}, \mathbb{G}$ be given as above and let $\mathcal{I}_c$ be the current set on X with conductance c, induced by $\mathcal{V}$. This set $\mathcal{I}_c$ satisfies that:

$$I_{v_1}, I_{v_2} \in \mathcal{I}_c \Longrightarrow I_{v_1} + I_{v_2} \in \mathcal{I}_c, \quad \text{and} \tag{4.6}$$

$$\alpha \in \mathbb{R}, \quad \text{and} \quad I_v \in \mathcal{I}_c \Longrightarrow \alpha I_v \in \mathcal{I}_c. \tag{4.7}$$

The statement (4.6) holds, because if $v_1, v_2 \in \mathcal{V}$, then $v_1 + v_2 \in \mathcal{V}$, and hence $I_{v_1} + I_{v_2}$ is identical to $I_{v_1+v_2}$. Also, the statement (4.7) holds, because αI_v is determined by $I_{|\alpha|v}$, for all $\alpha \in \mathbb{R}$, and $v \in \mathcal{V}$ (See Section 3.2 above).

By (4.6) and (4.7), the current set $\mathcal{I}_c$ is a vector space over $\mathbb{R}$. So, the complexification of $\mathcal{I}_c$ is well-defined as a vector space over $\mathbb{C}$. Let's denote this complexification of $\mathcal{I}_c$ by $\mathfrak{I}_c$. Define an inner product $\langle , \rangle_I$ on the vector space $\mathfrak{I}_c$ by a sesqui-linear form satisfying that

$$\langle I_{v_1}, I_{v_2} \rangle_I \overset{def}{=} \frac{1}{2} \sum_{(x,\ y) \in E(G)} \frac{1}{c_{xy}} \left(I_{v_1} \left((x,\ y) \right) \right) \left(I_{v_2} \left((x,\ y) \right) \right), \tag{4.8}$$

for all $I_{v_1}, I_{v_2} \in \mathcal{I}_c \subset \mathfrak{I}_c$. Recall that the currents I_v with conductance c are determined by

$$I_v \left((x,\ y) \right) = c_{xy} i_v \left((x,\ y) \right) = c_{xy} \left(v(x) - v(y) \right),$$

for all $(x, y) \in E(\widehat{G})$, and $v \in \mathcal{V}$ (See (3.22)). Then it has its corresponding norm $\|.\|_I$, satisfying

$$\|I_v\|_I^2 = \frac{1}{2} \sum_{(x,y) \in E(G)} \frac{1}{c_{xy}} \left| I_v \left((x,\ y) \right) \right|^2. \tag{4.9}$$

Remark here that the sum $\sum\limits_{(x,y) \in E(G)}$ in (4.8) is over $E(G)$ (not over $E(\widehat{G})$). By definition, we obtain that the formula (4.8) can be re-written by

$$\langle I_{v_1}, I_{v_2} \rangle_I = \frac{1}{2} \sum_{(x,y) \in E(G)} c_{xy} \left(i_{v_1} \left((x,\ y) \right) \right) \left(i_{v_2} \left((x,\ y) \right) \right),$$

where i_{v_1}, i_{v_2} are currents contained in $\mathcal{I}$ in the sense of Section 3.1. Thus we obtain that:

Lemma 4.6. *Let $v_1, v_2 \in \mathcal{V} \subset \mathfrak{V}$, and let $I_{v_1}, I_{v_2} \in \mathcal{I}_c \subset \mathfrak{I}_c$ be the corresponding currents with conductance c. Then*

$$\langle I_{v_1}, I_{v_2} \rangle_I = \langle v_1, v_2 \rangle_c, \quad \textit{in} \quad \mathbb{R} \quad \textit{or in} \quad \mathbb{C}. \tag{4.10}$$

The proof is straightforward. However, keep in mind that the inner product $\langle , \rangle_I$ is defined on $\mathfrak{I}$.

Definition 4.7. The Hilbert space $H_{\mathcal{D}}$, generated by the normed space $(\mathfrak{I}, \|.\|_I)$, is called the dissipation Hilbert space.

Let $H_{\mathcal{D}}$ be the dissipation Hilbert space and let $\mathbb{G}$ be the ERN-groupoid. Define a map

$$\pi : \mathbb{G} \to \mathcal{L}(H_{\mathcal{D}})$$

by a linear (unbounded) operator on $H_{\mathcal{D}}$

$$\pi : w \in \mathbb{G} \longmapsto \pi(w) = \pi_w : H_{\mathcal{D}} \to H_{\mathcal{D}},$$

where $\mathcal{L}(H_{\mathcal{E}})$ means the linear (bounded or unbounded) operators on $H_{\mathcal{E}}$, and where

$$\pi_w\left(I_v\right)(e) \overset{def}{=} I_v\left(we\right) \quad \text{for all} \quad e \in E(\widehat{G}), \tag{4.11}$$

i.e.,

$$\begin{aligned}
\pi_w(I_v)(x,y) &= \begin{cases} \delta_{x_k,x}\ I_v\left((x_1,\ \ldots,\ x_k,\ y)\right) & \text{if } w = (x_1, \ldots, x_k) \in FP_r(\widehat{G}), \\ \delta_{w,x}\ I_v\left((w,\ y)\right) & \text{if } w \in V(\widehat{G}), \\ 0 & \text{if } w = \emptyset, \end{cases} \\
&= \begin{cases} \delta_{x_k,x} \left(\sum_{j=1}^{k} c_{x_j x_{j+1}}\right) (v(x_1) - v(y)) & \text{if } w \in FP_r(\widehat{G}), \\ \delta_{w,x}\ c_{w,y}\left(v(w) - v(y)\right) & \text{if } w \in V(\widehat{G}), \\ 0 & \text{if } w = \emptyset, \end{cases}
\end{aligned}$$

for all $v \in \mathcal{V}$, and $(x,y) \in E(\widehat{G})$. (Recall the extended definitions for the conductance c and the current i (and I). See Section 3).

Then the morphism π of $\mathbb{G}$ is a well-defined groupoid action acting on the dissipation Hilbert space $H_{\mathcal{D}}$.

Theorem 4.8. *The map π defined in* (4.11) *is a well-defined groupoid action of the ERN-groupoid $\mathbb{G}$, acting on the dissipation space $H_{\mathcal{D}}$.*

Proof. It suffices to show that π satisfies:

$$\pi_{w_1}\pi_{w_2} = \pi_{w_1 w_2} \quad \text{on} \quad H_{\mathcal{D}}, \quad \text{for all} \quad w_1, w_2 \in \mathbb{G}. \tag{4.12}$$

Let's take $w_1, w_2 \in \mathbb{G}$. Then

$$\begin{aligned}
\pi_{w_1}\pi_{w_2}(I_v)\,(e) &= \pi_{w_1}\left(\pi_{w_2}(I_v)(e)\right) = \pi_{w_1}\left(I_v(w_2 e)\right) = \\
&= I_v\left(w_1 w_2 e\right) = I_v\left((w_1 w_2)e\right) = \pi_{w_1 w_2}(I_v)(e),
\end{aligned}$$

for all $I_v \in \mathfrak{I} \subseteq H_{\mathcal{D}}$, and for all $e \in E(\widehat{G})$. It shows that the statement (4.12) holds true.

Now, take $w \in \mathbb{G}$. Then

$$\begin{aligned}
\pi_w\left(I_{v_1} + I_{v_2}\right)(e) &= \pi_w\left(I_{v_1+v_2}\right)(e) = I_{v_1+v_2}(we) = \\
&= \left(I_{v_1} + I_{v_2}\right)(we) = I_{v_1}(we) + I_{v_2}(we) = \\
&= \left(\pi_w(I_{v_1}) + \pi_w(I_{v_2})\right)(e), \quad \text{for all} \quad v_1, v_2 \in \mathcal{V}, \text{ and } e \in E(\widehat{G}).
\end{aligned}$$

And

$$\pi_w(\alpha I_v)(e) = (\alpha I_v)(we) = \alpha I_v(we) = \alpha \pi_w(I_v)(e),$$

for all $v \in \mathcal{V}, e \in E(\widehat{G})$, and for $\alpha \in \mathbb{C}$. Therefore, each π_w is linear on $H_{\mathcal{D}}$, and hence, the map π is a well-defined groupoid action of $\mathbb{G}$ acting on $H_{\mathcal{D}}$. □

The above theorem shows that the ERN-groupoid $\mathbb{G}$ acts on $H_{\mathcal{D}}$. We call π the *dissipation action of* $\mathbb{G}$.

Definition 4.9. The pair $(H_{\mathcal{D}}, \pi)$ of the dissipation space $H_{\mathcal{D}}$ and the dissipation action π of $\mathbb{G}$ is called the dissipation representation of $\mathbb{G}$.

In this paper, we concentrate on energy representation of ERN-groupoids. However, it is definitely true that the dissipation representation is very interesting.

5. ERN-ACTIONS ON ENERGY HILBERT SPACE

Let X be a countable set and $\mathcal{V}$, the voltage set on X, and let G and $\widehat{G}$ be the direct network and ERN with their groupoid $\mathbb{G}$, the ERN-network. Also, let $H_{\mathcal{E}}$ be the energy Hilbert space. In Section 4.1, we showed that the groupoid $\mathbb{G}$ acts on $H_{\mathcal{E}}$ via a groupoid action

$$\lambda : w \in \mathbb{G} \longmapsto \lambda_w \in \mathcal{L}(H_{\mathcal{E}}),$$

satisfying

$$\lambda_w(v)(x) = v(wxw^{-1}) \quad \text{for all} \quad x \in X = V(\widehat{G}),$$

for all $v \in \mathcal{V} \subset H_{\mathcal{E}}$, for $w \in \mathbb{G}$, where $\mathcal{L}(H_{\mathcal{E}})$ is the set of all bounded or unbounded operators on $H_{\mathcal{E}}$. Also, recall the formula (4.5);

$$\langle \lambda_w v_1, v_2 \rangle_c = \frac{1}{2} v_1(s(w)) \sum_{r(w) \sim y} c_{r(w)y} \left(v_2(r(w)) - v_2(y)\right),$$

for all $v_1, v_2 \in \mathcal{V} \subset H_{\mathcal{E}}$, and $w \in \mathbb{G}$. In this section, we extend the energy groupoid action λ of $\mathbb{G}$ to the representation, also denoted by λ of algebra $\mathfrak{A}_G$.

Definition 5.1. Let $\mathbb{G}$ be the ERN-groupoid of an ERN $\widehat{G}$. Define the (pure algebraic) algebra $\mathfrak{A}_G$ generated by the formal series in $\mathbb{G}$. i.e.,

$$\mathfrak{A}_G \stackrel{def}{=} \mathbb{C}[[\mathbb{G}]].$$

We call this graph-groupoid algebra $\mathfrak{A}_G$, the ERN-algebra.

