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NEIGHBOURHOOD TOTAL DOMINATION IN GRAPHS

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Abstract. Let $G = (V, E)$ be a graph without isolated vertices. A dominating set S of G is called a *neighbourhood total dominating set* (*ntd-set*) if the induced subgraph $\langle N(S) \rangle$ has no isolated vertices. The minimum cardinality of a *ntd-set* of G is called the *neighbourhood total domination number* of G and is denoted by $\gamma_{nt}(G)$. The maximum order of a partition of V into *ntd*-sets is called the *neighbourhood total domatic number* of G and is denoted by $d_{nt}(G)$. In this paper we initiate a study of these parameters.

Keywords: neighbourhood total domination, total domination, connected domination, paired domination, neighbourhood total domatic number.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighbourhood and the closed neighbourhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$, then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$, then the private neighbour set of u with respect to S is defined by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$.

A subset S of V is called a dominating set of G if $N[S] = V$. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$ $(\Gamma(G))$. An excellent treatment of the fundamentals of domination is given in the book by Haynes *et al.* [6]. A survey of several advanced topics in domination is given in the book edited by Haynes *et al.* [7].

Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the Appendix of Haynes *et al.* [6].

Sampathkumar and Walikar [9] introduced the concept of connected domination in graphs. A dominating set S of a connected graph G is called a connected dominating set if the induced subgraph $\langle S\rangle$ is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$. Cockayne *et al.* [4] introduced the concept of total domination in graphs. A dominating set S of a graph G without isolated vertices is called a total dominating set of G if $\langle S\rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of G is called the total domination number of G and is denoted by $\gamma_t(G)$. Haynes and Slater [5] introduced the concept of paired domination in graphs. A dominating set S of a graph G without isolated vertices is called a paired dominating set if $\langle S\rangle$ has a perfect matching. The minimum cardinality of a paired dominating set of G is called the paired domination number of G and is denoted by $\gamma_{pr}(G)$.

For a dominating set S of G it is natural to look at how $N(S)$ behaves. For example, for the cycle $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_1, v_2, v_4\}$ are dominating sets, $\langle N(S_1)\rangle$ is not connected and $\langle N(S_2)\rangle$ is connected. Motivated by this example, in [1] we have introduced the concept of neighbourhood connected domination in graphs.

Definition 1.1 ([1]). A dominating set S of a connected graph G is called a neighbourhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S)\rangle$ is connected. A ncd-set S is said to be minimal if no proper subset of S is a ncd-set. The minimum cardinality of a ncd-set of G is called the neighbourhood connected domination number of G and is denoted by $\gamma_{nc}(G)$.

For the path $P_{10} = (v_1, v_2, \ldots, v_{10})$, $S_1 = \{v_2, v_5, v_7, v_9\}$ and $S_2 = \{v_1, v_4, v_6, v_7, v_{10}\}$ are dominating sets, $\langle N(S_1)\rangle$ has isolates and $\langle N(S_2)\rangle$ has no isolates. Motivated by this example, in this paper we introduce the concept of neighbourhood total domination and initiate a study of neighbourhood total domination number and neighbourhood total domatic number.

We need the following theorems.

Theorem 1.2 ([8]). *Let G be a nontrivial connected graph. Then $\gamma_c(G) + \kappa(G) = n$ if and only if $G = C_n$ or K_n or $K_{2a} - X$ where $a \geq 3$ and X is a 1-factor of K_{2a}.*

Theorem 1.3 ([1]). *Let G be any graph such that both G and $\overline{G}$ are connected. Then*

$$\gamma_{nc}(G) + \gamma_{nc}(\overline{G}) \leq \begin{cases} \lceil \frac{n}{2} \rceil + 2 & \text{if } diam\ G \geq 3, \\ \lceil \frac{n}{2} \rceil + 3 & \text{if } diam\ G = 2. \end{cases}$$

Theorem 1.4 ([1]). *Let T be any tree with $n > 2$. Then $\gamma_{nc}(T) = n - \Delta$ if and only if T can be obtained from a star by subdividing k of its edges, $k \geq 1$, once or by subdividing exactly one edge twice.*

2. MAIN RESULTS

We assume throughout that G is a graph without isolated vertices.

Definition 2.1. A dominating set S of a graph G is called a neighbourhood total dominating set (ntd-set) if the induced subgraph $\langle N(S)\rangle$ contains no isolated vertices. A ntd-set S is said to be minimal if no proper subset of S is a ntd-set. The minimum cardinality of a ntd-set of G is called the neighbourhood total domination number of G and is denoted by $\gamma_{nt}(G)$.

Remark 2.2. (i) Let S be a ntd-set of G. Since $\langle N(S)\rangle$ has no isolated vertices, it follows that $|N(S)| \geq 2$.
(ii) Clearly $\gamma_{nt} \geq \gamma$. Further if S is a total dominating set or a paired dominating set or a connected dominating set with $|S| > 1$, then $N(S) = V$ and hence $\gamma_{nt} \leq \gamma_t, \gamma_{nt} \leq \gamma_{pr}$ and $\gamma_{nt} \leq \gamma_c$ if $\gamma_c > 1$.
(iii) For any connected graph $G, \gamma_{nt} = 1$ if and only if there exists a vertex $v \in V(G)$ such that $deg\ v = n-1$ and $G - v$ has no isolated vertices.

