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RECURSIVELY ARBITRARILY VERTEX-DECOMPOSABLE SUNS

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Abstract. A graph $G = (V, E)$ is arbitrarily vertex decomposable if for any sequence τ of positive integers adding up to $|V|$, there is a sequence of vertex-disjoint subsets of V whose orders are given by τ, and which induce connected graphs. The aim of this paper is to study the recursive version of this problem on a special class of graphs called suns. This paper is a complement of [O. Baudon, F. Gilbert, M. Woźniak, *Recursively arbitrarily vertex-decomposable graphs*, research report, 2010].

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1. TERMINOLOGY AND PRELIMINARY RESULTS

In this paper, we deal only with simple graphs, that means, graphs without loops or multiple edges. We denote by n the number of vertices, also called *order* of the graph and by m the number of edges. If $G = (V, E)$ and $A \subseteq V$, $G[A]$ will denote the subgraph of G induced by A. For more definitions on graphs, please refer to [2].

1.1. ARBITRARILY VERTEX-DECOMPOSABLE GRAPHS

Let n, $\tau_1, \ldots, \tau_k$ be positive integers such that $\tau_1 + \ldots + \tau_k = n$. $\tau = (\tau_1, \ldots, \tau_k)$ is called a *decomposition* of n. If the size of the decomposition is pertinent, we would precise *k-decomposition*.

Let $G = (V, E)$ be a graph of order n, and τ a k-decomposition of n. G is *τ-vertex-decomposable* iff it exists a partition of V: $V_1, \ldots, V_k$ such that for each $i, 1 \leq i \leq k$

- $|V_i| = \tau_i$,
- $G[V_i]$ is connected.

A graph $G = (V, E)$ of order n is *arbitrarily vertex-decomposable* (in short *AVD*) iff for each decomposition τ of n, G is τ-vertex-decomposable.

1.2. RECURSIVELY ARBITRARILY VERTEX-DECOMPOSABLE GRAPHS

Definition 1.1. A graph $G = (V, E)$ with order n is *recursively arbitrarily vertex-decomposable* (in short *R-AVD)* iff:

- $G = K_1$ or
- G is connected and for each decomposition $\tau = (\tau_1, \ldots, \tau_k)$ of n, $k \geq 2$, it exists a partition of V: $V_1, \ldots, V_k$ such that for all $i, 1 \leq i \leq k$:
  - $|V_i| = \tau_i$
  - $G[V_i]$ is R-AVD.

Remark 1.2. A graph $G = (V, E)$ of order n is R-AVD iff for each integer $1 \leq \lambda \leq \lfloor \frac{n}{2} \rfloor$, it exists a subset V_λ of V such that:

- $|V_\lambda| = \lambda$,
- $G[V_\lambda]$ is R-AVD,
- $G[V \backslash V_\lambda]$ is R-AVD.

1.3. FAMILIES OF GRAPHS

We present here some families of graphs and their notations, used in the further sections.

Let a be a positive integer. P_a denotes the path of order a, C_a the cycle of order a (cp. Figures 1a and 1b).

A k-pode $T_k(t_1, \ldots, t_k)$ is a tree of order $1 + \sum_{i=1}^{k} t_i$ composed by k paths of order respectively $t_1, \ldots, t_k$, connected to a unique node, called the *root* of the k-pode (cp. Figure 1c).

Let a and b be two positive integers. A caterpillar $\mathrm{Cat}\,(a, b)$ is a tree of order $a + b$, composed by three paths of order a, b and 2, sharing exactly one node, called the *root* of the caterpillar. $\mathrm{Cat}\,(a, b)$ is isomorphic to $T_3(a-1, b-1, 1)$ (cp. Figure 1d).

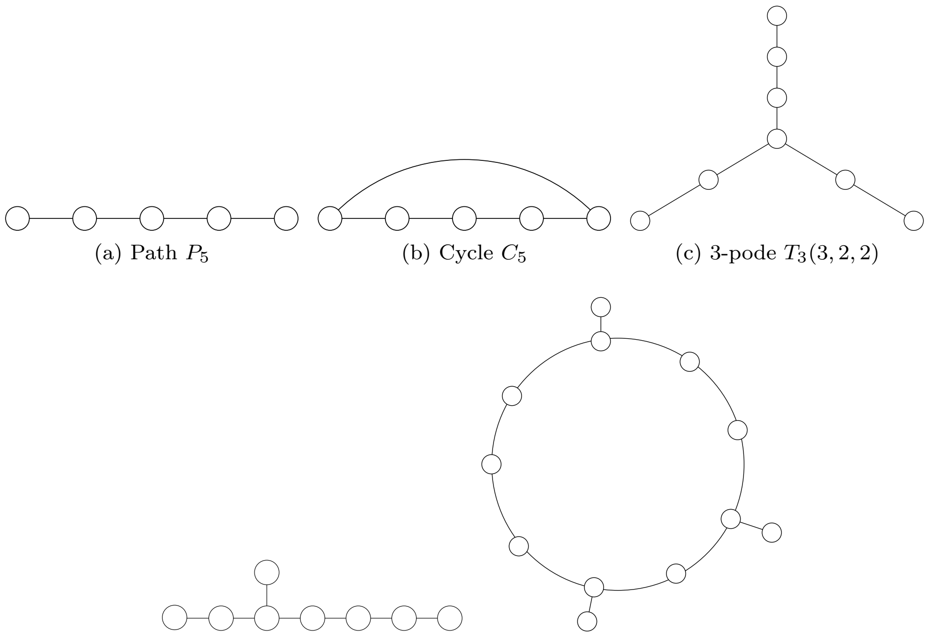
A sun with r rays is a graph of order $n \geq 2r$ with r hanging vertices $u_1, \ldots, u_r$ whose deletion yields a cycle C_{n-r}, and each vertex v_i adjacent to u_i is of degree three. If the sequence of vertices v_i is situated on the cycle C_{n-r} in such a way that there are exactly $a_i \geq 0$ vertices, each of degree two, between v_i and v_{i+1}, $i = 1, \ldots, r$ (the indices taken modulo r), then this sun is denoted by $\mathrm{Sun}\,(a_1, \ldots, a_r)$, and is unique up to isomorphism (cp. Figure 1e).

Note that the order of $\mathrm{Sun}\,(a_1, \ldots, a_r)$ equals $n = 2r + a_1 + \ldots + a_r$.

1.4. ON-LINE ARBITRARILY VERTEX-DECOMPOSABLE GRAPHS

The notion of on-line arbitrarily vertex decomposable graph has been introduced by Horňák and al. in [3].

Let $G = (V, E)$ be a graph. Imagine now the following decomposition procedure consisting of k stages, where k is a random variable attaining values from $[1, n]$. In the i^{th} stage, where $i \in [1, k]$, a positive integer τ_i arrives and we have to choose a subset V_i of V of order τ_i that is disjoint from all subsets of V chosen in previous stages (without a possibility of changing the choice in the future).



