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OPERATORS IN DIVERGENCE FORM AND THEIR FRIEDRICHS AND KREĬN EXTENSIONS

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Abstract. For a densely defined nonnegative symmetric operator $\mathcal{A} = L_2^* L_1$ in a Hilbert space, constructed from a pair $L_1 \subset L_2$ of closed operators, we give expressions for the Friedrichs and Kreĭn nonnegative selfadjoint extensions. Some conditions for the equality $(L_2^* L_1)^* = L_1^* L_2$ are obtained. Applications to 1D nonnegative Hamiltonians, corresponding to point interactions, are given.

Keywords: symmetric operator, divergence form, Friedrichs extension, Kreĭn extension.

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1. INTRODUCTION

Let H be a separable Hilbert space and let $\mathcal{A}$ be a densely defined closed, symmetric, and nonnegative operator, i.e., $(\mathcal{A}f, f) \geq 0$ for all $\operatorname{dom}(\mathcal{A})$. As is well known, the operator $\mathcal{A}$ admits at least one nonnegative self-adjoint extension $\mathcal{A}_F$ called the *Friedrichs extension*, which is defined as follows. Denote by $\mathcal{A}[\cdot,\cdot]$ the closure of the sesquilinear form

$$\mathcal{A}[f,g] = (\mathcal{A}f, g), \quad f, g \in \operatorname{dom}(\mathcal{A}),$$

and let $\mathcal{D}[\mathcal{A}]$ be the domain of this closure. According to the first representation theorem [16] there exists a nonnegative self-adjoint operator $\mathcal{A}_F$ associated with $\mathcal{A}[\cdot,\cdot]$, i.e.,

$$(\mathcal{A}_F h, \psi) = \mathcal{A}[h, \psi], \quad \psi \in \mathcal{D}[\mathcal{A}], \quad h \in \operatorname{dom}(\mathcal{A}_F).$$

Clearly $\mathcal{A} \subset \mathcal{A}_F \subset \mathcal{A}^*$, where $\mathcal{A}^*$ is adjoint to $\mathcal{A}$. It follows that

$$\operatorname{dom}(\mathcal{A}_F) = \mathcal{D}[\mathcal{A}] \cap \operatorname{dom}(\mathcal{A}^*).$$

By the second representation theorem the equalities

$$\mathcal{D}[\mathcal{A}] = \operatorname{dom}(\mathcal{A}_F^{1/2}) \quad \text{and} \quad \mathcal{A}[\phi, \psi] = (\mathcal{A}_F^{1/2}\phi, \mathcal{A}_F^{1/2}\psi), \quad \phi, \psi \in \mathcal{D}[\mathcal{A}]$$

hold.

M.G. Kreĭn in [19] via fractional-linear transformation and parametrization of all contractive self-adjoint extensions of a non-densely defined Hermitian contraction discovered one more nonnegative self-adjoint extension of $\mathcal{A}$ having extremal property to be a minimal (in the sense of corresponding quadratic forms) among others nonnegative self-adjoint extensions of $\mathcal{A}$. This extension we will denote by $\mathcal{A}_K$ and call it the *Kreĭn extension* of $\mathcal{A}$. When $\mathcal{A}$ is positively definite, i.e., the lower bound of $\mathcal{A}$ is a positive number, it is shown in [19, 20] that

$$\operatorname{dom}(\mathcal{A}_K) = \operatorname{dom}(\mathcal{A}) \dotplus \ker(\mathcal{A}^*).$$

Thus, in that case the Kreĭn extension coincides with selfadjoint extension constructed by J. von Neumann. That's why $\mathcal{A}_K$ is often called the Kreĭn–von Neumann extension of $\mathcal{A}$. For the case of zero lower bound of $\mathcal{A}$ Ando and Nishio [2] proved that $\mathcal{A}_K$ can be defined as follows. Let $P_{\overline{\operatorname{ran}}(\mathcal{A})}$ be the orthogonal projection onto $\overline{\operatorname{ran}}(\mathcal{A})$ in H, and let $\mathcal{Q}$ be the operator in $\overline{\operatorname{ran}}(\mathcal{A})$ given by

$$\mathcal{Q}(\mathcal{A}f) = P_{\overline{\operatorname{ran}}(\mathcal{A})}f, \ f \in \operatorname{dom}(\mathcal{A}).$$