Theorem 2.3. *For any connected graph G, $\gamma(G) \leq \gamma_{nt}(G) \leq \gamma_{nc}(G) \leq 2\gamma(G)$. Further given three positive integers a, b and c with $a \leq b \leq c \leq 2a$, there exists a graph G with $\gamma(G) = a$, $\gamma_{nt}(G) = b$ and $\gamma_{nc}(G) = c$.*

Proof. We have $\gamma(G) \leq \gamma_{nt}(G) \leq \gamma_{nc}(G) \leq \gamma_{pr}(G) \leq 2\gamma(G)$. Now, let a, b and c be positive integers with $a \leq b \leq c \leq 2a$. Let $b = a + r, 0 \leq r \leq a$, $c = a + k, r \leq k \leq 2a - r$. Consider the corona $K_a \circ K_1$ with $V(K_a) = \{v_1, v_2, \dots, v_a\}$ and let u_i be the pendant vertex adjacent to v_i. Take r copies $H_1, H_2, \dots, H_r$ of $\overline{K_2}$ and $k - r$ copies $G_{r+1}, G_{r+2}, \dots, G_k$ of P_4. Let G be the graph obtained from $K_a \circ K_1$ by joining u_i to all the vertices of H_i where $1 \leq i \leq r$ and by joining u_{r+j} to all the vertices of G_{r+j} where $1 \leq j \leq k - r$. Then $\gamma(G) = a$, $\gamma_{nt}(G) = a + r = b$ and $\gamma_{nc}(G) = a + k = c$. □

Theorem 2.4. *For the path P_n,*

$$\gamma_{nt}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \textit{if } n \equiv 1 (mod\ 3), \\ \lceil \frac{n}{3} \rceil + 1 & \textit{otherwise.} \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \dots, v_n)$. If $n \equiv 1(mod\ 3)$, then $S = \{v_i : i = 3k + 1, k = 0, 1, 2, \dots\}$ is a *ntd-set* of P_n. If $n \equiv 2(mod\ 3)$, then $S \cup \{v_n\}$ is a *ntd-set* of P_n. If $n \equiv 0(mod\ 3)$, then $S \cup \{v_{n-1}\}$ is a *ntd-set* of P_n. Hence

$$\gamma_{nt}(P_n) \leq \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 (mod\ 3), \\ \lceil \frac{n}{3} \rceil + 1 & \text{otherwise.} \end{cases}$$

Now, $\gamma_{nt}(P_n) \geq \gamma(P_n) = \lceil \frac{n}{3} \rceil$. Further if $n \not\equiv 1(mod\ 3)$, then for any γ-set S of P_n, $\langle N(S)\rangle$ has at least one isolated vertex and hence $\gamma_{nt}(P_n) \geq \lceil \frac{n}{3} \rceil + 1$. Hence the result follows. □

Corollary 2.5. For any nontrivial path P_n,

(i) $\gamma_{nt}(P_n) = \gamma(P_n)$ if and only if $n \equiv 1(mod\ 3)$.
(ii) $\gamma_{nt}(P_n) = \gamma_c(P_n)$ if and only if $n = 4$ or 5.
(iii) $\gamma_{nt}(P_n) = \gamma_t(P_n)$ if and only if $n = 2, 3, 4, 5$ or 8.
(iv) $\gamma_{nt}(P_n) = \gamma_{nc}(P_n)$ if and only if $n = 3, 4, 5, 6$ or 8.

Proof. Since $\gamma(P_n) = \lceil \frac{n}{3} \rceil$, $\gamma_c(P_n) = n - 2$,

$$\gamma_t(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0(mod\ 4), \\ \lfloor \frac{n}{2} \rfloor + 1 & \text{otherwise} \end{cases}$$

and $\gamma_{nc}(P_n) = \lceil \frac{n}{2} \rceil$ the corollary follows. □

Theorem 2.6. *For the cycle C_n,*

$$\gamma_{nt}(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & \textit{if } n \equiv 2(mod\ 3), \\ \lceil \frac{n}{3} \rceil & \textit{otherwise.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $n = 3k + r$, where $0 \leq r \leq 2$.
Let $S = \{v_i : i = 3j + 1, 0 \leq j \leq k\}$.
Let $S_1 = \begin{cases} S \cup \{v_n\} & \text{if } n \equiv 2(mod\ 3), \\ S & \text{otherwise.} \end{cases}$
Then S_1 is a *ntd-set* of C_n and hence

$$\gamma_{nt}(C_n) \leq \begin{cases} \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 2(mod\ 3), \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Now, $\gamma_{nt}(C_n) \geq \gamma(C_n) = \lceil \frac{n}{3} \rceil$. Further if $n \equiv 2(mod\ 3)$, then for any γ-set of S of C_n, $\langle N(S) \rangle$ has at least one isolated vertex and hence $\gamma_{nt}(C_n) \geq \lceil \frac{n}{3} \rceil + 1$. Hence the result follows. □

Corollary 2.7. (i) $\gamma_{nt}(C_n) = \gamma(C_n)$ if and only if $n \not\equiv 2(mod\ 3)$.
(ii) $\gamma_{nt}(C_n) = \gamma_c(C_n)$ if and only if $n = 3, 4$ or 5.
(iii) $\gamma_{nt}(C_n) = \gamma_t(C_n)$ if and only if $n = 4, 5$ or 8.
(iv) $\gamma_{nt}(C_n) = \gamma_{nc}(C_n)$ if and only if $n = 3, 4, 5$ or 7.