(a) Path P_5 (b) Cycle C_5 (c) 3-pode $T_3(3,2,2)$

(d) Cat $(3,5)$ (e) Sun $(2,3,1)$

Fig. 1. Examples of graphs

More precisely, for every partial sequence $(\tau_1, \dots \tau_i)$ whose sum is less than n, there is a sequence $(V_1, \dots, V_i)$ of disjoint subsets of V such that for $1 \leq j \leq i$, $|V_j| = \tau_j$, with the following property: for all sequences $(\tau'_1, \dots, \tau'_k)$ with $k \geq i$ and summing to n, such that $\tau'_r = \tau_r$ for $1 \leq r \leq i$, there is a decomposition of V into disjoint subsets $V'_1, \dots, V'_k$ with $|V'_j| = \tau'_j$ and $G[V'_j]$ connected, for all j, and $V'_j = V_j$ for $1 \leq j \leq i$.

Definition 1.3 ([3])**.** If the decomposition procedure can be accomplished for any (random) sequence of positive integers $(\tau_1, \dots, \tau_k)$ adding up to n, the graph G is said to be *on-line arbitrarily vertex-decomposable*, (in short *OL-AVD*).

Lemma 1.4 ([3])**.** *A graph $G = (V, E)$ of order n is OL-AVD iff for each integer $1 \leq \lambda \leq n-1$, it exists a subset V_λ of V such that*

- $|V_\lambda| = \lambda$,
- $G[V_\lambda]$ *is connected,*
- $G[V \backslash V_\lambda]$ *is OL-AVD.*

Remark 1.5. A straightforward consequence of Lemma 1.4 and Remark 1.2 is that every R-AVDgraph is OL-AVD.

The opposite is not true. For example, the caterpillar Cat $(8,11)$ is OL-AVD [3], but not R-AVD [1].

The next result gives a complete characterization of OL-AVD suns.

Theorem 1.6 ([4])**.** *A sun with one ray is always OL-AVD.*

A sun with two rays Sun (a, b) *is OL-AVD iff* a *and* b *take values given in Table 1a.*

A sun with three rays Sun (a, b, c) *is OL-AVD iff* a*,* b *and* c *take values given in Table 1b.*

A sun with four rays is OL-AVD iff it is isomorphic to Sun $(0, 0, 1, d)$*, where* $d \equiv 2, 4$ *(mod 6).*

A sun with five or more rays is never OL-AVD.

Table 1. Values for OL-AVD suns

a	b
0	arbitrary
1, 3	$\equiv 0$ (mod 2)
2	$\not\equiv 3$ (mod 6), 3, 9, 21
4	$\equiv 2, 4$ (mod 6), $[4, 19] \backslash \{15\}$
5	$\equiv 2, 4$ (mod 6), 6, 18
6	6, 7, 8, 10, 11, 12, 14, 16
7	8, 10, 12, 14, 16
8	8, 9, 10, 11,12
9	10, 12

(a) Values a, b ($b \geq a$), such that Sun (a, b) is OL-AVD

a	b	c
0	0	$\equiv 1, 2$ (mod 3)
	1	$\equiv 0$ (mod 2)
	2	$\equiv 2, 4$ (mod 6), 3, 6, 7, 11, 18, 19
	3	$\equiv 2, 4$ (mod 6)
	4	4, 5, 6, 8, 10, 11, 12, 14, 16
	5	6, 8, 16
	6, 7	8, 10
	8	8, 9
1	2	$\equiv 2, 4$ (mod 6), 6, 18
2	3	4, 8, 16

(b) Values a, b, c ($c \geq b \geq a$), such that Sun (a, b, c) is OL-AVD

1.5. RECURSIVELY ARBITRARILY VERTEX-DECOMPOSABLE TREES

Theorem 1.7 ([1])**.** *A tree* T *is R-AVD if and only if either* T *is a path or* T *is a caterpillar* Cat (a, b) *with* a *and* b *given in Table 2 or* T *is the 3-pode* $T_3(2, 4, 6)$*.*

Table 2. Values a, b ($b \geq a$), such that Cat (a, b) is R-AVD

a	b
2, 4	$\equiv 1$ (mod 2)
3	$\equiv 1, 2$ (mod 3)
5	6, 7, 9, 11, 14, 19
6	7
7	8, 9, 11, 13, 15

2. RECURSIVELY ARBITRARILY VERTEX-DECOMPOSABLE SUNS

This section presents the main result of this paper, a complete characterization of R-AVD suns.

Theorem 2.1.
A sun with one ray is always R-AVD.
A sun with two rays $\mathrm{Sun}\,(a, b)$ *is R-AVD if and only if* a *and* b *take values given in Table 3a.*
A sun with three rays $\mathrm{Sun}\,(a, b, c)$ *is R-AVD if and only if* a*,* b *and* c *take values given in Table 3b.*
A sun with four rays is R-AVD if and only if it is isomorphic to $\mathrm{Sun}\,(0, 0, 1, 2)$ *or to* $\mathrm{Sun}\,(0, 0, 1, 4)$*.*
A sun with five or more rays in never R-AVD.

Table 3. Values for R-AVD suns

a	b
0	arbitrary
1	$\equiv 0 \pmod 2$
2	$\not\equiv 0 \pmod 3$, 3, 6, 9, 12, 18, 21, 24, 36
3	$\equiv 0 \pmod 2$
4	$4 \le b \le 19$ except for $b = 15$,
	$\equiv 2, 4 \pmod 6$ with $20 \le b \le 46$
5	$\equiv 2, 4 \pmod 6$ with $8 \le b \le 32$, 6, 18
6	6, 7, 8, 10, 11, 12, 14, 16

(a) Values a, b ($b \ge a$), such that $\mathrm{Sun}\,(a, b)$ is R-AVD

a	b	c
0	0	$\equiv 1, 2 \pmod 3$
	1	$\equiv 0 \pmod 2$
	2	2, 3, 4, 6, 7, 8, 10, 11, 14, 16, 18, 19
	3	4, 8, 10
	4	4, 5, 6, 8, 10, 11, 12, 14, 16
	5	6
1	2	2, 4, 6, 8, 10, 14, 16, 18
2	3	4

(b) Values a, b, c ($c \ge b \ge a$), such that $\mathrm{Sun}\,(a, b, c)$ is R-AVD

Proof. Since every R-AVD graph is also OL-AVD, so, we shall use the complete characterization of OL-AVD suns given in Theorem 1.6, and Remark 1.2.

The labelling used in the proof follows that one from Figure 2.

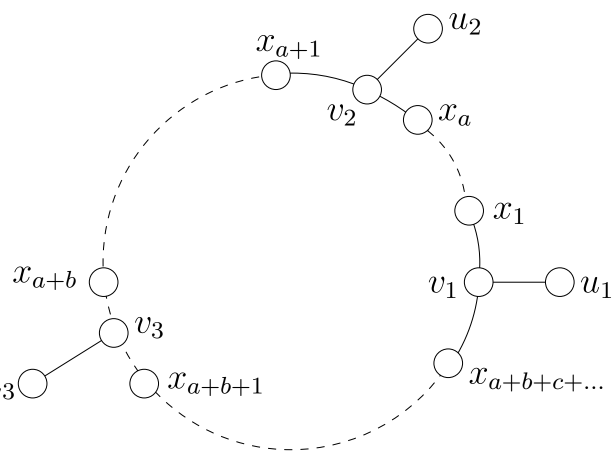


Fig. 2. $\mathrm{Sun}\,(a, b, \ldots)$

Sun with one ray. A sun with one ray is traceable. Thus, it is R-AVD.

Sun with two rays. Without loss of generality, we consider Sun (a,b) with $b \geq a$.