Then $\mathcal{Q}$ is symmetric, nonnegative and densely defined in $\overline{\operatorname{ran}}(\mathcal{A})$. Let $\mathcal{Q}_F$ be the Friedrichs extension of $\mathcal{Q}$. Its inverse $\mathcal{Q}_F^{-1}$ exists and the relation

$$\mathcal{A}_K = \mathcal{Q}_F^{-1} P_{\overline{\operatorname{ran}}(\mathcal{A})}$$

holds. One more intrinsic construction of the Kreĭn extension $\mathcal{A}_K$ by means of the Friedrichs extension $\mathcal{A}_F$ has been proposed in [9] and [10]. Another approach to nonnegative selfadjoint extensions is connected with boundary triplets (boundary value spaces) and corresponding Weyl functions [3, 12–14, 18, 23]. In concrete situations the intrinsic characterizations of the Friedrichs and Kreĭn extensions for symmetric operator with zero lower bound is a non-trivial problem. In this paper we study this problem for operators of the form

$$\mathcal{A} = L_2^* L_1, \tag{1.1}$$

where L_1 and L_2 are closed operators in H taking values in a Hilbert space $\mathfrak{H}$ and possessing the condition

$$L_1 \subset L_2. \tag{1.2}$$

Such kind of operators $\mathcal{A}$ we call *operators in divergence form.* A particular case is $\mathcal{A} = \mathcal{L}_0^2$, where $\mathcal{L}_0$ is symmetric operator in H. Here $L_1 = \mathcal{L}_0$, $L_2 = \mathcal{L}_0^*$. In [26] and [27] (see also [8,28]) it is shown that each nonnegative symmetric operator by artificial way can be represented in divergence form and descriptions of Friedrichs, Kreĭn, and all other extremal extensions have been obtained. Similar approach for representations of extremal extensions has been proposed in [15] for the case of nonnegative linear relations. Sturm-Liouville differential operators have natural representation in divergence form [11].

We establish here (see Theorem 3.1) that under the condition $\mathcal{A}^* = L_1^* L_2$ the Friedrichs and Kreĭn extensions of $\mathcal{A}$ are given by the operators $L_1^* L_1$ and

$L_2^* P_{\overline{\mathrm{ran}}\,(L_1)} L_2$, respectively. The equality $\mathcal{A}^* = L_1^* L_2$ holds true if, for example, $\dim(\mathrm{dom}\,(L_2)/\mathrm{dom}\,(L_1)) < \infty$ [7] or if $L_1 = \mathcal{L}_0$, $L_2 = \mathcal{L}_0^*$, where $\mathcal{L}_0$ maximal symmetric operator, cf. [22]. It is proved in [21, Theorems 6.4, 6.5] that $(\mathcal{L}_0^2)_F = \mathcal{L}_0^* \mathcal{L}_0$ for a closed symmetric operator $\mathcal{L}_0$ provided $\mathcal{L}_0^2$ is densely defined and $(\mathcal{L}_0^2)^* = \mathcal{L}_0^{*2}$, while $(\mathcal{L}_0^2)_K = \mathcal{L}_0 \mathcal{L}_0^*$, if, additionally, $\ker(\mathcal{L}_0^*) = \{0\}$. For a pair $\mathcal{L}_0 \subset \mathcal{L}_0^*$ we prove (see Theorem 3.4) that if $(\mathcal{L}_0^2)^* = \mathcal{L}_0^{*2}$, then $(\mathcal{L}\mathcal{L}_0)^* = \mathcal{L}_0^* \mathcal{L}$ and $(\mathcal{L}_0 \mathcal{L})^* = \mathcal{L}\mathcal{L}_0^*$ for an arbitrary selfadjoint extension $\mathcal{L}$ of $\mathcal{L}_0$. Our main results are applied to the following differential operators in the Hilbert space $L_2(\mathbb{R})$:

$$\mathrm{dom}\,(A_0) = \left\{f \in W_2^2(\mathbb{R}) : f(y) = 0,\ y \in Y\right\}, \qquad A_0 := -\frac{d^2}{dx^2}, \tag{1.3}$$

$$\mathrm{dom}\,(\mathring{A}) = \left\{g \in W_2^2(\mathbb{R}) : g'(y) = 0,\ y \in Y\right\}, \qquad \mathring{A} := -\frac{d^2}{dx^2}, \tag{1.4}$$

$$\mathrm{dom}\,(H_0) = \left\{f \in W_2^2(\mathbb{R}) : f(y) = 0,\ f'(y) = 0,\ y \in Y\right\}, \qquad H_0 := -\frac{d^2}{dx^2}. \tag{1.5}$$

Here $W_2^1(\mathbb{R})$ and $W_2^2(\mathbb{R})$ are Sobolev spaces, Y is finite or infinite monotonic sequence of points in $\mathbb{R}$ satisfying the condition

$$\inf\{|y' - y''|,\ y', y'' \in Y,\ y' \neq y''\} > 0. \tag{1.6}$$

The operators A_0, $\mathring{A}$, and H_0 are densely defined and nonnegative with finite (the set Y is finite) or infinite defect indices (the set Y is infinite) and are basic for investigations of Hamiltonians on the real line corresponding to the δ, δ' and $\delta - \delta'$ interactions, respectively [1]. Note that in [7,8] Theorem 3.1 has been applied to the case of one point interaction in $\mathbb{R}$.

2. PRELIMINARIES

2.1. UNIQUENESS AND TRANSVERSALNESS

Let

$$\mathfrak{N}_z(\mathcal{A}) = H \ominus \mathrm{ran}\,(\mathcal{A} - \bar{z}I) = \ker(\mathcal{A}^* - zI)$$

be the defect subspace of $\mathcal{A}$. By von Neumann formulas

$$\mathrm{dom}\,(\mathcal{A}^*) = \mathrm{dom}\,(\mathcal{A}) \dot{+} \mathfrak{N}_z(\mathcal{A}) \dot{+} \mathfrak{N}_{\bar{z}}(\mathcal{A}), \quad \mathrm{Im}\, z \neq 0.$$

M.G. Kreĭn [19] established that $\mathcal{A}$ has a unique nonnegative selfadjoint extension if and only if

$$\inf_{f \in \mathrm{dom}\,(\mathcal{A})} \frac{|(f, \varphi_{-a})|^2}{(\mathcal{A}f, f)} = \infty \quad \text{for all} \quad \varphi_{-a} \in \mathfrak{N}_{-a}(\mathcal{A}) \setminus \{0\}, \quad a > 0. \tag{2.1}$$

Let $\widetilde{\mathcal{A}}$ be a nonnegative selfadjoint extension of $\mathcal{A}$. It is established by M.G. Kreĭn [19] that the domain $\mathcal{D}[\widetilde{\mathcal{A}}] = \mathrm{dom}\,(\widetilde{\mathcal{A}}^{1/2})$ admits the decomposition

$$\mathrm{dom}\,(\widetilde{\mathcal{A}}^{1/2}) = \mathrm{dom}\,(\mathcal{A}_F^{1/2}) \dot{+} (\mathrm{dom}\,(\widetilde{\mathcal{A}}^{1/2}) \cap \mathfrak{N}_{-a}(\mathcal{A})) \tag{2.2}$$

for arbitrary $a > 0$ and

$$\|\widetilde{\mathcal{A}}^{1/2}v\|^2 = \|\mathcal{A}_F^{1/2}v\|^2, \quad v \in \mathrm{dom}\,(\mathcal{A}_F^{1/2}).$$

Since

$$(\widetilde{\mathcal{A}} - zI)(\widetilde{\mathcal{A}} - \lambda I)^{-1}\mathfrak{N}_z(\mathcal{A}) = \mathfrak{N}_\lambda(\mathcal{A}), \quad z, \lambda \in \mathbb{C} \setminus [0, \infty),$$

we get also the decomposition

$$\mathrm{dom}\,(\widetilde{\mathcal{A}}^{1/2}) = \mathrm{dom}\,(\mathcal{A}_F^{1/2}) \dot{+} (\mathrm{dom}\,(\widetilde{\mathcal{A}}^{1/2}) \cap \mathfrak{N}_z(\mathcal{A})), \quad z \in \mathbb{C} \setminus [0, \infty). \tag{2.3}$$