Proof. Since $\gamma(C_n) = \lceil \frac{n}{3} \rceil$, $\gamma_c(C_n) = n - 2$,

$$\gamma_t(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 2(mod\ 4), \\ \lceil \frac{n}{2} \rceil & \text{otherwise,} \end{cases}$$

and

$$\gamma_{nc}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 3(mod\ 4), \\ \lfloor \frac{n}{2} \rfloor & \text{if } n \equiv 3(mod\ 4) \end{cases}$$

the result follows. □

We now proceed to obtain a characterization of minimal ntd-sets.

Lemma 2.8. *A superset of a ntd-set is a ntd-set.*

Proof. Let S be a ntd-set of a graph G and let $S_1 = S \cup \{v\}$, where $v \in V - S$. Clearly, $v \in N(S_1)$ and S_1 is a dominating set of G. Suppose there exists an isolated vertex y in $\langle N(S_1)\rangle$. Then $N(y) \subseteq S - N(S)$ and hence y is an isolated vertex in $\langle N(S)\rangle$, which is a contradiction. Hence $\langle N(S_1)\rangle$ has no isolated vertices and S_1 is a ntd-set. □

Theorem 2.9. *A ntd-set S of a graph G is a minimal ntd-set if and only if for every $u \in S$, one of the following holds:*

(i) $pn[u, S] \neq \emptyset$.
(ii) *There exists a vertex $x \in N(S - \{u\})$ such that $N(x) \cap N(S - \{u\}) = \emptyset$.*

Proof. Let S be a minimal ntd-set of G. Let $u \in S$. Then either $S - \{u\}$ is not a dominating set of G or $S - \{u\}$ is a dominating set and $\langle N(S - \{u\})\rangle$ has an isolated vertex. If $S - \{u\}$ is not a dominating set of G, then $pn[u, S] \neq \emptyset$. If $S - \{u\}$ is a dominating set and if $x \in N(S - \{u\})$ is an isolated vertex in $\langle N(S - \{u\})\rangle$, then $N(x) \cap N(S - \{u\}) = \emptyset$. Conversely, if S is a ntd-set of G satisfying the conditions of the theorem, then S is a 1-minimal ntd-set and hence the result follows from Lemma 2.8. □

Remark 2.10. Let G be a graph with $\Delta = n - 1$. Then $\gamma_{nt}(G) = 1$ or 2. Further $\gamma_{nt}(G) = 2$ if and only if G has exactly one vertex v with $deg\ v = n - 1$ and v is adjacent to a vertex of degree 1. (A vertex which is adjacent to a vertex of degree 1 is called a support vertex).

Remark 2.11. Since any ntd-set of a spanning subgraph H of a graph G is a ntd-set of G, we have $\gamma_{nt}(G) \leq \gamma_{nt}(H)$.

Remark 2.12. If G is a disconnected graph with k components $G_1, G_2, \ldots, G_k$ then $\gamma_{nt}(G) = \gamma_{nt}(G_1) + \gamma_{nt}(G_2) + \cdots + \gamma_{nt}(G_k)$.

We now proceed to obtain bounds for γ_{nt}.

Observation 2.13. For any graph $G, \gamma_{nt}(G) = n$ if and only if $G = mK_2$.

Theorem 2.14. *For any graph G, $\gamma_{nt}(G) \leq n - \Delta + 1$. Further, $\gamma_{nt}(G) = n - \Delta + 1$ if and only if G is isomorphic to H or $sK_2 \cup H$ where H is any graph having a support vertex v with $deg\ v = |V(H)| - 1$.*

Proof. Let $v \in V(G)$ and $deg\ v = \Delta$. Let $S = N(v) - \{u\}$ where $u \in N(v)$. Then $V - S$ is a ntd-set of G and hence $\gamma_{nt}(G) \leq n - \Delta + 1$.

Now, let G be any graph with $\gamma_{nt}(G) = n - \Delta + 1$.

Case i. G is connected.

If $\Delta < n - 1$, then $V - S$ where $S = (N(v) - \{u\}) \cup \{w\}$, $u \in N(v)$, $w \notin N[v]$, is a ntd-set of G with $|V - S| = n - \Delta$ which is a contradiction. Hence $\Delta = n - 1$ and $deg\ v = n - 1$. If $n = 2$, then $H = K_2$. Suppose $n \geq 3$. If $deg\ u \geq 2$ for all $u \in N(v)$,

then $\{v\}$ is a ntd-set of G and hence $\gamma_{nt}(G) = 1$, which is a contradiction. Hence $deg\ u = 1$ for some $u \in N(v)$, so that v is a support vertex of H.
Case ii. G is disconnected.