Let a be an element of $\mathfrak{A}_G$. Then, by definition, it is expressed by

$$a = \sum_{w \in \mathbb{G}} t_w w, \quad \text{with} \quad t_w \in \mathbb{C}.$$

For any fixed $a \in \mathfrak{A}_G$, the *support* $Supp(a)$ *of* a is defined by a subset of $\mathbb{G}$,

$$Supp(a) \stackrel{def}{=} \{w \in \mathbb{G} : t_w \neq 0\}.$$

Thus, we can re-write that

$$a = \sum_{w \in Supp(a)} t_w w.$$

Remark 5.2. We define the ERN-algebra $\mathfrak{A}_G$ by the algebra $\mathbb{C}[[\mathbb{G}]]$ of "formal" series in $\mathbb{G}$, not the usual groupoid algebra $\mathbb{C}[\mathbb{G}]$. So, if $a \in \mathfrak{A}_G$ with its support $Supp(a)$, then

$$|Supp(a)| \leq \infty.$$

For instance, if there exists a reduced finite path $l = (x_0, x_1, \ldots, x_0)$ in $\mathbb{G}$, with $|l| \geq 2$, then we can have an element,

$$\sum_{k=-\infty}^{\infty} l^n = \lim_{n \to \infty} \sum_{k=-n}^{n} l^n, \quad \text{in} \quad \mathfrak{A}_G$$

with the identity $l^0 = x_0$ in $\mathbb{G}$. Clearly, the usual groupoid algebra $\mathbb{C}[\mathbb{G}]$ is the algebra consisting of all "finitely" supported elements in $\mathbb{G}$, and hence

$$\mathbb{C}[\mathbb{G}] \overset{\text{Subalgebra}}{\subseteq} \mathfrak{A}_G.$$

Suppose our directed network G is a finite tree. Then

$$\mathfrak{A}_G = \mathbb{C}[[\mathbb{G}]] = \mathbb{C}[\mathbb{G}].$$

However, we are working on the general case where G is either finite or infinite. Recall that our directed networks are just simplicial (finite or infinite) graphs.

Define now a morphism, also denoted by λ, on $\mathfrak{A}_G$ by a linear transformation,

$$\lambda : a \in \mathfrak{A}_G \longmapsto \lambda(a) = \lambda_a \in B(H_{\mathcal{E}}) \tag{5.1}$$

satisfying that

$$\lambda_a (v) (x) = \sum_{w \in Supp(a)} t_w \lambda_w(v)(x) = \sum_{w \in Supp(a)} t_w v \left(wxw^{-1}\right)$$

for all $v \in \mathcal{V} \subset H_{\mathcal{E}}$, and $x \in X = V(\widehat{G})$.

Proposition 5.3. *Define a unary operation*

$$(*) : \mathfrak{A}_G \to \mathfrak{A}_G$$

by

$$\left(\sum_{w \in Supp(a)} t_w \ w \right)^* \overset{def}{=} \sum_{w \in Supp(a)} \overline{t_w} w^{-1}, \tag{5.2}$$

for all $\sum_{w \in Supp(a)} t_w w \in \mathfrak{A}_G$. *Then the ERN-algebra* $\mathfrak{A}_G$ *is a* $*$*-algebra.*

Proof. To show that $\mathfrak{A}_G$ is a $*$-algebra, we need to check

$$(a+b)^* = a^* + b^*, \quad \text{for all} \quad a, b \in \mathfrak{A}_G, \tag{5.3}$$

$$a^{**} = (a^*)^* = a, \quad \text{for all} \quad a \in \mathfrak{A}_G, \tag{5.4}$$

and

$$(ab)^* = b^* a^*, \quad \text{for all} \quad a, b \in \mathfrak{A}_G. \tag{5.5}$$

By definition, the statement (5.3) holds immediately.

Now, let $w \in \mathbb{G} \subset \mathfrak{A}_G$. Then, by definition, $w^* = w^{-1}$. So,

$$w^{**} = (w^*)^* = (w^{-1})^* = (w^{-1})^{-1} = w.$$

Therefore, for any $a \in \mathfrak{A}_G, a^{**} = a$. i.e., the statement (5.4) holds true on $\mathfrak{A}_G$. Now, let $w_1, w_2 \in \mathbb{G} \subset \mathfrak{A}_G$. Then

$$(w_1 w_2)^* = (w_1 w_2)^{-1} = w_2^{-1} w_1^{-1} = w_2^* w_1^*.$$

Thus, for any $a, b \in \mathfrak{A}_G$, we get

$$(ab)^* = b^* a^*,$$

and hence the statement (5.5) holds on $\mathfrak{A}_G$. So, by (5.3), (5.4), and (5.5), the ERN-algebra $\mathfrak{A}_G$ is a (pure algebraic) $*$-algebra. □

From now on, we can regard our ERN-algebras as $*$-algebras.

Remark 5.4. On $\mathfrak{A}_G$, the unary operation (5.1) is well-defined, and hence $\mathfrak{A}_G$ becomes a $*$-algebra. However, we do not know the corresponding groupoid actions λ_w's, in the sense of (4.3), acting on the energy Hilbert space $H_{\mathcal{E}}$, satisfy

$$\lambda_w^* \overset{??}{=} \lambda_{w^{-1}} \quad \text{on} \quad B(H_{\mathcal{E}}), \quad \text{for} \quad w \in \mathbb{G}. \tag{5.6}$$

In fact, the relation (5.6) does "not" hold on $B(H_{\mathcal{E}})$, in general (See below)! This shows that the unary operation (5.1) is defined naturally as "adjoint" on $\mathfrak{A}_G$, but the adjoints λ_w^* of the corresponding actions (or representations) λ_w^* acting on $H_{\mathcal{E}}$ do "not" satisfy

$$\lambda_w^* = \lambda_{w^*} = \lambda_{w^{-1}}, \quad \text{for} \quad w \in \mathbb{G},$$

in general, on $B(H_{\mathcal{E}})$.

5.1. TRANSFER OPERATORS AND LAPLACIANS

Throughout this section, we will use the same notations used before. Define now an element T_G in the ERN-algebra $\mathfrak{A}_G$ by

$$T_G \overset{def}{=} \sum_{e \in E(\widehat{G})} \lambda_e = \sum_{e \in E(G)} (\lambda_e + \lambda_{e^{-1}}) = \sum_{e \in E(G)} (\lambda_e + \lambda_e^*). \tag{5.7}$$

This element T_G is called the *radial operator of the* ERN-*groupoid* $\mathbb{G}$ (*or the* ERN $\widehat{G}$). The radial operators in the "canonical" graph-groupoidal settings have been studied in [3,4], and [6].

The radial operator T_G of $\mathfrak{A}_G$ represents the admissibility (or the connection) induced by the ERN $\widehat{G}$. Now, consider how the radial operator T_G acts on the energy Hilbert space $H_{\mathcal{E}}$.

Theorem 5.5. *Let $T_G \in \mathfrak{A}_G$ be the radial operator of the ERN $\widehat{G}$. Consider the action λ_{T_G} on the energy Hilbert space $H_{\mathcal{E}}$. Then*

$$\lambda_{T_G}(v)(x) = \sum_{x \sim y} v(y), \tag{5.8}$$

for all $v \in \mathcal{V} \subset H_{\mathcal{E}}$, and $x \in X = V(\widehat{G})$.

Proof. Let T_G be the radial operator of $\widehat{G}$ in $\mathfrak{A}_G$. Then the groupoid action λ_{T_G} in the sense of (5.1) acts on $H_{\mathcal{E}}$ as follows:

$$\begin{aligned}\lambda_{T_G}(v)(x) &= \sum_{e \in E(\widehat{G})} \lambda_e(v)(x) = \\ &= \sum_{e \in E(\widehat{G})} v\left(exe^{-1}\right) = \sum_{e \in E(\widehat{G})} \delta_{r(e),x} v\left(s(e)\right) = \\ &= \sum_{e \in E(\widehat{G}),\ r(e) \sim y} v\left(s(e)\right) = \sum_{e \in E(\widehat{G}), e=(x,y),\ \text{or}\ e=(y,x)} v(y) =\end{aligned}$$

since $r(e) = x$

$$= \sum_{x \sim y} v(y).$$

□

Recall that Jorgensen and Pearse define a transfer operator T on the energy Hilbert space $H_{\mathcal{E}}$ by an operator,

$$T(v)(x) \overset{def}{=} \sum_{x \sim y} v(y)$$

in [6]. Thus, the following corollary is an immediate consequence of (5.8).

Corollary 5.6. *The transfer operator T in the sense of* [6] *is equivalent to the groupoid action λ_{T_G} of our radial operator T_G on $H_{\mathcal{E}}$, in the sense that*

$$T(v)(x) = \lambda_{T_G}(v)(x),$$

for all $v \in \mathcal{V} \subset H_{\mathcal{E}}$, and $x \in X = V(\widehat{G})$.

So, without loss of generality, we can say the groupoid action λ_{T_G} of the radial operator $T_G \in \mathfrak{A}_G$ is the transfer operator on $H_{\mathcal{E}}$.

Define now the conductance, also denoted by c, by the operator on the energy Hilbert space $H_{\mathcal{E}}$, such that

$$c(v)(x) \overset{def}{=} \sum_{x\sim y} c_{xy} v(x), \tag{5.9}$$

for all $v \in \mathcal{V}$, and for all $x \in X$. We will call c, the *conductance operator on* $H_{\mathcal{E}}$.

Definition 5.7. Let $T_G \in \mathfrak{A}_G$ be the radial operator of the ERN $\widehat{G}$, and let $c \in B(H_{\mathcal{E}})$ be the conductance operator on $H_{\mathcal{E}}$. Then the Laplacian (operator) Δ_G on $H_{\mathcal{E}}$ is defined by the operator

$$\Delta_G \overset{def}{=} c - \lambda_{T_G}. \tag{5.10}$$

Observe that

$$\Delta_G(v)(x) = (c - \lambda_{T_G})(v)(x) = c(v)(x) - \lambda_{T_G}(v)(x) = \sum_{x\sim y} c_{xy} v(x) - \sum_{x\sim y} c_{xy} v(y)$$

by (5.10) and (5.8)

$$= \sum_{x\sim y} c_{xy} \left(v(x) - v(y)\right) \tag{5.11}$$

for all $v \in \mathcal{V} \subset H_{\mathcal{E}}$, and $x \in X$. By (5.11), we obtain the following lemma.

Lemma 5.8. *Let Δ_G be the Laplacian on $H_{\mathcal{E}}$. Then*

$$\Delta_G(v)(x) = \sum_{x\sim y} c_{xy} i_v\left((x,\ y)\right) = \sum_{x\sim y} I_v\left((x,\ y)\right) \tag{5.12}$$

in $\mathbb{R}$.

By (5.11), and (5.12), we can obtain the following theorem.