The Kreĭn extension $\mathcal{A}_K$ of $\mathcal{A}$, possesses the properties [19]

$$\mathrm{dom}\,(\widetilde{\mathcal{A}}^{1/2}) \subseteq \mathrm{dom}\,(\mathcal{A}_K^{1/2}),\ \|\widetilde{\mathcal{A}}^{1/2}u\| \geq \|\mathcal{A}_K^{1/2}u\|^2, \quad u \in \mathrm{dom}\,(\widetilde{\mathcal{A}}^{1/2})$$

for each nonnegative selfadjoint extension $\widetilde{\mathcal{A}}$ of $\mathcal{A}$. The domain $\mathrm{dom}\,(\mathcal{A}_K^{1/2})$ can be characterized as follows [2]

$$\begin{gathered} \mathcal{D}[\mathcal{A}_K] = \mathrm{dom}\,(\mathcal{A}_K^{1/2}) = \left\{ u \in H : \sup_{\varphi \in \mathrm{dom}\,(\mathcal{A})} \frac{|(\mathcal{A}\varphi, u)|^2}{(\mathcal{A}\varphi, \varphi)} < \infty \right\}, \\ \|\mathcal{A}_K^{1/2}u\|^2 = \sup_{\varphi \in \mathrm{dom}\,(\mathcal{A})} \frac{|(\mathcal{A}\varphi, u)|^2}{(\mathcal{A}\varphi, \varphi)}, \quad u \in \mathcal{D}[\mathcal{A}_K]. \end{gathered} \tag{2.4}$$

Let B be an arbitrary nonnegative selfadjoint operator. As is well known

$$\begin{gathered} \mathrm{ran}\,(B^{1/2}) = \left\{ g \in H : \sup_{f \in \mathrm{dom}\,(B)} \frac{|(f, g)|^2}{(Bf, f)} < \infty \right\}, \\ \|\widehat{B}^{-1/2}g\|^2 = \sup_{f \in \mathrm{dom}\,(B)} \frac{|(f, g)|^2}{(Bf, f)}, \quad g \in \mathrm{ran}\,(B^{1/2}). \end{gathered} \tag{2.5}$$

In particular, from (2.5) it follows that

$$u \in \mathrm{dom}\,(\mathcal{A}^*) \cap \mathrm{dom}\,(\mathcal{A}_K^{1/2}) \iff \mathcal{A}^*u \in \mathrm{ran}\,(\mathcal{A}_F^{1/2}).$$

Nonnegative selfadjoint extension $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ is called *extremal* [3] if

$$\inf \left\{ (\widetilde{\mathcal{A}}(u - x), u - x),\ x \in \mathrm{dom}\,(\mathcal{A}) \right\} = 0 \quad \text{for all} \quad u \in \mathrm{dom}\,(\widetilde{\mathcal{A}}).$$

The Friedrichs and Kreĭn-von Neumann extensions are extremal and, as it is shown in [4], the closed forms associated with extremal extensions are closed restrictions of the form $\mathcal{A}_K[\cdot, \cdot]$ on the linear manifolds $\mathcal{M}$ such that

$$\mathcal{D}[\mathcal{A}] \subseteq \mathcal{M} \subseteq \mathcal{D}[\mathcal{A}_K].$$

Notice that investigations of all extremal extensions in more detail and their applications are presented in [8].

Due to (2.5), (2.4), and (2.1) the following conditions are equivalent:

(i) the operator $\mathcal{A}$ admits a unique nonnegative selfadjoint extension ($\mathcal{A}_F = \mathcal{A}_K$),

(ii)
$$\inf_{v\in \operatorname{dom}(\mathcal{A})} \frac{|(v,\varphi_{-a})|^2}{(\mathcal{A}v,v)} = \infty$$
for all nonzero vectors φ_{-a} from the defect subspace $\mathfrak{N}_{-a}(\mathcal{A})$, where $a > 0$,

(iii) $\operatorname{ran}(\mathcal{A}_F^{1/2}) \cap \mathfrak{N}_{-a}(\mathcal{A}) = \{0\}$,

(iv) $\operatorname{dom}(\mathcal{A}_K^{1/2}) \cap \mathfrak{N}_{-a}(\mathcal{A}) = \{0\}$.