Let $G_1, G_2, \ldots, G_k$ be the components of G and let $|V(G_i)| = n_i$. If $\Delta = 1$, then $\gamma_{nt} = n$ and each G_i is isomorphic to K_2. Suppose $\Delta \geq 2$. Let $v \in V(G_1)$ be such that $deg\ v = \Delta$. Since $\gamma_{nt}(G) = n - \Delta + 1$ it follows that $\gamma_{nt}(G_1) = n_1 - \Delta + 1$ and $\gamma_{nt}(G_i) = n_i$ for all $i \geq 2$. Hence by Case i, G_1 is isomorphic to H where H is any graph having a support vertex v with $deg\ v = |V(H)| - 1$ and G_i is isomorphic to K_2 for all $i \geq 2$. □

Theorem 2.15. *Let G be a connected graph with $\Delta < n - 1$. Then $\gamma_{nt}(G) \leq n - \Delta$. Further, for a tree T with $\Delta < n - 1$ the following are equivalent.*

(i) $\gamma_{nt}(T) = n - \Delta$.
(ii) $\gamma_{nc}(T) = n - \Delta$.
(iii) *T can be obtained from a star by subdividing k of its edges, $k \geq 1$ once or by subdividing exactly one edge twice.*

Proof. Let $v \in V(G)$ and $deg\ v = \Delta$. Since G is connected and $\Delta < n - 1$, there exist two adjacent vertices u and w such that $u \in N(v)$ and $w \notin N[v]$. Let $S = (N(v) - \{u\}) \cup \{w\}$. Then $V - S$ is a ntd-set of G and hence $\gamma_{nt}(G) \leq n - \Delta$.

Now, let T be a tree with $\Delta < n - 1$. Suppose $\gamma_{nt}(T) = n - \Delta$. Then $n - \Delta = \gamma_{nt}(T) \leq \gamma_{nc}(T) \leq n - \Delta$. Hence $\gamma_{nc}(T) = n - \Delta$, so that (i) implies (ii).

It follows from Theorem 1.4 that (ii) implies (iii). We now prove (iii) implies (i). Consider the star $K_{1,\Delta}$, where $V(K_{1,\Delta}) = \{v, v_1, v_2, \ldots, v_\Delta\}$ with $deg\ v = \Delta$.
Case i. T is obtained from $K_{1,\Delta}$ by subdividing the k edges $vv_1, vv_2, \ldots, vv_k$. Let u_i be the vertex subdividing uv_i, $1 \leq i \leq k$. Clearly, $n - \Delta = k + 1$. Also any ntd-set S of T contains either u_i or v_i for each $i, 1 \leq i \leq k$ and also contains the vertex v. Hence it follows that $|S| \geq k + 1 = n - \Delta$ and $\gamma_{nt}(T) = n - \Delta$.
Case ii. T is obtained from $K_{1,\Delta}$ by subdividing the edge vv_1 twice.

Let u_1, u_2 be the vertices subdividing vv_1. Then $n - \Delta = 3$ and $S = \{v, u_1, u_2\}$ is a minimum ntd-set of T. Thus $\gamma_{nt}(T) = n - \Delta$. □

Corollary 2.16. For a forest G, $\gamma_{nt}(G) = n - \Delta$ if and only if G is isomorphic to $K_2 \cup T$, where T is a tree with $\gamma_{nt}(T) = |V(T)| - \Delta(T)$.

Theorem 2.17. *For each γ_{nt}-set S of a connected graph G, let t_S denote the number of vertices v such that v is not a pendant vertex of G and v is isolated in $\langle S \rangle$. Let $t = \min\{t_S : S$ is a γ_{nt}-set of $G\}$. Then $\gamma_{nc}(G) \leq \gamma_{nt}(G) + t$.*

Proof. Let S be a γ_{nt}-set of G such that the number of vertices in S which are non-pendant vertices of G and are isolated in $\langle S \rangle$ is t.

Let $X = \{v \in S : d(v) = 0$ in $\langle S \rangle$ and $d(v) > 1$ in $G\}$ so that $|X| = t$. For each $v \in X$, choose a vertex $f(v) \in V(G)$ which is adjacent to v. Then $S_1 = S \cup \{f(v) : v \in X\}$ is a ncd-set of G and hence $\gamma_{nc}(G) \leq |S| \leq \gamma_{nt}(G) + t$. □

Theorem 2.18. *Let G be a connected graph with diam $G = 2$. Then $\gamma_{nt}(G) \leq 1 + \delta(G)$ and the bound is sharp.*

Proof. If $v \in V(G)$ and $deg\ v = \delta$, then $N[v]$ is an ntd-set of G and hence the result follows. The bound is attained for $K_{1,n}$ and C_5. □

Theorem 2.19. *Let G be a connected graph with diam $G = 2$ and $\gamma_{nt}(G) = 1+\delta(G)$. Then for every vertex $v \in V(G)$ with deg $v = \delta(G)$, $N(v)$ is an independent set and for all $u \in N(v)$ there exists a vertex $w \notin N(v)$ such that w is adjacent only to u.*

Proof. Let $S_1 = N(v)$. Clearly S_1 is a dominating set of G. Now, suppose $N(v)$ is not an independent set. Then $\langle N(v)\rangle$ contains an edge $e = xy$. Hence v is not isolated in $\langle N(S_1)\rangle$ and since *diam* $G = 2$, every vertex $w \notin N[v]$ is adjacent to either x or a neighbour of x. Thus w is not isolated in $\langle N(S_1)\rangle$. Hence S_1 is a ntd-set of G and $\gamma_{nt}(G) \leq \delta(G)$ which is a contradiction. Thus $N(v)$ is an independent set.