Theorem 5.9. *Let Δ_G be the Laplacian on the energy Hilbert space $H_{\mathcal{E}}$, as in* (5.10). *Then it is equivalent to the Laplacian Δ in the sense of Jorgensen and Pearse* (*See* [6]), *on $H_{\mathcal{E}}$.*

5.2. ENERGY FORM AND ERN-ACTIONS ON $H_{\mathcal{E}}$

In this section, we consider how the energy form $\langle \bullet, \bullet \rangle_c$ on the energy Hilbert space $H_{\mathcal{E}}$ is affected by the ERN-actions $\lambda(\mathbb{G})$, more in detail. i.e., we study the formula (4.5) in special cases.

In [6], Jorgensen and Pearse define the following special types of elements in the energy Hilbert space $H_{\mathcal{E}}$.

Definition 5.10. Let $v \in \mathcal{V}$ in $H_{\mathcal{E}}$, and let Δ_G be the Laplacian on $H_{\mathcal{E}}$.

(1) v is harmonic, if $\Delta_G v = 0$, the zero element in $H_{\mathcal{E}}$.

(2) v is a dipole, if there exists $x_1, x_2 \in X = V(\widehat{G})$, such that

$$\Delta_G(v) = \delta_{x_1} - \delta_{x_2},$$

where $\delta_x : X \to \{0, 1\}$, defined by $\delta_x(y) \overset{def}{=} \delta_{x,\, y}$, for all $y \in X$. (3) For $x \in X, v_x$ is a reproducing kernel of $H_{\mathcal{E}}$, induced by $o \in X$, if

$$\langle v_x, u \rangle_c = u(x) - u(o),$$

for an arbitrary "fixed" vertex $o \in X$, called the origin.

Remark here that the origin o is arbitrary chosen in X, and in the above lemma, the reproducing kernel $\mathcal{K}_o$ of $H_{\mathcal{E}}$ is defined for the chosen origin o. By (4.5), we obtain the following theorem.

Theorem 5.11. *Let $w \in \mathbb{G}$ be an element of the ERN-algebra $\mathfrak{A}_G$, and let $v_1, v_2 \in \mathcal{V}$ be elements of the energy Hilbert space $H_{\mathcal{E}}$. Then*

$$\langle \lambda_w v_1, v_2 \rangle_c = \frac{1}{2} \left(v_1 \left(s(w) \right) \right) \left(\Delta_G(v_2) \left(r(w) \right) \right). \tag{5.13}$$

Proof. By (4.5),

$$\langle \lambda_w v_1, v_2 \rangle_c = \frac{1}{2} v_1 \left(s(w) \right) \left(\sum_{r(w) \sim y} c_{r(w)y} \left(v_2 \left(r(w) \right) - v_2(y) \right) \right)$$

for all $v_1, v_2 \in \mathcal{V} \subset H_{\mathcal{E}}$, and $w \in \mathbb{G}$. And by definition,

$$\Delta_G(v)(x) = \sum_{x \sim y} c_{xy} \left(v(x) - v(y) \right),$$

for all $v \in \mathcal{V}$, and $w \in \mathbb{G}$. Therefore,

$$\langle \lambda_w v_1, v_2 \rangle_c = \frac{1}{2} \left(v_1(s(w)) \right) \left(\Delta_G(v_2)(r(w)) \right),$$

for all $v_1, v_2 \in \mathcal{V}$, and $w \in \mathbb{G}$. $\square$

Now, we will compute the energy forms for harmonic elements, dipoles, and reproducing kernels affected by ERN-actions (or by ERN-representations). Such computation will show how the ERN-groupoid $\mathbb{G}$ (or the ERN-algebra $\mathfrak{A}_G$) acts on the energy Hilbert space $H_{\mathcal{E}}$. First, consider the energy form for harmonic elements up to the ERN-actions.

Corollary 5.12. *Let $v \in \mathcal{V} \subset H_{\mathcal{E}}$ be harmonic. Then*

$$\langle \lambda_w v, v \rangle_c = 0 \quad \textit{for all} \quad w \in \mathbb{G}, \tag{5.14}$$

and hence

$$\langle \lambda_a v, v \rangle_c = 0 \quad \textit{for all} \quad a \in \mathfrak{A}_G.$$

Proof. By (5.13), we have

$$\langle \lambda_w v, v \rangle_c = \frac{1}{2} v\,(s(w))\,(\Delta_G(v)\,(r(w)))\,.$$

Since v is harmonic, $\Delta_G(v) = 0$ on X. Therefore, the energy form, which is the left-hand side of the above equality, becomes 0. Equivalently, the formula (5.14) holds, for all $w \in \mathbb{G}$. Thus, by definition

$$\langle \lambda_a v, v \rangle_c = 0$$

for all $a \in \mathfrak{A}_G$, whenever v is harmonic. □

The above corollary is a direct consequence of the general formula (5.13), by the definition of harmonic elements.

Now, let $v \in \mathcal{V} \subset H_{\mathcal{E}}$ be a dipole. We compute the energy form of v.

Corollary 5.13. *Let $v \in \mathcal{V} \subset H_{\mathcal{E}}$ be a dipole with respect to the fixed vertices x_1 and x_2. i.e., $\Delta_G(v) = \delta_{x_1} - \delta_{x_2}$, in $H_{\mathcal{E}}$. Then*

$$\langle \lambda_w v, v \rangle_c = \frac{1}{2}\,(v\,(s(w)))\,\big(\delta_{r(w),x_1} - \delta_{r(w),x_2}\big) \tag{5.15}$$

for all $w \in \mathbb{G}$, and hence

$$\langle \lambda_a v, v \rangle_c = \frac{1}{2}\,(v\,(s(w)))\,\big(\big|\mathfrak{S}_{x_1}^{a}\big| - |\mathfrak{S}_{a}^{x_2}|\big)$$

for all $a \in \mathfrak{A}_G$, where

$$\mathfrak{S}_{x_1}^{a} = \{w \in Supp(a) : r(w) = x_1\},$$

and

$$\mathfrak{S}_{a}^{x_2} = \{y \in Supp(a) : r(y) = x_2\}.$$

Proof. Let v be a given dipole with respect to the vertices x_1 and x_2 in X, and let $w \in \mathbb{G}$. Then, by (5.13), we obtain that

$$\begin{aligned}
\langle \lambda_w v, v \rangle_c &= \frac{1}{2}\,(v\,(s(w)))\,(\Delta_G(v)\,(r(w))) = \\
&= \frac{1}{2}\,(v\,(s(w)))\,((\delta_{x_1} - \delta_{x_2})\,(r(w))) = \\
&= \frac{1}{2}\,(v\,(s(w)))\,\big(\delta_{x_1,r(w)} - \delta_{x_2,r(w)}\big) = \qquad (5.16)\\
&= \begin{cases} \frac{1}{2} v\,(s(w)) & \text{if } x_1 = r(w), \\ -\frac{1}{2} v\,(s(w)) & \text{if } x_2 = r(w), \\ 0 & \text{otherwise,} \end{cases} \qquad (5.17)
\end{aligned}$$

for all $w \in \mathbb{G}$. So, now, let

$$a = \sum_{w \in Supp(a)} t_w w \in \mathfrak{A}_G, \quad \text{with} \quad t_w \in \mathbb{C}.$$

Then

$$\langle \lambda_a v, v \rangle_c = \sum_{w \in Supp(a)} \left(\frac{1}{2} \left(v \left((s(w)) \right) \right) \left(\delta_{x_1, r(w)} - \delta_{x_2, r(w)} \right) \right) =$$

by (5.16)

$$= \sum_{w \in Supp(a),\, r(w)=x_1} \frac{1}{2} \left(v \left(s(w) \right) \right) - \sum_{w \in Supp(a),\, r(w)=x_2} \frac{1}{2} \left(v \left(s(w) \right) \right) =$$

by (5.17)

$$= \frac{1}{2} v \left(s(w) \right) \left(\left| \mathfrak{S}_{x_1}^{a} \right| - \left| \mathfrak{S}_{a}^{x_2} \right| \right),$$

since the direct network G, and the ERN $\widehat{G}$ are simplicial, where

$$\mathfrak{S}_{x_1}^{a} = \{ w \in Supp(a) : r(w) = x_1 \},$$

and

$$\mathfrak{S}_{a}^{x_2} = \{ y \in Supp(a) : r(y) = x_2 \},$$

in the ERN-groupoid $\mathbb{G}$. □

Now, let's consider how the ERN-actions $\lambda(\mathfrak{A}_G)$ works on the reproducing kernels $\{v_x : x \in X\}$ of $H_{\mathcal{E}}$ for a fixed vertex (origin) $o \in X$.

Theorem 5.14. *Let $v_x \in \mathcal{V} \subset H_{\mathcal{E}}$ be a reproducing kernel with respect to a fixed origin $o \in X$. i.e., it satisfies*

$$\langle v_x, u \rangle_c = u(x) - u(o), \quad \textit{for all} \quad u \in \mathcal{V} \subset H_{\mathcal{E}}.$$

Then we obtain

$$\langle \lambda_w v_x, v_x \rangle_c = \frac{1}{c_{xo}} \left(\delta_{s(w),x} - \delta_{s(w),o} \right) \left(\delta_{r(w),x} - \delta_{r(w),o} \right) \tag{5.18}$$

for all $w \in \mathbb{G}$, and hence

$$\langle \lambda_a v_x, v_x \rangle_c = \sum_{w \in Supp(a)} \frac{t_w}{c_{xo}} \left(\delta_{s(w),x} - \delta_{s(w),o} \right) \left(\delta_{r(w),x} - \delta_{r(w),o} \right)$$

for all $a = \sum_{w \in Supp(a)} t_w w \in \mathfrak{A}_G$.

Proof. Let v_x be given as above, and let $w \in \mathbb{G}$. Different from the above two corollaries, we will not use our general formula (5.13), to prove (5.18). Observe that

$$\langle \lambda_w v_x, v_x \rangle_c = \overline{\langle v_x,\ \lambda_w v_x \rangle_c} =$$

since the energy inner product $\langle , \rangle_c$ is a sesqui-linear form on $H_{\mathcal{E}}$

$$= \overline{\lambda_w (v_x)(x) - \lambda_w (v_x)(o)}$$

since $\langle v_x, u\rangle_c = u(x) - u(o)$

$$= \overline{v_x(wxw^{-1}) - v_x(wow^{-1})} =$$
$$= v_x(wxw^{-1}) - v_x(wow^{-1}) =$$

since all elements of $\mathcal{V}$ are $\mathbb{R}$-valued

$$\begin{aligned} &= \delta_{r(w),x} v_x\left(s(w)\right) - \delta_{r(w),o} v_x\left(s(w)\right) = \\ &= v_x\left(s(w)\right)\left(\delta_{r(w),x} - \delta_{r(w),o}\right). \end{aligned} \tag{5.19}$$

Now, let's define the following two morphisms. First, define an operator $E_{w_0} : \mathfrak{A}_G \to \mathfrak{A}_G$ by

$$E_{w_0}\left(\sum_{w\in\mathbb{G}} t_w w\right) \overset{def}{=} t_{w_0} w_0 \tag{5.20}$$

for all $\sum\limits_{w\in\mathbb{G}} t_w w \in \mathfrak{A}_G$, for a fixed groupoid element $w_0 \in \mathbb{G}$. (Clearly, if $w_0 \notin Supp(a)$, then $E_{w_0}(a) = 0_{\mathfrak{A}_G} = \emptyset$.)