- Sun $(0, b)$ is traceable and then is R-AVD.
- Sun $(1, b)$ contains Cat (2, b+3) as partial graph. Thus, Sun $(1, b)$ with $b \equiv 0\ (mod\ 2)$ is R-AVD.
- Sun $(2, b)$ is OL-AVD only for $b \not\equiv 3 \pmod 6$ or $b = 3, 9, 21$.
  - Sun $(2, b)$ contains Cat $(3, b+3)$ as spanning tree and thus is R-AVD for $b \not\equiv 0 \pmod 3$.
  - If $b = 6k$ with $k = 5$ or $k \geq 7$, it is not possible to find a partition into two R-AVD subgraphs of size 18 and $n - 18$.
  - If $b \in \{3, 6, 9, 12, 18, 21, 24, 36\}$, then Sun $(2, b)$ is R-AVD and the values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 4.
- Sun $(3, b)$ contains Cat $(4, b+1)$ as a spanning tree. Thus, it is R-AVD for $b \equiv 0 \pmod 2$.
- Sun $(4, b)$ is OL-AVD only for $b \equiv 2, 4 \pmod 6$ or $b \in \{4, \ldots, 19\} \backslash \{15\}$.
  - Sun $(4, b)$ contains Cat $(5, b+3)$ as a spanning tree. Thus, it is R-AVD for $b \in \{4, 6, 8, 11, 16\}$.
  - Similarly, Sun $(4, b)$ contains Cat $(7, b{+}1)$ as a spanning tree. Thus, it is R-AVD for $b \in \{7, 10, 12, 14\}$.
  - Let us consider the case where $b \equiv 2, 4 \pmod 6$.
    * If $b \geq 50$, then $n = b + 8 \geq 58$. Then, we have to consider the case $\lambda = 30$ with $n - \lambda \geq 28$. Because there is no caterpillar with order 30, $G[V_\lambda]$ must be a path and $G[V \backslash V_\lambda]$ a caterpillar Cat $(5, x)$ or Cat $(7, x)$. But such a caterpillar has a maximum order 24. Thus, if $b \geq 50$, Sun $(4, b)$ cannot be R-AVD.
    * For $b \equiv 2, 4 \pmod 6, 20 \leq b \leq 46$, all the Sun $(4, b)$ are R-AVD and the values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 5.
  - For the last possible values of b, that is $b \in \{5, 9, 13, 17, 18, 19\}$, Sun $(4, b)$ is R-AVD and the values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 5.
- Sun $(5, b)$ is OL-AVD only for $b \equiv 2, 4 \pmod 6$ or $b \in \{6, 18\}$.
  - Consider the case $b \equiv 2, 4 \pmod 6$. For $\lambda = 18$, the only possibility is that $G[V_{18}] = P_{18}$. But in that case, $G[V \backslash V_{18}]$ must be a caterpillar Cat $(6, x)$ or Cat $(8, x)$, which is impossible for $n - 18 \geq 14$, that is $n \geq 32$ and $b \geq 23$. Thus Sun $(5, b)$ may be R-AVD only for $b \equiv 2, 4 \pmod 6$ with $8 \leq b \leq 22$ or $b \in \{6, 18\}$.
  - For all the remaining values of b, that is $b \in \{6, 8, 10, 14, 16, 18, 20, 22\}$, Sun $(5, b)$ is R-AVD and the values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 6.
- Sun $(6, b)$ is OL-AVD only for $b \in \{6, 7, 8, 10, 11, 12, 14, 16\}$.
  - Observe that Sun $(6, b)$ contains, as a spanning tree, the caterpillar Cat $(7, b{+}3)$. Thus, Sun (a, b) is R-AVD for $b \in \{6, 8, 10, 12\}$.
  - For all the remaining values of b, that is $b \in \{7, 11, 14, 16\}$, Sun $(6, b)$ is R-AVD and the values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 7.

- Sun $(7, b)$ is OL-AVD only for $b \in \{8, 10, 12, 14, 16\}$. For all of these values of b, it is not possible to find an edge $\{w_1, w_2\}$ such that $G[V \backslash \{w_1, w_2\}]$ is R-AVD.
- Sun $(8, b)$ is OL-AVD only for $b \in \{8, 9, 10, 11, 12\}$.
  - For $b \in \{8, 10, 11, 12\}$, it is not possible to find a set of size 3 V_3 such that both $G[V_3]$ and $G[V \backslash V_3]$ are R-AVD.
  - For $b = 9$, it is not possible to find an edge $\{w_1, w_2\}$ such that $G[V \backslash \{w_1, w_2\}]$ is R-AVD.
- Sun $(9, b)$ is OL-AVD only for $b \in \{10, 12\}$. For these two values of b, it is not possible to find an edge $\{w_1, w_2\}$ such that $G[V \backslash \{w_1, w_2\}]$ is R-AVD.

Table 4. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ for Sun $(2, b)$

b	λ	V_λ	$G[V_\lambda]$	$G[V \backslash V_\lambda]$
3, 6, 9, 12, 18, 21, 24, 36	1	$\{u_1\}$	P_1	sun with one ray
	2	$\{x_1, x_2\}$	P_2	P_{b+4}
3	3	$\{x_3, x_4, x_5\}$	P_3	P_6
6, 12, 18, 24, 36		$\{x_2, v_2, u_2\}$	P_3	Cat $(2, b+1)$
9, 21		$\{x_{a+b-2}, x_{a+b-1}, x_{a+b}\}$	P_3	Cat $(5, b-2)$
3, 6, 9, 12, 18, 21, 24, 36	4	$\{u_1, 1_2, x_1, x_2\}$	P_4	P_{b+2}
6, 9, 12, 18, 21, 24, 36	5	$\{x_1, x_2, v_2, u_2, x_3\}$	Cat $(2, 3)$	P_{b+1}
	6	$\{u_1, v_1, x_1, x_2, v_2, u_2\}$	P_6	P_b
9, 12, 18, 21, 24, 36	7	$\{u_1, v_1, x_1, x_2, v_2, u_2, x_3\}$	Cat $(2, 5)$	P_{b-1}
12, 18, 21, 24, 36	8	$\{u_1, v_1, x_1, x_2, v_2, u_2, x_3, x_4\}$	Cat $(3, 5)$	P_{b-2}
	9	$\{u_1, v_1, x_1, x_2, v_2, u_2, x_3, x_4, x_5\}$	Cat $(4, 5)$	P_{b-3}
18, 21, 24, 36	10	$\{x_1, x_2, v_2, u_2, x_3, \ldots, x_8\}$	Cat $(3, 7)$	P_{b-4}
	11	$\{x_1, x_2, v_2, u_2, x_3, \ldots, x_9\}$	Cat $(3, 8)$	P_{b-5}
	12	$\{u_1, v_1, x_1, x_2, v_2, u_2, x_3, \ldots, x_8\}$	Cat $(5, 7)$	P_{b-6}
21, 24, 36	13	$\{x_1, x_2, v_2, u_2, x_3, \ldots, x_{11}\}$	Cat $(3, 10)$	P_{b-7}
24, 36	14	$\{x_1, x_2, v_2, u_2, x_3, \ldots, x_{12}\}$	Cat $(3, 11)$	P_{b-8}
	15	$\{x_2, v_2, u_2, x_3, \ldots, x_{14}\}$	Cat $(2, 13)$	Cat $(2, b-11)$
36	16	$\{x_1, x_2, v_2, u_2, x_3, \ldots, x_{14}\}$	Cat $(3, 13)$	P_{26}
	17	$\{x_1, x_2, v_2, u_2, x_3, \ldots, x_{15}\}$	Cat $(3, 14)$	P_{25}
	18	$\{x_{21}, \ldots, x_{38}\}$	P_{18}	Cat $(5, 19)$
	19	$\{x_1, x_2, v_2, u_2, x_3, \ldots, x_{17}\}$	Cat $(3, 16)$	P_{23}
	20	$\{x_1, x_2, v_2, u_2, x_3, \ldots, x_{18}\}$	Cat $(3, 17)$	P_{22}
	21	$\{x_2, v_2, u_2, x_3, \ldots, x_{20}\}$	Cat $(2, 19)$	Cat $(2, 19)$

Sun with three rays. Without loss of generality, we consider Sun (a,b,c) with $c \geq b \geq a$.