Now, suppose there exists a vertex $u \in N(v)$ such that u has no private neighbour in $V - N[v]$. Then $N[v] - \{u\}$ is a ntd-set of G with cardinality $\delta(G)$ which is a contradiction. Hence the result follows. □

Remark 2.20. The converse of Theorem 2.19 is not true. Consider the graph G given in Figure 1.

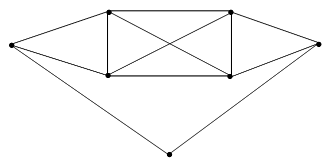

Fig. 1

Here $\delta(G) = 2$ and $\gamma_{nt}(G) = 2$. However, the unique vertex v with $deg\ v = \delta = 2$ satisfies the conditions given in Theorem 2.19.

Theorem 2.21. *Let G be a graph such that both G and $\overline{G}$ have no isolated vertices. Then $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq n + 2$. Further, equality holds if and only if G or $\overline{G}$ is isomorphic to sK_2, where $s > 1$.*

Proof. If G and $\overline{G}$ are both connected, then $\gamma_{nt}(G) \leq \gamma_{nc}(G) \leq \lceil \frac{n}{2} \rceil$ and $\gamma_{nt}(\overline{G}) \leq \lceil \frac{n}{2} \rceil$, so that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq n + 1$.

If G is disconnected, then $\gamma_{nt}(\overline{G}) = 2$ and hence $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq n + 2$.

Now, let G be any graph with $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = n + 2$. Then G or $\overline{G}$ is disconnected. Suppose G is disconnected. Then $\gamma_{nt}(G) = n$ and $\gamma_{nt}(\overline{G}) = 2$ and hence G is isomorphic to sK_2 where $s > 1$. The converse is obvious. □

The bound given by Theorem 2.21 can be substantially improved when G and $\overline{G}$ are both connected, as shown in the following theorem.

Theorem 2.22. *Let G be any graph such that both G and $\overline{G}$ are connected. Then*

$$\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \begin{cases} \lceil \frac{n}{2} \rceil + 2 & \text{if } diam\ G \geq 3, \\ \lceil \frac{n}{2} \rceil + 3 & \text{if } diam\ G = 2. \end{cases}$$

Proof. Since $\gamma_{nt} \leq \gamma_{nc}$ the result follows from Theorem 1.3 □

Remark 2.23. The bounds given in Theorem 2.22 are sharp. The graph $G = C_5$ has diameter 2, $\gamma_{nt}(G) = \gamma_{nt}(\overline{G}) = 3$ and $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = 6 = \lceil \frac{n}{2} \rceil + 3$. For the graph $G = C_k \circ K_1$ *diam* $G \geq 3$ and $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \lceil \frac{n}{2} \rceil + 2$.

Problem 2.24. Characterize graphs which attain the bounds given in Theorem 2.22.

Theorem 2.25. *For any connected graph G, $\gamma_{nt}(G) + \kappa(G) \leq n - \Delta + \delta + 1$ and equality holds if and only if G contains a support vertex v with deg $v = n - 1$.*

Proof. We have $\gamma_{nt} \leq n - \Delta + 1$ and $\kappa \leq \delta$. Hence $\gamma_{nt} + \kappa \leq n - \Delta + \delta + 1$.

Let G be a connected graph and let $\gamma_{nt}(G) + \kappa(G) = n - \Delta + \delta + 1$. Then $\gamma_{nt}(G) = n - \Delta + 1$ and $\kappa = \delta$ and the result follows from Theorem 2.14. □

Theorem 2.26. *For any graph G, $\gamma_{nt}(G) + \kappa(G) = n$ if and only if G is isomorphic to one of the graphs sK_2, $s > 1$, P_3 or C_5 or K_n or $K_{2a} - X, a \geq 3$ and X is a 1-factor of K_{2a}.*

Proof. Let G be a graph with $\gamma_{nt}(G) + \kappa(G) = n$.
Case i. G is connected.

Suppose $\Delta = n - 1$. Then $\gamma_{nt} = 1$ or 2. If $\gamma_{nt} = 1$, then $\kappa = n - 1$ and hence G is isomorphic to K_n. If $\gamma_{nt} = 2$ then G contains a support vertex of degree $n - 1$ and hence $\kappa = 1$, $n = 3$. Hence G is isomorphic to P_3.