Recall that two selfadjoint extensions $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ of a symmetric operator $\mathcal{A}$ are called disjoint if $\operatorname{dom}(\widetilde{\mathcal{A}}_1) \cap \operatorname{dom}(\widetilde{\mathcal{A}}_2) = \operatorname{dom}(\mathcal{A})$ and transversal if
$$\operatorname{dom}(\widetilde{\mathcal{A}}_1) + \operatorname{dom}(\widetilde{\mathcal{A}}_2) = \operatorname{dom}(\mathcal{A}^*).$$

The next statement provides equivalent transversalness conditions of Friedrichs and Kreĭn extensions (see [10, 24]).

Proposition 2.1. *The conditions:*

(i) *the Friedrichs and Kreĭn extensions $\mathcal{A}_F$ and $\mathcal{A}_K$ are transversal,*

(ii) $\operatorname{ran}(\mathcal{A}^*) \subset \operatorname{ran}(\mathcal{A}_F^{1/2})$,

(iii) $\operatorname{dom}(\mathcal{A}^*) \subset \operatorname{dom}(\mathcal{A}_K^{1/2})$,

(iv) $\mathfrak{N}_z(\mathcal{A}) \subset \operatorname{dom}(\mathcal{A}_K^{1/2})$ *at least for one (then for all)* $z \in \mathbb{C} \setminus [0,\infty)$,

(v) $\mathfrak{N}_z(\mathcal{A}) \subset \operatorname{ran}(\mathcal{A}_F^{1/2})$ *at least for one (then for all)* $z \in \mathbb{C} \setminus [0,\infty)$

are equivalent.

2.2. OPERATORS IN DIVERGENCE FORM

Assume that

(**A**) L_1 and L_2 are two closed densely defined operators in the Hilbert space H taking values in a Hilbert space $\mathfrak{H}$ and such that $L_1 \subset L_2$,

(**B**) $Q \in \mathcal{L}(\mathfrak{H})$ is a positive definite operator.

Consider two sesquilinear forms
$$S_j[u,v] = (QL_ju, L_jv)_{\mathfrak{H}},\ u,v \in \operatorname{dom}(L_j), j = 1,2.$$

On account of (**A**) and (**B**) these forms are closed and nonnegative, and moreover, $S_k = L_k^*QL_k$, $k = 1,2$, are associated with them by the first representation theorem [16] nonnegative selfadjoint operators in H, i.e.,
$$(L_j^*QL_ju, v)_H = (QL_ju, L_jv)_{\mathfrak{H}},\ u \in \operatorname{dom}(S_j),\ v \in \operatorname{dom}(L_j).$$

The operator $S := L_2^*QL_1$ is closed since its graph is the intersection of graphs for S_1 and S_2. We equip the linear manifolds $\operatorname{dom}(L_j)$, $j = 1,2$, by the graph norms. We will need the following statement [7].

Theorem 2.2. *Assume that conditions (**A**) and (**B**) are fulfilled.*

1) *If, in addition,*
   (**C**) *the lineal* $\mathrm{dom}\,(L_1) \cap \mathrm{dom}\,(S_2)$ *is dense in* $\mathrm{dom}\,(L_1)$,
   then:
   (i) *the operator*
   $$S = L_2^* Q L_1$$
   is a closed densely defined nonnegative operator in H, *the operators* S_1 *and* S_2 *are nonnegative selfadjoint extensions of* S, *and the operator* S_1 *is the Friedrichs extension of* S;
   (ii) $\mathcal{D}[S_K] \supseteq \mathrm{dom}\,(L_2)$ *and for all* $u, v \in \mathrm{dom}\,(L_2)$
   $$S_K\,[u,v] = (Q\mathcal{P}L_2 u, L_2 v)_{\mathfrak{H}}\,,$$
   where $\mathcal{P}$ *is the projection in* $\mathfrak{H}$ *onto* $\overline{\mathrm{ran}}\,(L_1)$ *with respect to the decomposition*
   $$\mathfrak{H} = \overline{\mathrm{ran}}\,(L_1) \dot{+} Q^{-1}\,(\ker(L_1^*))\,.$$
2) *If the condition*
   $$\dim\,(\mathrm{dom}\,(L_2)/\mathrm{dom}\,(L_1)) < \infty$$
   is satisfied, then (**C**) *holds and*
   $$\mathcal{D}[S_K] = \mathrm{dom}\,(L_2),\ S_K = L_2^* Q \mathcal{P} L_2, \quad S^* = L_1^* Q L_2.$$