- Sun (0,0,c) is OL-AVD only for $c \equiv 1, 2 \pmod 3$. Because Sun (0,0,c) contains Cat $(3, c+3)$ as a spanning tree, thus it is also R-AVD for $c \equiv 1, 2 \pmod 3$.
- Sun (0,1,c) is OL-AVD only for $c \equiv 0 \pmod 2$. Because Sun (0,1,c) contains Cat $(4, c+3)$ as a spanning tree, thus it is also R-AVD for $c \equiv 0 \pmod 2$.
- Sun (0,2,c) is OL-AVD only for $c \equiv 2, 4 \pmod 6$ or $c \in \{3, 6, 7, 11, 18, 19\}$.

Table 5. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ for some Sun $(4, b)$

b	λ	V_λ	$G[V_\lambda]$	$G[V \backslash V_\lambda]$
5, 9, 13, 17, 18, 19 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$	1	$\{u_1\}$	P_1	sun with one ray
5, 13, 17, 19 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$	2	$\{x_1, x_2\}$	P_2	Cat $(3, b+3)$
9		$\{x_5, x_6\}$	P_2	Cat $(7, 8)$
18		$\{u_1, v_1\}$	P_2	Cat $(5, 19)$
5, 13, 17	3	$\{x_5, x_6, x_7\}$	P_3	Cat $(7, b-2)$
9, 19		$\{u_2, v_2, x_5\}$	P_3	Cat $(5, b)$
18 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$		$\{x_1, x_2, x_3\}$	P_3	Cat $(2, b+3)$
5, 9, 13, 17, 18, 19 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$	4	$\{x_1, \ldots, x_4\}$	P_4	P_{b+4}
5, 13, 17, 19	5	$\{x_3, x_4, v_2, u_2, x_5\}$	Cat $(2, 3)$	Cat $(3, b)$
9		$\{x_5, \ldots, x_9\}$	P_5	Cat $(5, 7)$
18 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$		$\{x_4, v_2, u_2, x_5, x_6\}$	Cat $(2, 3)$	Cat $(4, b-1)$
5, 9, 13, 17, 18, 19 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$	6	$\{u_1, v_1, x_1, \ldots, x_4\}$	P_6	P_{b+2}
9, 13, 19	7	$\{x_3, x_4, v_2, u_2, x_5, x_6, x_7\}$	Cat $(3, 4)$	Cat $(3, b-2)$
17		$\{x_{15}, \ldots, x_{21}\}$	P_7	Cat $(7, 11)$
18 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$		$\{x_4, v_2, u_2, x_5, \ldots, x_8\}$	Cat $(2, 5)$	Cat $(4, b-3)$
9, 13, 17, 18, 19 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$	8	$\{u_1, v_1, x_1, \ldots, x_4, v_2, u_2\}$	P_8	P_b
13, 17, 18, 19 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$	9	$\{x_1, \ldots, x_4, v_2, u_2, x_5, x_6, x_7\}$	Cat $(4, 5)$	P_{b-1}
	10	$\{u_1, v_1, x_1, \ldots, x_4, v_2, u_2, x_5, x_6\}$	Cat $(3, 7)$	P_{b-2}
17, 18, 19 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$	11	$\{x_1, \ldots, x_4, v_2, u_2, x_5, \ldots, x_9\}$	Cat $(5, 6)$	P_{b-3}
	12	$\{x_1, \ldots, x_4, v_2, u_2, x_5, \ldots, x_{10}\}$	Cat $(5, 7)$	P_{b-4}
18, 19 $b \equiv 2, 4 \pmod 6, 20 \le b \le 46$	13	$\{u_1, v_1, x_1, \ldots, x_4, v_2, u_2, x_5, \ldots, x_9\}$	Cat $(6, 7)$	P_{b-5}
$b \equiv 2, 4 \pmod 6, 20 \le b \le 46$	14	$\{x_1, \ldots, x_4, v_2, u_2, x_5, \ldots, x_{12}\}$	Cat $(5, 9)$	P_{b-6}
$b \equiv 2, 4 \pmod 6, 22 \le b \le 46$	15	$\{x_4, v_2, u_2, x_5, \ldots, x_{16}\}$	Cat $(2, 13)$	Cat $(4, b-11)$
26, 28, 32, 34, 38, 40, 44, 46	16	$\{x_1, \ldots, x_4, v_2, u_2, x_5, \ldots, x_{14}\}$	Cat $(5, 11)$	P_{b-8}
	17	$\{x_4, v_2, u_2, x_5, \ldots, x_{18}\}$	Cat $(2, 15)$	Cat $(4, b-13)$
28, 32, 34, 38, 40, 44, 46	18	$\{u_1, v_1, x_1, \ldots, x_4, v_2, u_2, x_5, \ldots, x_{14}\}$	Cat $(7, 11)$	P_{b-10}
32, 34, 38, 40, 44, 46	19	$\{x_4, v_2, u_2, x_5, \ldots, x_{20}\}$	Cat $(2, 17)$	Cat $(4, b-15)$
	20	$\{u_1, v_1, x_1, \ldots, x_4, v_2, u_2, x_5, \ldots, x_{16}\}$	Cat $(7, 13)$	P_{b-12}
34, 38, 40, 44, 46	21	$\{x_4, v_2, u_2, x_5, \ldots, x_{22}\}$	Cat $(2, 19)$	Cat $(4, b-17)$
38, 40, 44, 46	22	$\{u_1, v_1, x_1, \ldots, x_4, v_2, u_2, x_5, \ldots, x_{18}\}$	Cat $(7, 15)$	P_{b-14}
	23	$\{x_4, v_2, u_2, x_5, \ldots, x_{24}\}$	Cat $(2, 21)$	Cat $(4, b-19)$
40, 44, 46	24	$\{x_1, \ldots, x_4, v_2, u_2, x_5, \ldots, x_{22}\}$	Cat $(5, 19)$	P_{b-16}
44, 46	25	$\{x_4, v_2, u_2, x_5, \ldots, x_{26}\}$	Cat $(2, 23)$	Cat $(4, b-21)$
	26	$\{x_3, x_4, v_2, u_2, x_5, \ldots, x_{26}\}$	Cat $(3, 23)$	Cat $(3, b-21)$
46	27	$\{x_4, v_2, u_2, x_5, \ldots, x_{28}\}$	Cat $(2, 25)$	Cat $(4, 23)$