Suppose $\Delta < n - 1$. Then $\gamma_{nt} \leq \gamma_c$ and $\gamma_{nt} + \kappa \leq \gamma_c + \kappa$ so that $\gamma_c + \kappa \geq n$. Since $\gamma_c + \kappa \leq n$ we get $\gamma_c + \kappa = n$ and $\gamma_{nt} = \gamma_c$. Therefore by Theorem 1.2 G is isomorphic to C_5 or $K_{2a} - X$ where X is a 1-factor in K_{2a}.
Case ii. G is disconnected.

Then $\kappa = 0$. Hence $\gamma_{nt} = n$ so that G is isomorphic to $sK_2, s > 1$. The converse is obvious. □

3. NEIGHBOURHOOD TOTAL DOMATIC NUMBER

The maximum order of a partition of the vertex set V of a graph G into dominating sets is called the domatic number of G and is denoted by $d(G)$. For a survey of results on domatic number and their variants we refer to Zelinka [10]. In [2] we have initiated a study of the neighbourhood connected domatic number of a graph. In this section we present a few basic results on the neighbourhood total domatic number of a graph.

Definition 3.1. Let G be a graph without isolated vertices. A neighbourhood total domatic partition (nt-domatic partition) of G is a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ in which each V_i is a ntd-set of G. The maximum order of an nt-domatic partition of G is called the neighbourhood total domatic number (nt-domatic number) of G and is denoted by $d_{nt}(G)$.

Observation 3.2. Since any domatic partition of K_n, where $n \geq 3$, is also a nt-domatic partition, we have $d_{nt}(K_n) = d(K_n) = n$. Similarly $d_{nt}(K_{r,s}) = d(K_{r,s}) = min\{r, s\}$. Also for the wheel W_n, $d_{nt}(W_n) = d(W_n) = \begin{cases} 4 & \text{if } n \equiv 1 (mod\ 3), \\ 3 & \text{otherwise.} \end{cases}$

Observation 3.3. Since any total domatic partition of G is a nt-domatic partition and any nc-domatic partition is a nt-domatic partition, we have $d_t(G) \leq d_{nc}(G) \leq d_{nt}(G) \leq d(G)$.

Observation 3.4. Let $v \in V(G)$ and $deg\ v = \delta$. Since any ntd-set of G must contain either v or a neighbour of v, it follows that $d_{nt}(G) \leq \delta(G) + 1$.

Definition 3.5. A graph G is called nt-domatically full if $d_{nt}(G) = \delta(G) + 1$.

Example 3.6. The graph G given in Figure 2 is nt-domatically full. In fact $\{\{v_1\}, \{v_2, v_4, v_6, v_8\}, \{v_3, v_5, v_7, v_9\}\}$ is a nt-domatic partition of G of maximum order and $d_{nt}(G) = 3 = 1 + \delta(G)$.

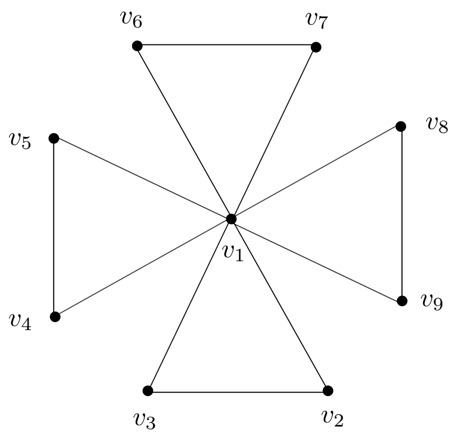

Fig. 2. nt-domatically full graph

Observation 3.7. Given two positive integers n and k with $n \geq 4$ and $1 \leq k \leq n$, there exists a graph G with n vertices such that $d_{nt}(G) = k$. We take

$$G = \begin{cases} K_n & \text{if } k = n, n \geq 3, \\ K_{1,n-1} & \text{if } k = 1, \\ B(n_1, n-2-n_1) & \text{if } k = 2, \\ K_{k-1} + \overline{K_{n-k+1}} & \text{otherwise.} \end{cases}$$

Theorem 3.8. *For the path $P_n, n \geq 2$, we have*

$$d_{nt}(P_n) = \begin{cases} 1 & \textit{if } n = 2, 3 \textit{ or } 5, \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. The result is trivial for $n = 2, 3$ or 5. Suppose $n \neq 2, 3, 5$. It follows from Observation 3.4 that $d_{nt}(P_n) \leq 2$.