Second, define a functional $\chi_{w_0} : \mathfrak{A}_G \to \mathbb{C}$ by

$$\chi_{w_0}\left(\sum_{w\in\mathbb{G}} t_w w\right) \overset{def}{=} t_{w_0} \tag{5.21}$$

for all $\sum\limits_{w\in\mathbb{G}} t_w w \in \mathfrak{A}_G$, for a fixed groupoid element $w_0 \in \mathbb{G}$. (Clearly, if $w_0 \notin Supp(a)$, then $\chi_{w_0}(a) = 0$ in $\mathbb{C}$.)

Then, for a fixed groupoid element $w_0 \in \mathbb{G}$, we define a new functional $d_{w_0} : \mathfrak{A}_G \to \mathbb{C}$ by

$$d_{w_0} \overset{def}{=} \chi_{w_0} \circ E_{w_0}, \tag{5.22}$$

where χ_{w_0} is defined in (5.21), and E_{w_0} is defined in (5.20).

For the rest of this proof, we will show that a reproducing kernel v_x satisfies

$$v_x = \frac{1}{c_{xo}}\left(d_x - d_o\right), \quad \text{for} \quad x \in X, \tag{5.23}$$

where

$$d_x = d_x \mid_X, \quad \text{and} \quad d_o = d_o \mid_X,$$

where d_x and d_o are defined in (5.22). By definition, a reproducing kernel v_{x_0} satisfies

$$\langle v_{x_0}, u\rangle_c = u(x_0) - u(o),$$

for all $u \in \mathcal{V} \subset H_{\mathcal{E}}$, and for an arbitrary fixed $x_0 \in X$, with respect to an arbitrary chosen origin $o \in X$. Observe that

$$\begin{aligned}
\langle v_x, u\rangle_c &= \frac{1}{2}\sum_{(x,y)\in E(G)} c_{xy}\left(v_{x_0}(x) - v_{x_0}(o)\right)\left(u(x) - u(o)\right) = \\
&= \frac{1}{2}\sum_{(x,y)\in E(G)} c_{xy}\left(\frac{1}{c_{x_0o}}(d_{x_0} - d_o)(x) - \frac{1}{c_{x_0o}}(d_{x_0} - d_o)(o)\right)\left(u(x) - u(o)\right) = \\
&= \frac{1}{2}c_{x_0o}\left(\frac{1}{c_{x_0o}}d_{x_0}(x_0) - \frac{1}{c_{x_0o}}(-d_o(o))\right)\left(u(x_o) - u(o)\right) = \\
&= \frac{1}{2}\left(d_{x_0}(x_0) + d_o(o)\right)\left(u(x_0) - u(o)\right) = \\
&= \frac{1}{2}(1+1)\left(u(x_0) - u(o)\right) = \\
&= u(x_0) - u(o).
\end{aligned}$$

Therefore, a reproducing kernel v_{x_0} is identified with the function on X,

$$\frac{1}{c_{x_0o}}\left(d_{x_0} - d_o\right).$$

Since x_0 is arbitrary in X, a reproducing kernel v_x satisfies the identity (**??**), for $x \in X$. Therefore, by (5.19) and (5.23), we obtain that

$$\begin{aligned}
\langle \lambda_w v_x, v_x\rangle_c &= v_x\left(s(w)\right)\left(\delta_{r(w),x} - \delta_{r(w),o}\right) = \\
&= \left(\frac{1}{c_{xo}}\left(d_x - d_o\right)\left(s(w)\right)\right)\left(\delta_{r(w),x} - \delta_{r(w),o}\right) = \\
&= \frac{1}{c_{xo}}\left(d_x\left(s(w)\right) - d_o\left(s(w)\right)\right)\left(\delta_{r(w),x} - \delta_{r(w),o}\right) = \\
&= \frac{1}{c_{xo}}\left(\delta_{s(w),x} - \delta_{s(w),o}\right)\left(\delta_{r(w),x} - \delta_{r(w),o}\right).
\end{aligned} \tag{5.24}$$

Therefore, the formula (5.18) holds true, for all $w \in \mathbb{G}$, and $x \in X$. So, if we take $a = \sum_{w\in Supp(a)} t_w w$ in the ERN-algebra $\mathfrak{A}_G$, then it acts on a reproducing kernel v_x, for $x \in X$, with the following energy form:

$$\langle \lambda_a v_x, v_x\rangle_c = \sum_{w\in Supp(a)} t_w \langle \lambda_w v_x, v_x\rangle_c = \sum_{w\in Supp(a)} \frac{t_w}{c_{xo}}\left(\delta_{s(w),x} - \delta_{s(w),o}\right)\left(\delta_{r(w),x} - \delta_{r(w),o}\right), \tag{5.25}$$

by (5.24).

Therefore, by (5.24) and (5.25), we can get the formula (5.18). □

The formula (5.18) is important to compute arbitrary energy form affected by the ERN-groupoid-actions, because of the following proposition, proven in [6].

Jorgensen and Pearse showed that:

Theorem 5.15 ([6])**.** *Let $\mathcal{K}_o = \{v_x : x \in X\}$ be the subset of $H_{\mathcal{E}}$, consisting of all reducing kernel with respect to a fixed vertex $o \in X$. Then the energy Hilbert space $H_{\mathcal{E}}$ is generated by $\mathcal{K}_o$. i.e.,*

$$H_{\mathcal{E}} = \overline{span\mathcal{K}_o}^{\|\cdot\|_c}, \tag{5.26}$$

where $\overline{S}^{\|\cdot\|_c}$ means the $\|.\|_c$-norm closure of $S \subseteq H_{\mathcal{E}}$, and where $spanY$ means the vector space spanned by a set Y.

The above theorem shows that if $g \in H_{\mathcal{E}}$, then

$$g = \sum_{x \in X} r_x v_x = \lim_{n \to \infty} \sum_{x \in S_n} r_x v_x,$$

such that

$$\|g\|_c < \infty,$$

where S_n are the finite subsets of X, for $n \in \mathbb{N}$, with

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots.$$

So, thanks to (5.26), we understand arbitrary elements g of the energy Hilbert space $H_{\mathcal{E}}$ as (infinite) linear sum of reproducing kernels in $\mathcal{K}_o = \{v_x\}_{x \in X}$, where we fix a suitable origin o in X. And this shows that our formula (5.18) would be the key computation to "measure" how the ERN-groupoid $\mathbb{G}$ (or the ERN-algebra $\mathfrak{A}_G$) acts on $H_{\mathcal{E}}$ (for a fixed origin o in $H_{\mathcal{E}}$). Also, conversely, the formula (5.18) provides a noncommutative probability on the ERN-algebra $\mathfrak{A}_G$ (See Section 6 below).

Corollary 5.16. *Let $a = \sum_{w \in Supp(a)} t_w w$ be an element of the ERN-algebra $\mathfrak{A}_G$, and let $g = \sum_{x \in X} r_x v_x$ be a finite linear combination in the energy Hilbert space $H_{\mathcal{E}}$, where $\mathcal{K}_o = \{v_x\}_{x \in X}$ are the reproducing kernels of $H_{\mathcal{E}}$, with respect to a fixed origin $o \in X \subset H_{\mathcal{E}}$. Then the action λ_a of a acts on g as follows:*

$$\langle \lambda_a g, g \rangle_c = \sum_{(x_1, x_2) \in X^2,\, w \in Supp(a)} \frac{t_w r_{x_1} \overline{r_{x_2}}}{c(x_2, o)} \left(\delta_{s(w), x_1} - \delta_{s(w), o}\right) \left(\delta_{r(w), x_2} - \delta_{r(w), o}\right), \tag{5.27}$$

via energy form.

Proof. By (5.18), we have that

$$\langle \lambda_w v_{x_1}, v_{x_2} \rangle_c = \frac{1}{c(x_2, o)} \left(\delta_{s(w), x_1} - \delta_{s(w), o}\right) \left(\delta_{r(w), x_2} - \delta_{r(w), o}\right) \tag{5.28}$$

for all $w \in \mathbb{G}$, and hence

$$\langle \lambda_a v_{x_1}, v_{x_2} \rangle_c = \sum_{w \in Supp(a)} \frac{t_w}{c(x_2, o)} \left(\delta_{s(w), x_1} - \delta_{s(w), o}\right) \left(\delta_{r(w), x_2} - \delta_{r(w), o}\right)$$

for $a = \sum_{w \in Supp(a)} t_w w \in \mathfrak{A}_G$, for $x_1, x_2 \in X$. Thus, if $g = \sum_{x \in X} r_x v_x$ in $H_{\mathcal{E}}$, then

$$\langle \lambda_a g, g \rangle_c = \left\langle \lambda_a \left(\sum_{x \in X} r_x v_x \right), \sum_{x \in X} r_x v_x \right\rangle_c =$$
$$= \sum_{(x_1,x_2) \in X^2} r_{x_1} \overline{r_{x_2}} \langle \lambda_a v_{x_1}, \; v_{x_2} \rangle_c$$

where $X^2 = X \times X$

$$= \sum_{(x_1,x_2) \in X^2} r_{x_1} \overline{r_{x_2}} \left(\sum_{w \in Supp(a)} \frac{t_w}{c(x_2, \, o)} \left(\delta_{s(w),x_1} - \delta_{s(w),o} \right) \left(\delta_{r(w),x_2} - \delta_{r(w),o} \right) \right) =$$
$$= \sum_{(x_1,x_2) \in X^2} \sum_{w \in Supp(a)} \frac{t_w r_{x_1} \overline{r_{x_2}}}{c(x_2, o)} \left(\delta_{s(w),x_1} - \delta_{s(w),o} \right) \left(\delta_{r(w),x_2} - \delta_{r(w),o} \right) =$$
$$= \sum_{(x_1,x_2) \in X^2, \, w \in Supp(a)} \frac{t_w r_{x_1} \overline{r_{x_2}}}{c(x_2, o)} \left(\delta_{s(w),x_1} - \delta_{s(w),o} \right) \left(\delta_{r(w),x_2} - \delta_{r(w),o} \right),$$

for all $a = \sum_{w \in Supp(a)} t_w w \in \mathfrak{A}_G$. □

6. FREE PROBABILITY ON ERN-ALGEBRAS

Throughout this section, we also use the same notations used in previous sections. In Section 5, we study how the ERN-algebra $\mathfrak{A}_G$ act on the energy Hilbert space $H_{\mathcal{E}}$, in terms of the energy form $\langle , \rangle_c$, via the representation λ of $\mathfrak{A}_G$. Conversely, in this section, we consider how the *energy measure* $\varepsilon(\bullet)$, satisfying

$$\varepsilon_h(w) = \langle \lambda_w h, h \rangle_c,$$

for all $h \in H_{\mathcal{E}}$, for $w \in \mathbb{G}$, acts on $\mathfrak{A}_G$.