Table 6. Values of $G[V_\lambda]$ and $G[V\backslash V_\lambda]$ for Sun $(5, b)$

b	λ	V_λ	$G[V_\lambda]$	$G[V\backslash V_\lambda]$
6, 8, 10, 14, 16, 18, 20, 22	1	$\{u_1\}$	P_1	sun with one ray
	2	$\{x_1, x_2\}$	P_2	Cat $(4, b+7)$
6, 18	3	$\{u_1, v_1, x_1\}$	P_3	Cat $(5, b+1)$
8, 10, 14, 16, 20, 22		$\{x_1, x_2, x_3\}$	P_3	Cat $(3, b+3)$
6, 8, 10, 14, 16, 18, 20, 22	4	$\{x_1, x_2, x_3, x_4\}$	P_4	Cat $(2, b+5)$
	5	$\{x_1 \ldots, x_5\}$	P_5	P_{b+4}
	6	$\{x_2 \ldots, x_5, v_2, u_2\}$	P_6	Cat $(2, b+1)$
	7	$\{u_1, v_1, x_1 \ldots, x_5\}$	P_7	P_{b+2}
8, 10, 14, 16, 18, 20, 22	8	$\{x_2 \ldots, x_5, v_2, u_2, x_6, x_7\}$	Cat $(3, 5)$	Cat $(2, b-1)$
10, 14, 16, 18, 20, 22	9	$\{u_1, v_1, x_1 \ldots, x_5, v_2, u_2\}$	P_9	P_b
14, 16, 18, 20, 22	10	$\{x_4, x_5, v_2, u_2, x_6, \ldots, x_{11}\}$	Cat $(3, 7)$	Cat $(4, b-5)$
	11	$\{u_1, v_1, x_1 \ldots, x_5, v_2, u_2, x_6, x_7\}$	Cat $(3, 8)$	P_{b-2}
16, 18, 20, 22	12	$\{x_2, \ldots, x_5, v_2, u_2, x_6, \ldots, x_{11}\}$	Cat $(5, 7)$	Cat $(2, b-5)$
18, 20, 22	13	$\{x_1, \ldots, x_5, v_2, u_2, x_6, \ldots, x_{11}\}$	Cat $(6, 7)$	P_{b-4}
20, 22	14	$\{x_2, \ldots, x_5, v_2, u_2, x_6, \ldots, x_{13}\}$	Cat $(5, 9)$	Cat $(2, b-7)$
22	15	$\{u_1, v_1, x_1 \ldots, x_5, v_2, u_2, x_6, \ldots, x_{11}\}$	Cat $(7, 8)$	P_{16}

Table 7. Values of $G[V_\lambda]$ and $G[V\backslash V_\lambda]$ for some Sun $(6, b)$

b	λ	V_λ	$G[V_\lambda]$	$G[V\backslash V_\lambda]$
7, 11, 14, 16	1	$\{u_1\}$	P_1	sun with one ray
7, 14	2	$\{u_1, v_1\}$	P_2	Cat $(7, b+1)$
11, 16		$\{x_1, x_2\}$	P_2	Cat $(5, b+3)$
7	3	$\{x_7, x_8, x_9\}$	P_3	Cat $(5, 9)$
11		$\{v_2, u_2, x_7\}$	P_3	Cat $(7, b)$
14, 16		$\{x_1, x_2, x_3\}$	P_3	Cat $(4, b+3)$
7, 11, 14, 16	4	$\{x_1, \ldots, x_4\}$	P_4	Cat (3, b+3)
7, 11	5	$\{x_5, x_6, v_2, u_2, x_7\}$	Cat $(2, 3)$	Cat $(5, b)$
14, 16		$\{x_1, \ldots, x_5\}$	P_5	Cat $(2, b+3)$
7, 11, 14, 16	6	$\{x_1, \ldots, x_6\}$	P_6	P_{b+4}
7, 11	7	$\{x_3, \ldots, x_6, v_2, u_2, x_7\}$	Cat $(2, 5)$	Cat $(3, b)$
14, 16		$\{x_2, \ldots, x_6, v_2, u_2\}$	P_7	Cat $(2, b+1)$
7, 11, 14, 16	8	$\{x_1, \ldots, x_6, v_2, u_2\}$	P_8	P_{b+2}
11, 14, 16	9	$\{x_1, \ldots, x_6, v_2, u_2, x_7, \ldots, x_{\lambda-2}\}$	Cat $(\lambda - 7, 7)$	$P_{b+10-\lambda}$
	10			
14, 16	11			
	12			
16	13			

- Sun (0,2,c) contains Cat $(5, c+3)$ as a spanning tree, thus it is R-AVD for $c \in \{3, 6, 11\}$.
- For $c \in \{7, 18, 19\}$, Sun $(0, 2, c)$ is R-AVD and the values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 8.
- For $c \equiv 2, 4 \pmod 6$, we first eliminate values of c such that Sun (0,2,c) is not R-AVD.
  * $c \equiv 2 \pmod 6, c \neq 2, 8, 14, 26$
    Consider $\lambda = 10$. The two possibilities for $G[V_{10}]$ to be R-AVD are
    - Cat $(3, 7)$ with $V_{10} = \{u_2, v_2, v_1, u_1, x_{c+2}, \dots, x_{c-3}\}$, and thus $G[V \backslash V_{10}] =$ Cat $(3, c-5)$. Cat $(3, c-5)$ is not R-AVD for $c \equiv 2 \pmod 6$.
    - P_{10} with $V_{10} = \{u_1, v_1, x_{c+2}, \dots, x_{c-5}\}$. Then $G[V \backslash V_{10}] =$ Cat $(5, c-7)$. If we consider only the cases where $c \geq 20$ and $c \equiv 2 \pmod 6$, $G[V \backslash V_{10}]$ is R-AVD only if $c = 26$.
  * $c = 26$
    Consider $\lambda = 13$. The possibilities for $G[V_{13}]$ to be R-AVD are
    - Cat $(3, 10)$ with $V_{13} = \{u_2, v_2, v_1, u_1, x_{c+2}, \dots, x_{c-6}\}$. If $c = 26$, then $G[V \backslash V_{13}] =$ Cat $(3, 18)$ which is not R-AVD.
    - P_{13} with $V_{13} = \{u_1, v_1, x_{c+2}, \dots, x_{c-8}\}$. If $c = 26$, $G[V \backslash V_{13}] =$ Cat $(5, 16)$ which is not R-AVD.
  * $c \equiv 4 \pmod 6, c \geq 22$
    First, observe that because $n = c + 8$ and $c \equiv 4 \pmod 6$, we have $n \equiv 0 \pmod 6$ and then $n \equiv 0 \pmod 3$.
    We consider $\lambda = 15$. Both 15 and $n - 15 \equiv 0 \pmod 3$. Therefore, both G_{15} and $G[V \backslash V_{15}]$ cannot be realized as a R-AVD caterpillar of the form Cat $(3, b)$. Because Cat $(5, 10)$ is not R-AVD the only remaining possibility is that G_{15} is a path P_{15} and $G[V \backslash V_{15}]$ is a caterpillar Cat $(5, c-12)$. But Cat $(5, c-12)$ is not R-AVD for $c = 22, 28$ or $c \geq 34$.