Now let $S = \{v_i : i \equiv 1(mod\ 3)\}$ and let

$$V_1 = \begin{cases} S & \text{if } n \equiv 1(mod\ 3), \\ S \cup \{v_{n-2}\} & \text{if } n \equiv 2(mod\ 3), \\ S \cup \{v_{n-1}\} & \text{if } n \equiv 0(mod\ 3). \end{cases}$$

Then $\{V_1, V - V_1\}$ is a nt-domatic partition of P_n and hence $d_{nt}(P_n) = 2$. □

Theorem 3.9. *For the cycle C_n with $n \geq 4$ we have*

$$d_{nt}(C_n) = \begin{cases} 1 & \text{if } n = 5, \\ 3 & \text{if } n \equiv 0(mod\ 3), \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = (v_0, v_1, \ldots, v_{n-1}, v_0)$. The result is trivial for $n = 5$. Suppose $n \neq 5$. It follows from Observation 3.4 that $d_{nt}(C_n) \leq 3$. If $n \equiv 0(mod\ 3)$, let $n = 3k$ and let $S_i = \{v_j : 0 \leq j \leq n-1 \text{ and } j \equiv i(mod\ 3)\}, i = 0, 1, 2$. Then $\{S_0, S_1, S_2\}$ is a nt-domatic partition of C_n and hence $d_{nt}(C_n) = 3$. Now, suppose $n \not\equiv 0(mod\ 3)$. Let $n = 3k + r$ where $r = 1$ or 2.

Let $S_1 = \begin{cases} \{v_i : i \equiv 1(mod\ 3)\} & \text{if } n \equiv 1(mod\ 3), \\ \{v_i : i \equiv 2 \text{ or } 3(mod\ 4)\} & \text{if } n \equiv 2(mod\ 3). \end{cases}$

Then $\{S_1, V - S_1\}$ is a nt-domatic partition of C_n and hence $d_{nt}(C_n) \geq 2$. Also it follows from Theorem 2.6 that $d_{nt}(C_n) \leq 2$ and hence $d_{nt}(C_n) = 2$. □

Observation 3.10. If $\{V_1, V_2, \ldots, V_{d_{nt}}\}$ is a nt-domatic partition of G, then $|V_i| \geq \gamma_{nt}$ for each i and hence $\gamma_{nt}(G)d_{nt}(G) \leq n$.

Example 3.11.
(i) If $G \cong sK_r$ $r \geq 3, s \geq 1$, then $d_{nt}(G) = r$ and $\gamma_{nt}(G) = s$ and hence $d_{nt}(G)\gamma_{nt}(G) = sr = n$.
(ii) If $G \cong sK_{r,r}$ $r \geq 2, s \geq 1$, then $d_{nt}(G) = r$, $\gamma_{nt}(G) = 2s$ and hence $d_{nt}(G)\gamma_{nt}(G) = 2sr = n$.
(iii) If $G \cong G_1 \circ K_1$ where G_1 is any connected graph, then $d_{nt}(G) = 2$ and $\gamma_{nt}(G) = \frac{n}{2}$ and hence $d_{nt}(G)\gamma_{nt}(G) = n$.

Problem 3.12. Characterize the class of graphs for which $d_{nt}(G)\gamma_{nt}(G) = n$.

Theorem 3.13. *Let G be a graph of order $n \geq 5$ with $\Delta = n-1$ and let k denote the number of vertices of degree $n-1$. Then $d_{nt}(G) \leq \frac{1}{2}(n+k)$. Further $d_{nt}(G) = \frac{1}{2}(n+k)$ if and only if $G = K_k + H$ where either H is isomorphic to $2K_{\frac{n-k}{2}}$ or H is a connected graph with $V(H) = X_1 \cup X_2 \cup \cdots \cup X_r$, $r = \frac{n-k}{2}$, $|X_i| = 2$, $X_i \cap X_j = \emptyset$ for all $i \neq j$ and the subgraph induced by the edges of H with one end in X_i and the other end in X_j has a perfect matching.*

Proof. Let $\{V_1, V_2, \ldots, V_s\}$ be any nt-domatic partition of G with $|V_i| = 1, 1 \leq i \leq k$. Since $|V_j| \geq 2$ for all j with $k+1 \leq j \leq s$, it follows that $s \leq k + \frac{n-k}{2} = \frac{n+k}{2}$. Hence $d_{nt}(G) \leq \frac{1}{2}(n+k)$.

Now, let G be a graph with $d_{nt}(G) = \frac{1}{2}(n+k)$. Then there exists a nt-domatic partition $\{V_1, V_2, \ldots, V_k, V_{k+1}, \ldots, V_{\frac{n+k}{2}}\}$ such that $|V_i| = 1$ if $1 \leq i \leq k$ and $|V_j| = 2$ if $k+1 \leq j \leq \frac{n+k}{2}$. Clearly, $\langle V_1 \cup V_2 \cup \cdots \cup V_k \rangle \cong K_k$. Let $H = \left\langle V_{k+1} \cup \cdots \cup V_{\frac{n+k}{2}} \right\rangle$.

Case i. H is disconnected.

Since $|V_j| = 2$ for all j with $k+1 \leq j \leq \frac{n+k}{2}$, it follows that H has exactly two components H_1, H_2 and each V_j contains one vertex from H_1 and one vertex from H_2. Since V_j is a ntd-set of G, it follows that H_1 and H_2 are complete graphs and

$|V(H_1)| = |V(H_2)| = \frac{n-k}{2}$. Hence H is isomorphic to $2K_{\frac{n-k}{2}}$. If $k = 1$, then each H_1 and H_2 must contain at least two vertices. Hence $n \geq 5$.

Case ii. H is connected.