3. MAIN RESULTS

Theorem 3.1. *Let* $L_1, L_2 : H \to \mathfrak{H}$ *be closed and densely defined operators, satisfying condition* (1.2).

1) *If the operator* $\mathcal{A} = L_2^* L_1$ *is densely defined and its adjoint is given by*
   $$\mathcal{A}^* = L_1^* L_2, \tag{3.1}$$
   then:
   (i)
   $$\mathcal{D}[\mathcal{A}] = \mathrm{dom}\,(L_1),\ \mathcal{A}[u,v] = (L_1 u, L_1 v),\ u, v \in \mathrm{dom}\,(L_1),$$
   (ii) *the Friedrichs extension of* $\mathcal{A}$ *is given by the operator* $L_1^* L_1$, *i.e.,*
   $$\mathrm{dom}\,(\mathcal{A}_F) = \{f \in \mathrm{dom}\,(L_1) : L_1 f \in \mathrm{dom}\,(L_1^*)\},$$
   $$\mathcal{A}_F f = L_1^* L_1 f = L_1^* L_2 f, \quad f \in \mathrm{dom}\,(\mathcal{A}_F),$$
   (iii) *the Kreĭn extension of* $\mathcal{A}$ *is the operator* $\mathcal{A}_K = L_2^* P_{\overline{\mathrm{ran}}\,(L_1)} L_2$, *i.e.,*
   $$\mathrm{dom}\,(\mathcal{A}_K) = \{f \in \mathrm{dom}\,(L_2) : P_{\overline{\mathrm{ran}}\,(L_1)} L_2 f \in \mathrm{dom}\,(L_2^*)\},$$
   $$\mathcal{A}_K f = L_2^* P_{\overline{\mathrm{ran}}\,(L_1)} L_2 f,\ f \in \mathrm{dom}\,(\mathcal{A}_K),$$
   and
   $$\mathcal{D}[\mathcal{A}_K] = \mathrm{dom}\,(L_2),\ \mathcal{A}_K[u,v] = (P_{\overline{\mathrm{ran}}\,(L_1)} L_2 u, P_{\overline{\mathrm{ran}}\,(L_1)} L_2 v),\ u, v \in \mathrm{dom}\,(L_2), \tag{3.2}$$
   (iv) *the Friedrichs and Kreĭn extensions of* $\mathcal{A}$ *are transversal.*

2) *If the operator* $\mathcal{A} = L_2^* L_1$ *is densely defined, the operator* $L_1^* L_1$ *is the Friedrichs extension of* $\mathcal{A}$*, and the linear manifold*

$$\mathfrak{M} := \ker(L_1^* L_2 + I) \tag{3.3}$$

is a subspace in H*. Then* $\mathcal{A}^* = L_1^* L_2$*.*

Proof. 1) Let us proof that $\mathrm{dom}\,(\mathcal{A})$ is dense in $\mathrm{dom}\,(L_1)$ with respect to the graph inner product. If $h \in \mathrm{dom}\,(L_1)$ is orthogonal to $\mathrm{dom}\,(\mathcal{A})$, then

$$(L_1 f, L_1 h)_{\mathfrak{H}} + (f,h)_H = 0 \quad \text{for all} \quad f \in \mathrm{dom}\,(\mathcal{A}) = \mathrm{dom}\,(L_2^* L_1).$$