  In conclusion, for $c \equiv 2, 4 \pmod 6$, the only remaining values are 2, 4, 8, 10, 14 and 16. For all of these values, Sun (0,2,c) is R-AVD and the values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 8.

- Sun (0,3,c) is OL-AVD only for $c \equiv 2, 4 \pmod 6$.
  Consider first $\lambda = 6$. Because there is no R-AVD caterpillar of order 6, $G[V_6]$ must be a path of length 6. The two possibilities are that $V_6 = \{u_1, v_1, x_{c+3}, \dots, x_c\}$ or $\{u_3, v_3, x_4, \dots x_7\}$.
  If $V_6 = \{u_3, v_3, x_4, \dots x_7\}$, $G[V \backslash V_6]$ is R-AVD if and only if $G[V \backslash V_6]$ is a caterpillar Cat $(3, 4)$ and $c = 4$.
  If $V_6 = \{u_1, v_1, x_{c+3}, \dots, x_c\}$, $G[V \backslash V_6]$ is R-AVD if and only if $G[V \backslash V_6]$ is a caterpillar Cat $(5, 6)$ or Cat $(6, 7)$ and then $c = 8$ or $c = 10$.
  For $c \in \{4, 8, 10\}$, Sun $(0, 3, c)$ is R-AVD and the values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 9.
- Sun (0,4,c) is OL-AVD only for $c \in \{4, 5, 6, 8, 10, 11, 12, 14, 16\}$.
  For all of these values of c, Sun (0,4,c) is also R-AVD and values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 10.
- Sun (0,5,c) is OL-AVD only for $c \in \{6, 8, 16\}$.

Table 8. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ for some Sun $(0, 2, c)$

c	λ	V_λ	$G[V_\lambda]$	$G[V \backslash V_\lambda]$
2, 4, 7, 8, 10, 14, 16, 18, 19	1	$\{u_3\}$	P_1	P_{c+7}
2, 4, 7, 8, 10, 14, 16, 19	2	$\{u_1, v_1\}$	P_2	Cat $(3, c+3)$
18		$\{x_1, x_2\}$	P_2	Cat $(5, 19)$
2, 4, 8, 10, 14, 16, 18	3	$\{u_2, v_2, x_1\}$	P_3	Cat $(2, c+3)$
7, 19		$\{u_1, v_1, x_{c+2}\}$	P_3	Cat $(5, c)$
2, 4, 7, 8, 10, 14, 16, 18, 19	4	$\{u_2, v_2, x_1, x_2\}$	P_4	P_{c+4}
2, 4, 8, 10, 14, 16, 18	5	$\{u_1, v_1, v_2, u_2, x_1\}$	Cat $(2, 3)$	Cat $(2, c+1)$
7, 19		$\{x_{c+2}, v_1, u_1, v_2, u_2\}$	Cat $(2, 3)$	Cat $(3, c)$
4, 7, 8, 10, 14, 16, 18, 19	6	$\{u_2, v_2, x_1, x_2, v_3, u_3\}$	P_6	P_{c+2}
7, 8, 10, 14, 16, 18, 19	7	$\{u_2, v_2, x_1, x_2, v_3, u_3, x_3\}$	Cat $(2, 5)$	P_{c+1}
8, 10, 14, 16, 18, 19	8	$\{u_1, v_1, v_2, u_2, x_1, x_2, v_3, u_3\}$	Cat $(3, 5)$	P_c
10, 14, 16, 18, 19	9	$\{u_2, v_2, x_1, x_2, v_3, u_3, x_3, x_4, x_5\}$	Cat $(4, 5)$	P_{c-1}
14	10	$\{u_1, v_1, x_{16}, \ldots, x_9\}$	P_{10}	Cat $(5, 7)$
16, 18, 19		$\{u_2, v_2, v_1, u_1, x_{c+2}, \ldots, x_{c-3}\}$	Cat $(3, 7)$	Cat $(3, c-5)$
14, 16, 18, 19	11	$\{u_2, v_2, x_1, x_2, v_3, u_3, x_3, \ldots, x_7\}$	Cat $(5, 6)$	P_{c-3}
16, 18, 19	12	$\{u_2, v_2, x_1, x_2, v_3, u_3, x_3, \ldots, x_8\}$	Cat $(5, 7)$	P_{c-4}
18, 19	13	$\{u_2, v_2, v_1, u_1, x_{c+2}, \ldots, x_{c-6}\}$	Cat $(3, 10)$	Cat $(3, c-8)$

Table 9. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ for Sun $(0, 3, c)$

c	λ	V_λ	$G[V_\lambda]$	$G[V \backslash V_\lambda]$
4, 8, 10	1	$\{u_3\}$	P_1	P_{c+8}
	2	$\{u_2, v_2\}$	P_2	Cat $(4, c+3)$
	3	$\{u_2, v_2, x_1\}$	P_3	Cat $(3, c+3)$
	4	$\{u_2, v_2, x_1, x_2\}$	P_4	Cat $(2, c+3)$
	5	$\{u_2, v_2, x_1, x_2, x_3\}$	P_5	P_{c+4}
4	6	$\{u_3, v_3, x_4, \ldots x_7\}$	P_6	Cat $(3, 4)$
8, 10		$\{u_1, v_1, x_{c+3}, \ldots, x_c\}$	P_6	Cat $(6, c-3)$
8, 10	7	$\{u_2, v_2, x_1, x_2, x_3, v_3, u_3\}$	P_7	P_{c+2}
8, 10	8	$\{u_2, v_2, v_1, u_1, x_{c+3}, \ldots, x_c\}$	Cat $(3, 5)$	Cat $(4, c-3)$
10	9	$\{x_1, x_2, x_3, v_3, u_3, x_4, \ldots, x_7\}$	Cat $(4, 5)$	Cat $(3, 7)$

Consider $\lambda = 2$. There is only two possibilities for V_2, either $V_2 = \{u_2, v_2\}$, or $V_2 = \{u_1, v_1\}$.
If $V_2 = \{u_2, v_2\}$, then $G[V \backslash V_2] = \text{Cat}\,(6, c+3)$ which is not R-AVD for any $c \in \{6, 8, 16\}$.
If $V_2 = \{u_1, v_1\}$, then $G[V \backslash V_2] = \text{Cat}\,(8, c+1)$ which is R-AVD for $c = 6$ but not for $c = 8$ or $c = 16$.
In fact, Sun (0, 5, 6) is R-AVD and values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 11.

- Sun (0,6,c) is OL-AVD only for $c \in \{8, 10\}$.
  Consider $\lambda = 3$. The two possibilities for V_3 are $\{u_2, v_2, x_1\}$ and $\{u_1, v_1, x_{c+6}\}$.