6.1. FREE PROBABILITY

Let $\mathfrak{A}$ be an arbitrary (pure algebraic) algebra, and let $\varepsilon : \mathfrak{A} \to \mathbb{C}$ be a linear functional on $\mathfrak{A}$. Then the pair $(\mathfrak{A}, \varepsilon)$ is called a (*noncommutative*) *free probability space* (e.g., see [30]). By definition, the so-called free probability on a given algebra $\mathfrak{A}$ is completely dependent upon a fixed linear functional ε. Each element a of a free probability space $(\mathfrak{A}, \varepsilon)$ are called (*noncommutative*) *free random variables.*

Let $a \in (\mathfrak{A}, \varepsilon)$ be a free random variable. Then the *n-th* (*free*) *moments of a* is defined by

$$\varepsilon(a^n) \quad \text{for} \quad n \in \mathbb{N}.$$

Similarly, if $a_1, \ldots, a_s \in \mathfrak{A}$ are chosen elements, for $s \in \mathbb{N}$, then the $(j_1, \ldots, j_n)$-*th joint (free) moments of* $a_1, \ldots, a_s$ is defined by

$$\varepsilon\left(a_{j_1} a_{j_2} \ldots a_{j_n}\right),$$

for all $(j_1, \ldots, j_n) \in \{1, \ldots, s\}^n$, for all $n \in \mathbb{N}$. Define a set Θ by a collection of certain formal series in (a variable) z,

$$\Theta_z \overset{def}{=} \{\sum_{n=1}^{\infty} k_n z^n : k_n \in \mathbb{C}, \forall n \in \mathbb{N}\}.$$

The set $\{\varepsilon(a^n)\}_{n=1}^{\infty}$ of n-th moments of a represents the free-distributional data of a in $(\mathfrak{A}, \varepsilon)$ i.e., the *(noncommutative) free distribution* μ_a *of* a is a linear functional,

$$\mu_a : \Theta(z) \to \mathbb{C}$$

defined by

$$\mu_a\left(\sum_{n=1}^{\infty} k_n z^n\right) = \varepsilon\left(\sum_{n=1}^{\infty} k_n a^n\right) = \sum_{n=1}^{\infty} k_n \left(\varepsilon(a^n)\right),$$

for all $\sum_{n=1}^{\infty} k_n z^n \in \Theta(z)$, with $k_n \in \mathbb{C}$. So, by definition, indeed, the free moments $\{\varepsilon(a^n)\}_{n=1}^{\infty}$ of $a \in \mathfrak{A}$ represent the free distributional data μ_a of the free random variable a. Similarly, for a set $\Theta_{z_1,\ldots,z_s}$ of the multi-variable formal series

$$\Theta_{z_1,\ldots,z_s} = \overset{\infty}{\underset{n=1}{\cup}} \{ \sum_{(i_1,\ldots,i_n) \in \{1,\ldots,s\}^n} k_{i_1,\ldots,i_n} (z_{i_1} \ldots z_{i_n}) : k_{i_1,\ldots,i_n} \in \mathbb{C}\},$$

in noncommutative variables $z_1, \ldots, z_s$, the set

$$\overset{\infty}{\underset{n=1}{\cup}} \{\varepsilon(a_{i_1} \ldots a_{i_n}) : (i_1, \ldots, i_n) \in \{1, \ldots, s\}^n\}$$

of joint free moments of the random variables $a_1, \ldots, a_s$ of $(\mathfrak{A}, \varepsilon)$ represent the joint free distributional data of $a_1, \ldots, a_s$. Indeed, the *joint free distribution* $\mu_{a_1,\ldots,a_s}$ *of* $a_1, \ldots, a_s$ is defined by a linear functional,

$$\mu_{a_1,\ldots,a_s} : \Theta_{z_1,\ldots,z_s} \to \mathbb{C}$$

satisfying

$$\mu_{a_1,\ldots,a_s}(g) = g(a_1, \ldots, a_s) = \varepsilon\left(\sum_{(i_1,\ldots,i_n) \in \{1,\ldots,s\}^n} k_{i_1,\ldots,i_n} a_{i_1} \ldots a_{i_n}\right),$$

whenever

$$g(z_1, \ldots, z_s) = \sum_{(i_1,\ldots,i_n) \in \{1,\ldots,s\}^n} k_{i_1,\ldots,i_n} (z_{i_1} \ldots z_{i_n}) \quad \text{in} \quad \Theta_{z_1,\ldots,z_s},$$

with $k_{i_1,\ldots,i_n} \in \mathbb{C}$. So, the study of joint free distributions $\mu_{a_1,\ldots,a_s}$ (or free distributions μ_a) of random variables $a_1, \ldots, a_s$ (resp., random variables a) is to investigate the joint free moments $\varepsilon(a_{i_1} a_{i_2} \ldots a_{i_n})$ (resp., the free moments $\varepsilon(a^n)$) of the random variables.

Now, let h be an arbitrary fixed element of our energy Hilbert space $H_{\mathcal{E}}$, and let $\mathfrak{A}_G$ be the ERN-algebra induced by the ERN $\widehat{G}$. Define a linear functional

$$\varepsilon_h : \mathfrak{A}_G \to \mathbb{C}$$

by

$$\begin{aligned} \varepsilon_h\,(a) &= \sum_{w \in Supp(a)} t_w \varepsilon_h(w) \\ &\overset{def}{=} \sum_{w \in Supp(a)} t_w \langle \lambda_w h, h \rangle_c, \end{aligned} \tag{6.1}$$

for all $a = \sum_{w \in Supp(a)} t_w w \in \mathfrak{A}_G$. Then, for a fixed $h \in H_{\mathcal{E}}$, the linear functional ε_h is well-defined on $\mathfrak{A}_G$.

Definition 6.1. Let $\mathfrak{A}_G$ be the ERN-algebra, and $h \in H_{\mathcal{E}}$. Let ε_h be the linear functional on $\mathfrak{A}_G$, defined in (6.1). Then the pair $(\mathfrak{A}_G, \varepsilon_h)$ is called a energy (noncommutative) probability space induced by h.

Let's go back to the general setting. Suppose $(\mathfrak{A}, \varepsilon)$ is an arbitrary noncommutative probability space. Then the n -th moments $\varepsilon(a^n)$ of a free random variable $a \in (\mathfrak{A}, \varepsilon)$ have their equivalent free distributional data, called *free cumulants* $k_n(a, \ldots, a)$ *of* a (See [30] and the cited references in [30]).

The *n-th cumulant* $k_n(a, \ldots, a)$ of a free random variable $a \in (\mathfrak{A}, \varepsilon)$ is defined by

$$k_n\left(\underbrace{a,\ \ldots,\ a}_{n\text{-times}}\right) \overset{def}{=} \sum_{\pi \in NC(n)} \varepsilon_\pi \left(\underbrace{a,\ \ldots\ldots,\ a}_{n\text{-times}}\right) \mu(\pi, 1_n),$$

for all $n \in \mathbb{N}$, where $NC(n)$ is the lattice consisting of all *noncrossing partitions over* $\{1, \ldots, n\}$, with its minimal element

$$0_n = \{(1), (2), \ldots, (n)\},$$

and its maximal element

$$1_n = \{(1, \ldots, n)\},$$

(Here, the parenthesis means *blocks* of the partitions.) and $\varepsilon_\pi(\ldots)$ means the *partition-depending moments*, and where

$$\mu : NC(n) \times NC(n) \to \mathbb{C}$$

is the *Moebius functional* in the *incident algebra* $\mathcal{I}$.

Noncrossing partitions over $\{1, \ldots, n\}$ mean the partitions without crossings. For example,

$$\pi = \{(1, 2, 5), (3, 4), (6, 7)\}$$

is a noncrossing partition in $NC(7)$, however

$$\theta = \{(1, 2, 5), (3, 4, 6), (7)\}$$

is a "crossing" partition, because the blocks $(1, 2, 5)$ and $(3, 4, 6)$ of the partition θ are crossing from each other.

Let $NC(n)$ be the collection of all noncrossing partitions over $\{1, \ldots, n\}$, for all $n \in \mathbb{N}$. Then this set $NC(n)$ is a lattice under the partial ordering $\leq$,

$$\pi_1 \leq \pi_2 \overset{def}{\Longleftrightarrow} \forall V_1 \in \pi_1, \exists V_2 \in \pi_2, \text{ s.t.}, V_1 \subseteq V_2,$$

where "$V \in \pi$" means "V is a block of π," and $\subseteq$ means the usual set-inclusion.

The *incident algebra* $\mathcal{I}$ is an algebra consisting of all multiplicative functionals

$$\psi : NC(n) \times NC(n) \to \mathbb{C}$$

satisfying that $\psi(\pi_1, \pi_2) = 0$, whenever $\pi_1 > \pi_2$, equipped with the usual functional addition $(+)$, and the functional convolution $(*)$, defined by

$$(\psi_1 * \psi_2)(\pi_1,\ \pi_2) \overset{def}{=} \sum_{\pi_1 \leq \theta \leq \pi_2} \psi(\pi_1, \theta)\psi(\theta, \pi_2)$$

for all $\psi_k \in \mathcal{I}$, for $k = 1, 2$. Here, a "multiplicative" functional ψ means that

$$\psi(\pi, 1_n) = \underset{V \in \pi}{\Pi} \psi\left(0_{|V|},\ 1_{|V|}\right),$$

where "$V \in \pi$" means that "V is a block of π," and $|V|$ means the cardinality of V. There exists the *zeta functional* ζ in $\mathcal{I}$, defined by

$$\zeta(\pi, \theta) \overset{def}{=} \begin{cases} 1 & \text{if } \pi \leq \theta, \\ 0 & \text{otherwise.} \end{cases}$$

As the convolution inverse of ζ, we can define the Moebius functional μ in $\mathcal{I}$. Since it is the inverse of ζ, it satisfies that

$$\mu(0_n, 1_n) = (-1)^{n-1} c_{n-1} \quad \text{and} \quad \sum_{\pi \in NC(n)} \mu(\pi, 1_n) = 0, \tag{6.2}$$

where $c_k = \frac{1}{k+1}\begin{pmatrix} 2k \\ k \end{pmatrix}$ are the k-th Catalan numbers, for all $k \in \mathbb{N}$.