Let $X_i = V_{k+i}, 1 \leq i \leq r = \frac{n-k}{2}$. Then $V(H) = X_1 \cup X_2 \cup \cdots \cup X_r$ and $X_i \cap X_j = \emptyset$ when $i \neq j$. Now, since each X_i is a dominating set of G, it follows that the subgraph induced by the edges of H with one end in X_i and the other end in X_j has a perfect matching.

Conversely, suppose G is of the form given in the theorem. Let $u_1, u_2, \ldots, u_k$ be the vertices of G with $deg\ u_i = n-1, 1 \leq i \leq k$.

Suppose $G = K_k + H$ where H is isomorphic to $2K_{\frac{n-k}{2}}$ with $n \geq 5$ when $k = 1$.

Let H_1 and H_2 be the two components of H with $V(H_1) = \{x_i : k+1 \leq i \leq \frac{n+k}{2}\}$ and $V(H_2) = \{y_i : k+1 \leq i \leq \frac{n+k}{2}\}$. Let

$$V_i = \begin{cases} \{u_i\} & \text{if } 1 \leq i \leq k, \\ \{x_i, y_i\} & \text{where } x_i \in V(H_1) \text{ and } y_i \in V(H_2), \text{ if } k+1 \leq i \leq \frac{n+k}{2}. \end{cases}$$

Then $\{V_1, V_2, \ldots, V_{\frac{n+k}{2}}\}$ is a nt-domatic partition of G. Also if $G = K_k + H$, where H is a connected graph satisfying the conditions stated in the theorem, then $\{\{u_1\}, \{u_2\}, \ldots, \{u_k\}, X_1, X_2, \ldots, X_r\}$ is a nt-domatic partition of G. Thus $d_{nt}(G) \geq k + r = \frac{n+k}{2}$ and hence $d_{nt}(G) = \frac{n+k}{2}$. □

Corollary 3.14. Let G be a graph with $\Delta < n-1$. Then $d_{nt}(G) \leq \frac{n}{2}$. Further $d_{nt}(G) = \frac{n}{2}$ if and only if $V = X_1 \cup X_2 \cup \cdots \cup X_{\frac{n}{2}}$, where $|X_i| = 2$ for all i, $X_i \cap X_j = \emptyset$ if $i \neq j$, the subgraph induced by the edges of G with one end in X_i and the other end in X_j has a perfect matching and $\langle V - X_i \rangle$ has no isolated vertex if X_i is independent.

Theorem 3.15. *Let G be any graph such that both G and $\overline{G}$ are connected. Then $d_{nt}(G) + d_{nt}(\overline{G}) \leq n$. Further equality holds if and only if $V(G) = X_1 \cup X_2 \cup \cdots \cup X_{\frac{n}{2}}$, where $X_i \cap X_j = \emptyset$ and $\langle X_i \cup X_j \rangle$ is C_4 or P_4 or $2K_2$ for all $i \neq j$.*

Proof. Since both G and $\overline{G}$ are connected, it follows that $\Delta < n-1$. Hence $d_{nt}(G) \leq \frac{n}{2}$ and $d_{nt}(\overline{G}) \leq \frac{n}{2}$, so that $d_{nt}(G) + d_{nt}(\overline{G}) \leq n$.

Now, suppose $d_{nt}(G) + d_{nt}(\overline{G}) = n$. Then $d_{nt}(G) = \frac{n}{2}$ and $d_{nt}(\overline{G}) = \frac{n}{2}$. Since $d_{nt}(G) \leq \delta(G) + 1$, it follows that $\delta(G) \geq \frac{n}{2} - 1$ and $\delta(\overline{G}) \geq \frac{n}{2} - 1$ and hence $deg\ v = \frac{n}{2} - 1$ or $\frac{n}{2}$ for all $v \in V(G)$.

Now, let $V = X_1 \cup X_2 \cup \cdots \cup X_{\frac{n}{2}}$ be a nt-domatic partition of G. Then the subgraph induced by the edges of G with one end in X_i and the other end in X_j has a perfect matching. Further, if $\langle X_i \cup X_j \rangle$ has more than four edges, then at least one vertex v of $\langle X_i \cup X_j \rangle$ has degree at least 3. Since there are $\frac{n}{2} - 2$ ntd-sets other than X_i and X_j, $deg\ v \geq \frac{n}{2} + 1$ which is a contradiction. Thus $\langle X_i \cup X_j \rangle$ contains at most four edges and hence is isomorphic to C_4 or P_4 or $2K_2$. The converse is obvious. □

4. CONCLUSION AND SCOPE

In this paper we have introduced a new type of domination, namely, neighbourhood total domination. We have also discussed the corresponding neighbour total domatic partition. The following are some interesting problems for further investigation.

Problem 4.1. Characterize the class of graphs for which $\gamma_{nt}(G) = n - \Delta$.

Problem 4.2. Characterize graphs for which $\gamma_{nt}(G) = \lceil \frac{n}{2} \rceil$.

Problem 4.3. Characterize the class of graphs for which $\gamma_{nt}(G) = n - 1$ or $n - 2$.

Problem 4.4. Characterize nt-domatically full graphs.

Problem 4.5. Characterize graphs for which $d_{nt}(G) = d(G)$.

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