Table 10. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ for Sun $(0, 4, c)$

c	λ	V_λ	$G[V_\lambda]$	$G[V\backslash V_\lambda]$
4, 5, 6, 8, 10, 11, 12, 14, 16	1	$\{u_3\}$	P_1	P_{c+9}
4, 6, 11, 16	2	$\{u_2, v_2\}$	P_2	Cat $(5, c+3)$
5, 8, 10, 12, 14		$\{u_1, v_1\}$	P_2	Cat $(7, c+1)$
4, 6, 8, 10, 12, 14, 16	3	$\{u_2, v_2, x_1\}$	P_3	Cat $(4, c+3)$
5, 11		$\{u_1, v_1, x_{c+4}\}$	P_3	Cat $(7, c)$
4, 5, 8, 10, 11, 14, 16	4	$\{x_1, \ldots, x_4\}$	P_4	Cat $(3, c+3)$
6, 12		$\{u_1, v_1, x_{c+4}, x_{c+3}\}$	P_4	Cat $(7, c-1)$
4, 6, 8, 10, 12, 14, 16	5	$\{u_2, v_2, x_1, x_2, x_3\}$	P_5	Cat $(2, c+3)$
5		$\{u_1, v_1, x_9, x_8, x_7\}$	P_5	Cat $(3, 7)$
11		$\{u_2, v_2, v_1, u_1, x_{15}\}$	Cat $(2, 3)$	Cat $(5, 11)$
4, 5, 6, 8, 10, 11, 12, 14, 16	6	$\{u_2, v_2, x_1, \ldots, x_4\}$	P_6	P_{c+4}
4, 6, 8, 10, 12, 14, 16	7	$\{u_1, v_1, v_2, u_2, x_1, x_2, x_3\}$	Cat $(3, 4)$	Cat $(2, c+1)$
5, 11		$\{u_2, v_2, v_1, u_1, x_{c+4}, x_{c+3}, x_{c+2}\}$	Cat $(3, 4)$	Cat $(5, c-2)$
6, 8, 10, 11, 12, 14, 16	8	$\{u_2, v_2, x_1, \ldots, x_4, v_3, u_3\}$	P_8	P_{c+2}
8, 10, 11, 12, 14, 16	9	$\{u_2, v_2, x_1, \ldots, x_4, v_3, u_3, x_5\}$	Cat $(2, 7)$	P_{c+1}
10, 11, 12, 14, 16	10	$\{u_2, v_2, x_1, \ldots, x_4, v_3, u_3, x_5, x_6\}$	Cat $(3, 7)$	P_c
12, 14, 16	11	$\{u_2, v_2, x_1, \ldots, x_4, v_3, u_3, x_5, x_6, x_7\}$	Cat $(4, 7)$	P_{c-1}
14, 16	12	$\{u_2, v_2, x_1, \ldots, x_4, v_3, u_3, x_5, \ldots, x_8\}$	Cat $(5, 7)$	P_{c-2}
16	13	$\{u_2, v_2, x_1, \ldots, x_4, v_3, u_3, x_5, \ldots, x_9\}$	Cat $(6, 7)$	P_{c-3}

Table 11. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ for Sun $(0, 5, 6)$

λ	V_λ	$G[V_\lambda]$	$G[V\backslash V_\lambda]$
1	$\{u_3\}$	P_1	P_{16}
2	$\{u_1, v_1\}$	P_2	Cat $(7, 8)$
3	$\{u_2, v_2, x_1\}$	P_3	Cat $(5, 9)$
4	$\{u_2, v_2, x_1, x_2\}$	P_4	Cat $(4, 9)$
5	$\{u_1, v_1, v_2, u_2, x_1\}$	Cat $(2, 3)$	Cat $(5, 7)$
6	$\{u_2, v_2, x_1, \ldots, x_4\}$	P_6	Cat $(2, 9)$
7	$\{u_2, v_2, x_1, \ldots, x_5\}$	P_7	P_{10}
8	$\{u_1, v_1, v_2, u_2, x_1, \ldots, x_4\}$	Cat $(3, 5)$	Cat $(2, 7)$

In the first case, $G[V\backslash V_3] = \text{Cat}\,(6, c+3)$, in the second case $G[V\backslash V_3] = \text{Cat}\,(9, c)$. In both cases, $G[V\backslash V_3]$ is not R-AVD for $c = 8$ or $c = 10$.

- Sun (0,7,c) is OL-AVD only for $c \in \{8, 10\}$.
  Consider $\lambda = 2$. There is only two possibilities for V_2, either $V_2 = \{u_2, v_2\}$, or $V_2 = \{u_1, v_1\}$.
  If $V_2 = \{u_2, v_2\}$, then $G[V\backslash V_2] = \text{Cat}\,(8, c+3)$. If $V_2 = \{u_1, v_1\}$, then $G[V\backslash V_2] = \text{Cat}\,(10, c+1)$. Both Cat $(8, c+3)$ and Cat $(10, c+1)$ are not R-AVD for $c = 8$ and $c = 10$.
- Sun (0,8,c) is OL-AVD only for $c \in \{8, 9\}$.

Consider again $\lambda = 2$ and the two possibilities for V_2: $V_2 = \{u_2, v_2\}$, or $V_2 = \{u_1, v_1\}$.
If $V_2 = \{u_2, v_2\}$, then $G[V \backslash V_2] = \text{Cat}\,(9, c+3)$. If $V_2 = \{u_1, v_1\}$, then $G[V \backslash V_2] = \text{Cat}\,(11, c+1)$. Both Cat $(9, c+3)$ and Cat $(11, c+1)$ are not R-AVD for $c = 8$ and $c = 9$.

- Sun (1,2,c) is OL-AVD only for $c \equiv 2, 4 \pmod 6$ or $c \in \{6, 18\}$.
  Consider first $\lambda = 11$. That means that $n \geq 22$ and thus $c \geq 13$. We consider four possibilities to obtain a R-AVD graph with order 11:
  - $V_{11} = \{u_2, v_2, x_1, v_1, u_1, x_{c+3}, \ldots, x_{c-2}\}$. In that case, $G[V \backslash V_{11}] = \text{Cat}\,(3, c-5)$.
  - $V_{11} = \{u_2, v_2, x_2, x_3, v_3, u_3, x_4, \ldots, x_8\}$. Thus, $G[V \backslash V_{11}] = \text{Cat}\,(2, c-4)$.
  - $V_{11} = \{x_2, x_3, v_3, u_3, x_4, \ldots, x_10\}$. Thus, $G[V \backslash V_{11}] = \text{Cat}\,(4, c-6)$.
  - $V_{11} = \{x_1, v_1, u_1, x_{c+3}, \ldots, x_{c-4}\}$. Thus, $G[V \backslash V_{11}] = \text{Cat}\,(5, c-7)$.

  For all these cases, $G[V \backslash V_{11}]$ is not R-AVD for $c \geq 13, c \equiv 2 \pmod 6$, except for $G[V \backslash V_{11}] = \text{Cat}\,(5, 7)$ or Cat $(5, 19)$ and $c = 14$ or 26.
  Consider now $\lambda = 13$. That means that $n \geq 26$ and thus $c \geq 17$. We consider three possibilities to obtain a R-AVD graph with order 13:
  - $V_{13} = \{x_2, x_3, v_3, u_3, x_4, \ldots, x_13\}$. Thus, $G[V \backslash V_{13}] = \text{Cat}\,(4, c-8)$.
  - $V_{13} = \{u_2, v_2, x_1, v_1, u_1, x_{c+3}, \ldots, x_{c-4}\}$. In that case, $G[V \backslash V_{13}] = \text{Cat}\,(3, c-7)$.
  - $V_{13} = \{x_1, v_1, u_1, x_{c+3}, \ldots, x_{c-6}\}$. Thus, $G[V \backslash V_{13}] = \text{Cat}\,(5, c-9)$.