Finally, the partition-depending moments $\varepsilon_\pi(a, \ldots, a)$ is computed as follows:

$$\varepsilon_\pi(a, \ldots, a) = \underset{V \in \pi}{\Pi} \varepsilon(a^{|V|}),$$

for all $\pi \in NC(n)$, for all $n \in \mathbb{N}$. For example, if

$$\pi = \{(1, 3), (2), (4, 5)\} \quad \text{in} \quad NC(5),$$

then

$$\varepsilon_{\pi}(a,a,a,a,a) = \varepsilon(a\varepsilon(a)a)\varepsilon(aa) = \varepsilon(a^2)\varepsilon(a)\varepsilon(a^2).$$

By considering the cumulant computation on $(\mathfrak{A}, \varepsilon)$, we can check the free structure on $\mathfrak{A}$ (with respect to ε). By Speicher, two *subalgebras* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *of* $\mathfrak{A}$ *are free in* $(\mathfrak{A}, \varepsilon)$, if all "mixed" free cumulants of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ vanish. Also, two *subsets* A_1 *and* A_2 *of* $\mathfrak{A}$ *are free in* $(\mathfrak{A}, \varepsilon)$, if the algebras $\mathfrak{A}_k$ generated by A_k are free in $(\mathfrak{A}, \varepsilon)$, for $k = 1, 2$. Similarly, two *free random variables* a_1 *and* a_2 *of* $\mathfrak{A}$ *are free in* $(\mathfrak{A}, \varepsilon)$, if two subsets $\{a_1\}$ and $\{a_2\}$ are free in $(\mathfrak{A}, \varepsilon)$. Assume now that $\mathfrak{A}_j$ are subalgebras of $\mathfrak{A}$, and suppose they are free from each other in $(\mathfrak{A}, \varepsilon)$. The we can construct a subalgebra $\mathfrak{A}_0$ generated by $\mathfrak{A}_j$'s in $\mathfrak{A}$. We denote this subalgebra $\mathfrak{A}_0$ by $\underset{j}{*}\mathfrak{A}_j$ to emphasize that it is generated by free subalgebras. If $\mathfrak{A}_0$ is identical to $\mathfrak{A}$, itself, then we call $\mathfrak{A}$ a *free product algebra of* $\mathfrak{A}_j$'s.

6.2. FREE-MOMENT COMPUTATIONS IN $(\mathfrak{A}_G, \varepsilon_X)$

Throughout this section, we keep using the same notations. In Section 6.1, we showed that there exists a well-defined noncommutative probability space $(\mathfrak{A}_G, \varepsilon_h)$, consisting of the ERN-algebra $\mathfrak{A}_G$, and a linear functional ε_h induced by h, for any fixed $h \in H_{\mathcal{E}}$. This shows that the energy Hilbert space $H_{\mathcal{E}}$ acts on the *algebraic dual* $\mathfrak{A}'_G$ *of* $\mathfrak{A}_G$, via a Hilbert-space-action ε,

$$\varepsilon : H_{\mathcal{E}} \to \mathfrak{A}'_G$$

such that

$$\varepsilon(h) \overset{def}{=} \varepsilon_h \quad \text{for all} \quad h \in H_{\mathcal{E}}, \tag{6.3}$$

where a linear functional $\varepsilon_h \in \mathfrak{A}'_G$ is defined by (6.1).

Proposition 6.2. *The energy Hilbert space $H_{\mathcal{E}}$ acts on the ERN-algebra $\mathfrak{A}_G$, in the sense of* (6.3).

Recall that the *algebraic dual* $\mathfrak{A}'$ of an arbitrary algebra $\mathfrak{A}$ is defined by

$$\mathfrak{A}' \overset{def}{=} \{f : \mathfrak{A} \to \mathbb{C} : f \text{ is linear}\}.$$

Remark here that, since our ERN-algebra $\mathfrak{A}_G$ is a pure algebraic algebra, its algebraic dual $\mathfrak{A}'_G$ is topology-free. i.e., the elements of $\mathfrak{A}'_G$ are simply linear (without boundedness, equivalently continuity).

Recall that the energy Hilbert space $H_{\mathcal{E}}$ is spanned by reproducing kernels $\mathcal{K}_o = \{v_x\}_{x \in X}$, for an arbitrary fixed origin $o \in X$. So, we can determine the linear functionals

$$\varepsilon_x = \varepsilon_{v_x} \in \mathfrak{A}'_G, \quad \text{for} \quad v_x \in \mathcal{K}_o. \tag{6.4}$$

Definition 6.3. Let $\mathfrak{A}_G$ be the ERN-algebra and let ε_x be a linear functional on $\mathfrak{A}_G$, defined in (6.4). Then the noncommutative probability space $(\mathfrak{A}_G, \varepsilon_x)$ is called the energy (noncommutative) probability space (centered at x).

i.e., a linear functional ε_x measures the quantity of elements of $\mathfrak{A}_G$, in terms of the energy form depending on the location $x \in X = V(\widehat{G})$ of the ERN $\widehat{G}$. Consider the following computation:

$$\begin{aligned}
\varepsilon_x(w) =& \langle \lambda_w v_x, v_x \rangle_c = \\
=& \frac{1}{c\,(r(w),\, o)} \left(\delta_{s(w),x} - \delta_{s(w),o}\right) \left(\delta_{s(w),x} - \delta_{r(w),o}\right) = \text{ by (5.28) (or by (5.18))} \\
=& \frac{1}{c\,(r(w), o)} \left(\delta_{s(w),x}\delta_{s(w),x} - \delta_{s(w),x}\delta_{r(w),o} - \delta_{s(w),o}\delta_{s(w),x} + \delta_{s(w),o}\delta_{r(w),o}\right) \\
=& \begin{cases} \frac{1}{c(x,\,o)} = \frac{1}{c_{xo}} & \text{if } w \text{ is a loop with } s(w) = x = r(w), \\ & \text{equivalently, } (s(w),\, r(w)) = (x,\ x), \\ & \\ -\frac{1}{c(o,o)} = -\frac{1}{c_{oo}} & \text{if } (s(w),\ r(w)) = (x, o), \\ -\frac{1}{c(x,\,o)} = -\frac{1}{c_{xo}} & \text{if } (s(w),\ r(w)) = (o, x), \\ \frac{1}{c(o,o)} = \frac{1}{c_{oo}} & \text{if } (s(w),\ r(w)) = (o, o), \end{cases}
\end{aligned} \tag{6.5}$$

where $s(w) = ww^{-1}$, and $r(w) = ww^{-1}$ in $V(\widehat{G}) = X$. In (6.5), we assume $x \neq o$ in X, for convenience. However, we can easily verify the case where $x = o$ in X, again by (5.18). Assume that $x = o$ in X. Then we obtain

$$\varepsilon_o(w) = 0 \quad \text{for all} \quad w \in \mathbb{G}, \tag{6.6}$$

equivalently,

$$\varepsilon_o = 0, \quad \text{the zero functional on} \quad \mathfrak{A}_G.$$

Indeed, we have that

$$\varepsilon_o(w) = \begin{cases} \frac{1}{c_{xo}}\,(1 - 1 - 1 + 1) & \text{if } (s(w), r(w)) = (o, o) \\ 0 & \text{otherwise,} \end{cases}$$

for all $w \in \mathbb{G} \subset \mathfrak{A}_G$. So, from now on, if we mention a energy functional ε_x, then we automatically assume $x \neq o$ in X. From the formula (6.5), we also verify that the quantity of $\varepsilon_x(w)$ is related to the resistance $R = \frac{1}{c}$, i.e., we can re-write (6.5) by

$$\varepsilon_x(w) = \begin{cases} R(x,\ o) & \text{if } w \text{ is a loop with } s(w) = x = r(w), \\ & \text{equivalently, } (s(w),\, r(w)) = (x,\ x) \\ -R(o,\ o) & \text{if } (s(w),\ r(w)) = (x,\ o) \\ -R(x,\ o) & \text{if } (s(w),\ r(w)) = (o,\ x) \\ R(o,\ o) & \text{if } (s(w),\ r(w)) = (o, o), \end{cases} \tag{6.7}$$

for all $w \in \mathbb{G} \subset \mathfrak{A}_G$. By physics, we can have

$$R(x, x) = 0 \quad \text{for all} \quad x \in X,$$

because the current has the zero resistance to flow from x to x. In such a sense, the formula (6.7) becomes

$$\varepsilon_x(w) = \begin{cases} R(x,\ o) = \frac{1}{c_{xo}} & \text{if } w \text{ is a loop with } s(w) = x = r(w), \\ & \text{equivalently, } (s(w),\ r(w)) = (x, x), \\ 0 & \text{if } (s(w),\ r(w)) = (x, o), \\ -R(x, o) = \frac{-1}{c_{xo}} & \text{if } (s(w),\ r(w)) = (o, x), \\ 0 & \text{if } (s(w),\ r(w)) = (o, o), \end{cases} \tag{6.8}$$

and hence

$$\varepsilon_x(w) = \begin{cases} R(x,\ o) = \frac{1}{c_{xo}} & \text{if } (s(w), r(w)) = (x, x), \\ -R(x,\ o) = \frac{-1}{c_{xo}} & \text{if } (s(w), r(w)) = (o, x), \\ 0 & \text{otherwise,} \end{cases}$$

for all $w \in \mathbb{G} \subset \mathfrak{A}_G$. The following lemma is the direct consequence of the computation (6.8).

Lemma 6.4. *Let $w \in \mathbb{G}$ be a groupoid element in the ERN-algebra $\mathfrak{A}_G$, and let ε_x be an energy functional induced by a reproducing kernel v_x, defined in* (6.4)*, in the energy Hilbert space $H_{\mathcal{E}}$. Then*

$$\varepsilon_x(w) = \begin{cases} \frac{1}{c_{xo}} & \textit{if } (s(w), r(w)) = (x, x), \\ -\frac{1}{c_{xo}} & \textit{if } (s(w), r(w)) = (o, x), \\ 0 & \textit{otherwise.} \end{cases}$$

Let w be a reduced finite path in the (arbitrary) graph groupoid $\mathbb{G}$ of a graph G. We say that w is a *loop* (*finite path*) in $\mathbb{G}$, if $w = xwx$, with $x \in V(G)$. Now, let G be our directed network with its ERN $\widehat{G}$. As we have seen in Section 3, every directed network is simplicial, and hence every ERN is simplicial in the sense that $\widehat{G}$ has neither loop-edges nor multi-edges. However, it is possible that $\widehat{G}$ may / can have loops, which are not loop-edges! For example, let

$$G = \begin{array}{ccc} {}^{x_1} & \bullet & \\ & \uparrow & \searrow \\ {}_{x_3} & \bullet & \leftarrow & \bullet\ {}_{x_2} \end{array} .$$

Then this graph G has neither loop-edges nor multi-edges. However, it has its loops

$$(x_1, x_2, x_3)^n, (x_2, x_3, x_1)^n, \quad \text{and} \quad (x_3, x_1, x_2)^n$$

for all $n \in \mathbb{N}$. Thus, the above lemma has the following combinatorial equivalency:

Corollary 6.5. *Let $w \in \mathbb{G}$ be a groupoid element in $\mathfrak{A}_G$, and let ε_x be an energy functional in the sense of* (6.4)*. Then*

$$\varepsilon_x(w) = \begin{cases} \frac{1}{c_{xo}} & \textit{if } w \textit{ is a loop with } w = xwx, \\ -\frac{1}{c_{xo}} & \textit{if } w = owx, \\ 0 & \textit{otherwise.} \end{cases} \tag{6.9}$$

More precisely, we have that:

(1) *if w is an edge in $\mathbb{G}$, then $\varepsilon_x(w) = -\frac{1}{c_{xo}}$, only when $w = (o, x)$ in $E(\widehat{G})$.*
(2) *if w is a non-loop reduced finite path in $\mathbb{G}$, which is not an edge, then $\varepsilon_x(w) = -\frac{1}{c_{xo}}$, only when $w = owx$.*
(3) *if w is a loop in $\mathbb{G}$, with $|w| > 1$, then $\varepsilon_x(w) = \frac{1}{c_{xo}}$, only when $w = xwx$.*

The above corollary shows that, if $w \in \mathbb{G}$ in $\mathfrak{A}_G$ is "non-loop," then the nonzero energy form becomes a negative quantity, and if w is loop, then the nonzero energy form becomes a positive quantity.