  For all these cases, $G[V \backslash V_{13}]$ is not R-AVD for $c \geq 17, c \equiv 4 \pmod 6$, except when $G[V \backslash V_{13}] = \text{Cat}\,(5, 19)$ and $c = 28$.
  At last, consider an induced subgraph with order 18. Because the only caterpillar with this order is Cat $(7, 11)$, the only way to have a R-AVD subgraph of Sun $(1, 2, c)$ with order 18 is a path P_{18}. In the cases of $c = 26$ or $c = 28$, the remaining subgraph contains four leaves and then, cannot be R-AVD.
  Thus, the only remaining values for c are 2, 4, 6, 8, 10, 14, 16 and 18. For all these values of c, Sun $(1, 2, c)$ is R-AVD and values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 12.
- Sun (2,3,c) is OL-AVD only for $c \in \{4, 8, 16\}$.
  Let us consider $\lambda = 2$. If $V_2 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}$ or $V_2 = \{u_3, v_3\}$, then $G[V \backslash V_2]$ has four leaves and then is not R-AVD. The only remaining possibility is $V_2 = \{x_1, x_2\}$, and thus $G[V \backslash V_2] = \text{Cat}\,(6, c+3)$. Then, Sun $(2, 3, c)$ cannot be R-AVD with $c = 8$ or $c = 16$.
  Sun $(2, 3, 4)$ is R-AVD and values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ are given in Table 13.

Sun with four rays. A sun with four rays is OL-AVD if and only if it is isomorphic to Sun $(0, 0, 1, d)$ with $d \equiv 2, 4 \pmod 6$.

Consider $\lambda = 6$. Since an R-AVD graph with order 6 must be a path, the only possibility is to have:

- $d = 2$,$V_6 = \{u_1, v_1, x_3, x_2, v_4, u_4\}$ and $G[V \backslash V_6] = \text{Cat}\,(2, 3)$
  or
- $d = 4$, $V_6 = \{u_1, v_1, x_5, x_4, x_3, x_2\}$ and $G[V \backslash V_6] = \text{Cat}\,(4, 3)$.

Table 12. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ for $\mathrm{Sun}\,(1,2,c)$

c	λ	V_λ	$G[V_\lambda]$	$G[V\backslash V_\lambda]$
2, 4, 6, 8, 10, 14, 16, 18	1	$\{u_1\}$	P_1	$\mathrm{Sun}\,(2,c+2)$
2, 4, 6, 8, 10, 14, 16, 18	2	$\{x_2,x_3\}$	P_2	$\mathrm{Cat}\,(4,c+3)$
2, 4, 8, 10, 14, 16		$\{x_1,v_2,u_2\}$	P_3	$\mathrm{Cat}\,(3,c+3)$
6, 18	3	$\{u_1,v_1,x_1\}$	P_3	$\mathrm{Cat}\,(5,c+1)$
2, 4, 6, 8, 10, 14, 16, 18	4	$\{u_2,v_2,x_2,x_3\}$	P_4	$\mathrm{Cat}\,(2,c+3)$
2, 4, 6, 8, 10, 14, 16, 18	5	$\{x_1,v_2,u_2,x_2,x_3\}$	$\mathrm{Cat}\,(2,3)$	P_{c+4}
4, 6, 8, 10, 14, 16, 18	6	$\{u_2,v_2,x_2,x_3,v_3,u_3\}$	P_6	$\mathrm{Cat}\,(2,c+1)$
6, 8, 10, 14, 16, 18	7	$\{x_1,v_2,u_2,x_2,x_3,v_3,u_3\}$	$\mathrm{Cat}\,(2,5)$	P_{c+2}
8, 10, 14, 16, 18	8	$\{u_2,v_2,x_2,x_3,v_3,u_3,x_4,x_5\}$	$\mathrm{Cat}\,(3,5)$	$\mathrm{Cat}\,(2,c-1)$
10, 14, 16, 18	9	$\{u_1,v_1,x_1v_2,u_2,x_2,x_3,v_3,u_3\}$	$\mathrm{Cat}\,(4,5)$	P_c
14, 16, 18	10	$\{x_2,x_3,v_3,u_3,x_4,\ldots,x_9\}$	$\mathrm{Cat}\,(3,7)$	$\mathrm{Cat}\,(4,c-5)$
14		$\{x_1,v_1,u_1,x_{c+3},\|dots,x_{c-4}\}$	$\mathrm{Cat}\,(2,9)$	$\mathrm{Cat}\,(5,c-7)$
16, 18	11	$\{u_2,v_2,x_1,v_1,u_1,x_{c+3},\|dots,x_{c-2}\}$	$\mathrm{Cat}\,(4,7)$	$\mathrm{Cat}\,(3,c-5)$
16, 18	12	$\{u_2,v_2,x_2,x_3,v_3,u_3,x_4,\ldots,x_9\}$	$\mathrm{Cat}\,(5,7)$	$\mathrm{Cat}\,(2,c-5)$
18	13	$\{u_2,v_2,x_1,v_1,u_1,x_{21},\ldots,x_9\}$	$\mathrm{Cat}\,(4,9)$	$\mathrm{Cat}\,(3,11)$

Table 13. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ for $\mathrm{Sun}\,(2,3,4)$

λ	V_λ	$G[V_\lambda]$	$G[V\backslash V_\lambda]$
1	$\{u_3\}$	P_1	$\mathrm{Sun}\,(2,8)$
2	$\{x_1,x_2\}$	P_2	$\mathrm{Cat}\,(6,7)$
3	$\{x_3,x_4,x_5\}$	P_3	$\mathrm{Cat}\,(5,7)$
4	$\{x_1,x_2,v_2,u_2\}$	P_4	$\mathrm{Cat}\,(4,7)$
5	$\{x_1,x_2,v_2,u_2,x_3\}$	$\mathrm{Cat}\,(2,3)$	$\mathrm{Cat}\,(3,7)$
6	$\{u_1,v_1,x_1,x_2,v_2,u_2\}$	P_6	$\mathrm{Cat}\,(4,5)$
7	$\{u_2,v_2,x_3,x_4,x_5,v_3,u_3\}$	P_7	$\mathrm{Cat}\,(3,5)$

We prove that both $\mathrm{Sun}\,(0,0,1,2)$ and $\mathrm{Sun}\,(0,0,1,4)$ are R-AVD, by giving the values of $G[V_\lambda]$ and $G[V\backslash V_\lambda]$ in Table 14.

Table 14. Values of $G[V_\lambda]$ and $G[V \backslash V_\lambda]$ for $\mathrm{Sun}\,(0,0,1,d)$

d	λ	V_λ	$G[V_\lambda]$	$G[V\backslash V_\lambda]$
2, 4	1	$\{u_1\}$	P_1	$\mathrm{Sun}\,(0,1,d+1)$
2, 4	2	$\{u_2,v_2\}$	P_2	$\mathrm{Cat}\,(4,d+3)$
2, 4	3	$\{u_3,v_3,x_1\}$	P_3	$\mathrm{Cat}\,(3,d+3)$
2, 4	4	$\{u_2,v_2,v_3,u_3\}$	P_4	$\mathrm{Cat}\,(2,d+3)$
2, 4	5	$\{u_2,v_2,v_3,u_3,x_1\}$	$\mathrm{Cat}\,(2,3)$	P_{d+4}
4	6	$\{u_1,v_1,x_5,x_4,x_3,x_2\}$	P_6	$\mathrm{Cat}\,(4,3)$

$\square$

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