By (6.8) (or (6.9)), we obtain the following distributional data.

Theorem 6.6. *Let $w \in \mathbb{G}$ be a random variable in the energy probability space $(\mathfrak{A}_G, \varepsilon_x)$ centered at $x \in X$. Then*

$$\varepsilon_x(w^n) = \begin{cases} \frac{1}{c_{xo}} & \text{if } w \text{ is a loop with } w = xwx, \\ & \text{for all } n \in \mathbb{N}, \\ -\frac{1}{c_{xo}} & \text{if } w = owx, \text{ and } n = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{6.10}$$

for all $n \in \mathbb{N}$.

Proof. If $n = 1$, then the formula (6.10) holds, by (6.9). Assume now that $n > 1$ in $\mathbb{N}$. If w is non-loop, then $w^n = \emptyset$, for all $n > 1$. So,

$$\varepsilon_x(w^n) = \varepsilon_x(\emptyset) = \varepsilon_x(0_{\mathfrak{A}_G}) = 0,$$

whenever $n > 1$, if w is non-loop. Now, assume that w is loop, moreover, $w = xwx$ in $\mathbb{G}$. Then $w^n = xw^n x$, for all $n \in \mathbb{N}$. i.e.,

$$(s(w^n),\ r(w^n)) = (x, x), \quad \text{in} \quad X^2,$$

for all $n \in \mathbb{N}$. Therefore, by (6.8), we can obtain that

$$\varepsilon_x(w^n) = \frac{1}{c_{xo}} \quad \text{for all} \quad n \in \mathbb{N}.$$

□

The above theorem provides the noncommutative probabilistic distributional data of groupoid elements $w \in \mathbb{G}$ in the energy probability space $(\mathfrak{A}_G, \varepsilon_x)$. Based on the above theorem, we establish a calculus on the ERN-algebra $\mathfrak{A}_G$ with respect to energy forms.

6.3. FREE-CUMULANT COMPUTATIONS IN $(\mathfrak{A}_G, \varepsilon_X)$

Let $(\mathfrak{A}_G, \varepsilon_x)$ be an energy probability space centered at $x \in X$, and let $w \in \mathbb{G}$ be an element in the ERN-algebra $\mathfrak{A}_G$. In Section 6.2, we considered the free distributional data of w, by computing the free moments $\{\varepsilon_x(w^n)\}_{n\in\mathbb{N}}$ (See (6.10)). In this section, we study the equivalent free-distributional data of w by computing the free cumulants

$$\{k_n(w, \dots, w)\}_{n\in\mathbb{N}}.$$

By doing that, we obtain the free structure of $\mathfrak{A}_G$, in terms of a fixed vertex x. Let $w \in \mathbb{G}$ be a free random variable in the energy probability space $(\mathfrak{A}_G, \varepsilon_x)$. Observe

$$\begin{aligned} k_n\Big(\underbrace{w, \; \dots\dots, \; w}_{n\text{-times}}\Big) &= \sum_{\pi\in NC(n)} \varepsilon_{x:\pi}(w, \; \dots, \; w)\, \mu(\pi, 1_n) = \\ &= \sum_{\pi\in NC(n)} \left(\prod_{V\in\pi} \varepsilon_x\left(w^{|V|}\right)\right) \mu(\pi, 1_n) = \\ &= \sum_{\pi\in NC(n)} \left(\prod_{V\in\pi} \left(\varepsilon_x(w^{|V|}) \; \mu(0_{|V|}, \; 1_{|V|})\right)\right) \end{aligned} \tag{6.11}$$

for all $n \in \mathbb{N}$. By (6.11), we obtain the following lemma without proof.

Lemma 6.7. *Let $w \in \mathbb{G}$ be a free random variable in $(\mathfrak{A}_G, \varepsilon_x)$. Then*

$$k_n\Big(\underbrace{w, \; \dots, \; w}_{n\text{-times}}\Big) = \sum_{\pi\in NC(n)} \left(\prod_{V\in\pi} \left(\varepsilon_x(w^{|V|})\mu\left(0_{|V|}, \; 1_{|V|}\right)\right)\right)$$

for all $n \in \mathbb{N}$.

By the above lemma (or (6.11)) and (6.10), we get that

$$k_1(w) = \varepsilon_x(w) = \begin{cases} \frac{1}{c_{xo}} & \text{if } w = xwx, \\ -\frac{1}{c_{xo}} & \text{if } w = owx, \\ 0 & \text{otherwise,} \end{cases}$$

and, for $n > 1$,

$$\begin{aligned} k_n(w) &= \begin{cases} \sum\limits_{\pi\in NC(n)} \left(\prod\limits_{V\in\pi} \left(\frac{(-1)^{|V|}}{c_{xo}|V|} \begin{pmatrix} 2\,(|V|-1) \\ |V|-1 \end{pmatrix}\right)\right) & \text{if } w = xwx, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \sum\limits_{\pi\in NC(n)} \left(\prod\limits_{V\in\pi} \left(\frac{(-1)^{|V|}}{c_{xo}|V|} \frac{(2(|V|-1))!}{(|V|-1)!(|V|-1)!}\right)\right) & \text{if } w = xwx, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{6.12}$$

by (6.2).

Let $\pi \in NC(n)$ be a noncrossing partition. Then we define the *size* $|\pi|$ *of* π by the cardinality of the set of blocks in π. For example, if

$$\pi = \{(1, 3, 6, 7), (2), (4, 5)\},$$

with blocks $(1,3,6,7),(2)$, and $(4,5)$, in $NC(7)$, then

$$|\pi| = 3.$$

By (6.12), we can obtain the following theorem, which is equivalent to (6.10) (combinatorially).

Theorem 6.8. *Let $w \in \mathbb{G}$ be a free random variable in the energy probability space $(\mathfrak{A}_G, \varepsilon_x)$ centered at x. Then the n-th free cumulants of w are determined by the formula,*

$$k_1(w) = \varepsilon_x(w) = \begin{cases} \frac{1}{c_{xo}} & \textit{if } w = xwx, \\ -\frac{1}{c_{xo}} & \textit{if } w = owx, \\ 0 & \textit{otherwise,} \end{cases} \tag{6.13}$$

and

$$k_n\Big(\underbrace{w, \ \ldots\ldots, \ w}_{n\text{-times}}\Big) = \begin{cases} \sum\limits_{\pi \in NC(n)} \left(\frac{1}{c_{xo}}\right)^{|\pi|} \mu(\pi, \ 1_n) & \textit{if } w = xwx, \\ 0 & \textit{otherwise} \end{cases}$$

for all $n \in \mathbb{N} \setminus \{1\}$, where $|\pi|$ means the size of π.

Proof. Let $n = 1$. Then, by definition, $k_1(w) = \varepsilon_x(w)$. Assume now that $n > 1$ in $\mathbb{N}$. Then, by (6.12), we obtain that

$$\begin{aligned} k_n(w, \ldots, w) &= \begin{cases} \sum\limits_{\pi \in NC(n)} \left(\prod\limits_{V \in \pi} \left(\frac{(-1)^{|V|}}{c_{xo}|V|} \frac{(2(|V|-1))!}{(|V|-1)!(|V|-1)!} \right) \right) & \text{if } w = xwx, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \sum\limits_{\pi \in NC(n)} \left(\frac{1}{c_{xo}}\right)^{|\pi|} \left(\prod\limits_{V \in \pi} \left(\frac{(-1)^{|V|}(2(|V|-1))!}{|V|((|V|-1)!)^2} \right) \right) & \text{if } w = xwx, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \sum\limits_{\pi \in NC(n)} \left(\frac{1}{c_{xo}}\right)^{|\pi|} (\mu(\pi, \ 1_n)) & \text{if } w = xwx, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since

$$\mu(\pi, 1_n) = \prod_{V \in \pi} \mu\left(0_{|V|}, \ 1_{|V|}\right),$$

for all $\pi \in NC(n)$, for $n \in \mathbb{N}$ (See Section 6.1 or [30]). $\square$

Now, let's consider the mixed cumulants of the distinct groupoid elements w_1 and w_2, as free random variables in the energy probability space $(\mathfrak{A}_G, \varepsilon_x)$ "centered at x". First, compute the followings;

$$\begin{aligned} k_2(w_1, \ w_2) &= \varepsilon_x(w_1 w_2)\mu(1_2, 1_2) + \varepsilon_x(w_1)\varepsilon_x(w_2)\mu(0_2, 1_2) = \\ &= \varepsilon_x(w_1 w_2) - \varepsilon_x(w_1)\varepsilon_x(w_2), \end{aligned} \tag{6.14}$$

since

$$\mu(1_2, 1_2) = 1, \quad \text{and} \quad \mu(0_2, 1_2) = (-1)^{2-1} c_1 = -1.$$

Assume that $w_k = x w_k x$ in $\mathbb{G}$, for all $k = 1, 2$. Then the formula (6.14) goes to

$$k_2(w_1, w_2) = \frac{1}{c_{xo}} - \frac{1}{c_{xo}} \frac{1}{c_{xo}} = \frac{c_{xo} - 1}{c_{xo}^2},$$

by (6.13). Suppose now that w_1 and w_2 are admissible, i.e., $w_1 w_2 \neq \emptyset$ in $\mathbb{G}$, and assume that $w_1 = o w_1 x$ and $w_2 = x w_2 x$. Then the formula (6.14) goes to

$$k_2(w_1, w_2) = -\frac{1}{c_{xo}} - \left(-\frac{1}{c_{xo}}\right)\left(\frac{1}{c_{xo}}\right) = \frac{1 - c_{xo}}{c_{xo}^2}, \tag{6.15}$$

by (6.13). Similarly, let $w_1 = x w_1 x$ and $w_2 = x w_2 o$. Then

$$k_2(w_1, w_2) = 0, \tag{6.16}$$

again by (6.13). This shows that even though w_1 and w_2 are admissible in $\mathbb{G}$, the cumulants induced by the energy form can vanish. Since we assume $x \neq o$ in $X = V(\widehat{G}) \subset \mathbb{G}$, from (6.15) and (6.16) we can verify as follows:

Theorem 6.9. *Let $w_1, w_2 \in \mathbb{G}$ be free random variables in the energy probability space $(\mathfrak{A}_G, \varepsilon_x)$ centered at x. Then the "mixed" cumulants satisfy*

$$k_n\left(w_{j_1},\ w_{j_2},\ \ldots,\ w_{j_n}\right) =$$
$$= \begin{cases} \sum\limits_{\pi \in NC(n)} \left(\frac{1}{c_{xo}}\right)^{|\pi|} \mu(\pi,\ 1_n) & \text{if } w_k = x w_k x,\ \forall k = 1,\ 2, \\ \sum\limits_{\pi \in NC(n)} \left(\frac{1}{c_{xo}}\right)^{|\mathfrak{W}_\pi|} \mu(\pi,\ 1_n) & \text{if } w_1 = o w_1 x,\ w_2 = x w_2 x, \\ 0 & \text{otherwise,} \end{cases} \tag{6.17}$$

for all "mixed" n-tuples $(j_1, \ldots, j_n) \in \{1, 2\}^n$, for $n \in \mathbb{N} \setminus \{1\}$, where

$$\mathfrak{W}_\pi = \left\{ V \in \pi \,\middle|\, \begin{array}{c} V = (i_1,\ \ldots,\ i_{|V|}) \text{ in } \pi, \\ i_1 \leq i_2 \leq \ldots \leq i_{|V|} \\ \text{in } \{1,\ 2\} \end{array} \right\},$$

for each $\pi \in NC(n)$, for all $n \in \mathbb{N} \setminus \{1\}$.

Proof. Assume first that $w_k = x w_k x$ in $\mathbb{G}$, for all $k = 1, 2$. Then the mixed cumulants

$$k_n(w_{j_1}, \ldots, w_{j_n}) = \sum_{\pi \in NC(n)} \varepsilon_{x:\pi}\left(w_{j_1},\ \ldots,\ w_{j_n}\right) \mu(\pi, 1_n) =$$
$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \varepsilon_V\left(w_{j_1},\ \ldots,\ w_{j_n}\right) \right) \mu(\pi, 1_n),$$

where

$$\varepsilon_V(w_{j_1},\ldots,w_{j_n}) \overset{def}{=} \varepsilon\left(w_{i_1}w_{i_2}\ldots w_{i_{|V|}}\right),$$

whenever $V=(i_1,\ldots,i_{|V|})\in\pi$. And we have

$$\varepsilon_V(w_{j_1},\ldots,w_{j_n}) = \varepsilon(w_{i_1}\ldots w_{i_{|V|}}) = \frac{1}{c_{xo}}, \tag{6.18}$$

for all $V\in\pi\in NC(n)$, for $n\in\mathbb{N}\setminus\{1\}$, by (6.10), because the groupoid element $w_{i_1}\ldots w_{i_n}$ is again a loop connecting x to x in $\mathbb{G}$. By (6.18), if w_k are loop with their initial and terminal vertices x, then the mixed cumulants of w_1 and w_2 satisfy

$$k_n(w_{j_1},\ldots,w_{j_n}) = \sum_{\pi\in NC(n)}\left(\frac{1}{c_{xo}}\right)^{|\pi|}\mu(\pi,1_n). \tag{6.19}$$

Assume now that $w_1 = ow_1x$ and $w_2 = xw_2x$. Then w_1w_2 is nonempty in $\mathbb{G}$, and it satisfies

$$w_1w_2 = o(w_1w_2)x,$$

and hence, we obtain the nonzero quantities for it, in (6.10) and (6.15). So, in general, we can get that

$$\begin{aligned} k_n\left(w_{j_1},\ \ldots,\ w_{j_n}\right) &= \sum_{\pi\in NC(n)}\left(\varepsilon_{x:\pi}(w_{j_1},\ \ldots,\ w_{j_n})\right)\mu(\pi,1_n) = \\ &= \sum_{\pi\in NC(n)}\left(\prod_{V\in\pi}\varepsilon_{x:V}\left(x_{j_1},\ \ldots,\ x_{j_n}\right)\right)\mu(\pi,1_n) = \\ &= \sum_{\pi\in NC(n)}\left(-\frac{1}{c_{xo}}\right)^{|\mathfrak{W}_\pi|}\mu(\pi,1_n), \end{aligned} \tag{6.20}$$

where

$$\mathfrak{W}_\pi \overset{def}{=} \left\{ V\in\pi \;\middle|\; \begin{array}{c} V=(i_1,\ \ldots,\ i_{|V|}),\ \text{and} \\ i_1\le i_2\le\ldots\le i_{|V|} \\ \text{in } \{1,\ 2\} \end{array} \right\},$$

since $\varepsilon_x(w_2w_1)=0$, by (6.10).

Besides (6.19) and (6.20), all other mixed cumulants vanish, by (6.13) and (6.10). $\square$

By the previous theorem we can obtain the following free structure on the energy probability space $(\mathfrak{A}_G,\varepsilon_x)$.

Theorem 6.10. *Let $w\in\mathbb{G}$ be a random variable in $(\mathfrak{A}_G,\varepsilon_x)$, either*

$$w = xwx \quad \text{or} \quad w = oxw \quad \text{in} \quad \mathbb{G},$$

and let $w'\in\mathbb{G}\subset\mathfrak{A}_G$. Then w and w' are $*$-free in $(\mathfrak{A}_G,\varepsilon_x)$, if and only if the subsets

$$\{w, w^{-1}=w^*\} \quad \text{and} \quad \{w',(w')^{-1}=(w')^*\}$$

of $\mathfrak{A}_G$ *are free in* $(\mathfrak{A}_G, \varepsilon_x)$*, if and only if*

$$s(w'), r(w') \in X \setminus \{x, o\}.$$

Proof. ($\Leftarrow$) Assume that a free random variable $w' \in \mathbb{G}$ in $(\mathfrak{A}_G, \varepsilon_x)$ satisfies

$$s(w'), r(w') \in X \setminus \{x, o\}.$$

Then, by (6.17), the mixed free cumulants of

$$\{w, w^{-1} = w^*\} \quad \text{and} \quad \{w', (w')^{-1} = (w')^*\}$$

vanish. Equivalently, the free random variables w and w' are free in $(\mathfrak{A}_G, \varepsilon_x)$.

($\Rightarrow$) Suppose the free random variables w and w' are free in $(\mathfrak{A}_G, \varepsilon_x)$, equivalently,

$$\{w, w^{-1} = w^*\} \quad \text{and} \quad \{w', (w')^{-1} = (w')^*\}$$

have vanishing mixed cumulants. Now, assume that either

$$s(w') \in \{x, o\} \quad \text{or} \quad r(w') \in \{x, o\}.$$

Then, again by (6.17), there exists (at least one) nonvanishing mixed cumulants (for example, like in (6.14) or in (6.15)). This contradicts our assumption that w and w' are free in $(\mathfrak{A}_G, \varepsilon_x)$. $\square$

The above theorem characterize the free structure on the ERN-algebra $\mathfrak{A}_G$, in terms of the energy form ε_x (centered at $x \in X$). The following corollary is the direct consequence of the above theorem.

Corollary 6.11. *Let* $\mathfrak{A}_G$ *be our ERN-algebra and let* $(\mathfrak{A}_G, \varepsilon_x)$ *be an energy probability space centered at* $x \in X \setminus \{o\}$*. Then there exists* $*$*-subalgebras* $\mathfrak{A}_o$ *and* $\mathfrak{A}_o^c$ *of* $\mathfrak{A}_G$*, such that*

$$\mathfrak{A}_G = \mathfrak{A}_o * \mathfrak{A}_o^c, \tag{6.21}$$

in the sense of Section 6.1, where

$$\mathfrak{A}_o \overset{def}{=} A\lg\left(\mathbb{C}\left[\left[\left\{w \in \mathbb{G} \,\middle|\, \begin{array}{c} s(w) \in \{x,\ o\} \text{ or} \\ r(w) \in \{x,\ o\} \end{array}\right\}\right]\right]\right),$$

and

$$\mathfrak{A}_o^c \overset{def}{=} A\lg\left(\mathbb{C}\left[\left[\left(w \in \mathbb{G} \,\middle|\, \begin{array}{c} s(w) \in X \setminus \{x,\ o\} \\ \text{and} \\ r(w) \in X \setminus \{x,\ o\} \end{array}\right)\right]\right]\right),$$

where $A\lg(Y)$ *means an algebra generated by a set* Y*. i.e., the ERN-algebra* $\mathfrak{A}_G$ *is a free product* $*$*-algebra of* $\mathfrak{A}_o$ *and* $\mathfrak{A}_o^c$*.*

Proof. The above corollary is proven by the above theorem and by the following observation: if we let

$$\mathbb{G}_o = \{w \in \mathbb{G} : s(w) \in \{x, o\} \quad \text{or} \quad r(w) \in \{x, o\}\},$$

then

$$\mathbb{G} = \mathbb{G}_o \cup \mathbb{G}_o^c,$$

where $\mathbb{G}_o^c$ is the compliment of $\mathbb{G}_o$ in $\mathbb{G}$, set-theoretically. Remark here that, by definition,

$$\mathbb{G}_o^{-1} = \mathbb{G}_o,$$

where $Y^{-1} \overset{def}{=} \{y^{-1} : y \in Y\}$, for all subsets Y of $\mathbb{G}$. So, the $*$-subalgebra

$$\mathfrak{A}_o = A\lg\left(\mathbb{C}[[\mathbb{G}_o]]\right) = * - A\lg\left(\mathbb{C}[[\mathbb{G}_o]]\right) \subseteq \mathfrak{A}_G$$

is well-defined and hence $\mathfrak{A}_o^c$, too. Also, by definition,

$$\mathfrak{A}_G = \mathbb{C}[[\mathbb{G}]] = \mathbb{C}[[\mathbb{G}_o \cup \mathbb{G}_o^c]],$$

and hence

$$\mathfrak{A}_G = A\lg\left(\mathfrak{A}_o \ \cup \ \mathfrak{A}_o^c\right). \tag{6.22}$$

By the above theorem, the $*$-subalgebras $\mathfrak{A}_o$ and $\mathfrak{A}_o^c$ are $*$-free in $(\mathfrak{A}_G, \varepsilon_x)$. Therefore, by (6.22),

$$\mathfrak{A}_G = \mathfrak{A}_o * \mathfrak{A}_o^c. \qquad \square$$

The relation (6.21) characterizes the free structure of $\mathfrak{A}_G$ in terms of ε_x, for $x \in X$.

Acknowledgements
The authors thank Daniel Alpay, Anna Paolucci, Erin Pearse, and Myung-Sin Song for helpful discussions at various stages in our work on the paper.
The second named author is supported by the U.S. National Science Foundation.

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Received: January 6, 2011.
Revised: March 29, 2011.
Accepted: March 29, 2011.