Since $L_1 f \in \mathrm{dom}\,(L_2^*)$, $L_1 h = L_2 h$, and $\mathcal{A} = L_2^* L_1$ we get

$$(\mathcal{A}f, h)_H + (f,h)_H = 0 \quad \text{for all} \quad f \in \mathrm{dom}\,(\mathcal{A}).$$

It follows that $h \in \mathrm{dom}\,(\mathcal{A}^*)$ and $\mathcal{A}^* h = -h$. Due to the assumption we have $\mathcal{A}^* = L_1^* L_2$. But $h \in \mathrm{dom}\,(L_1)$. Hence, $\mathcal{A}^* h = L_1^* L_1 h = -h$. Because the operator $L_1^* L_1$ is nonnegative, we obtain $h = 0$.

Since the form

$$(L_1 u, L_1 v)_{\mathfrak{H}}, \quad u, v \in \mathrm{dom}\,(L_1)$$

is closed,

$$(\mathcal{A}f, g)_H = (L_1 f, L_1 g)_{\mathfrak{H}}, \; f, g \in \mathrm{dom}\,(\mathcal{A}),$$

and $\mathrm{dom}\,(\mathcal{A})$ is dense in $\mathrm{dom}\,(L_1)$ w.r.t. the graph norm, we get that

$$\mathcal{A}_F = L_1^* L_1.$$

Clearly, $\mathrm{ran}\,(\mathcal{A}_F^{1/2}) = \mathrm{ran}\,(L_1^*)$, and $\mathrm{ran}\,(\mathcal{A}^*) = \mathrm{ran}\,(L_1^* L_2) \subseteq \mathrm{ran}\,(L_1^*)$. Applying Proposition 2.1 we get that $\mathcal{A}_F$ and $\mathcal{A}_K$ are transversal.

The operator $\widetilde{\mathcal{A}} = L_2^* L_2$ is a selfadjoint and nonnegative extension of $\mathcal{A}$ and $\mathcal{D}[\widetilde{\mathcal{A}}] = \mathrm{dom}\,(L_2)$. Let $\mathfrak{N}_{-1}(\mathcal{A})$ be the defect subspace of $\mathcal{A}$, i.e., $\mathfrak{N}_{-1}(\mathcal{A}) = \ker(\mathcal{A}^* + I)$. Then, clearly, $\mathfrak{N}_{-1}(\mathcal{A}) = \mathfrak{M}$, where $\mathfrak{M}$ is given by (3.3). Hence, $\mathfrak{N}_{-1}(\mathcal{A}) \subset \mathrm{dom}\,(L_2)$. Therefore, from $\mathcal{D}[\widetilde{\mathcal{A}}] \subseteq \mathcal{D}[\mathcal{A}_K]$ and (2.2) it follows that $\mathcal{D}[\mathcal{A}_K] \supset \mathfrak{N}_{-1}(\mathcal{A})$ and

$$\mathcal{D}[\mathcal{A}_K] = \mathcal{D}[\widetilde{\mathcal{A}}] = \mathrm{dom}\,(L_2).$$

Applying Theorem 2.2 we get that

$$\mathcal{A}_K[u,v] = (P_{\overline{\mathrm{ran}}\,(L_1)} L_2 u, P_{\overline{\mathrm{ran}}\,(L_1)} L_2 v), \quad u, v \in \mathcal{D}[\mathcal{A}_K] = \mathrm{dom}\,(L_2).$$

Now the first representation theorem yields that

$$\begin{aligned}
\mathrm{dom}\,(\mathcal{A}_K) &= \mathrm{dom}\,(L_2^* P_{\overline{\mathrm{ran}}\,(L_1)} L_2) = \{f \in \mathrm{dom}\,(L_2) : P_{\overline{\mathrm{ran}}\,(L_1)} L_2 f \in \mathrm{dom}\,(L_2^*)\},\\
\mathcal{A}_K f &= L_2^* P_{\overline{\mathrm{ran}}\,(L_1)} L_2 f = L_2^*(L_2 f - P_{\ker(L_1^*)} L_2 f) =\\
&= L_1^*(L_2 f - P_{\ker(L_1^*)} L_2 f) = L_1^* L_2 f, \; f \in \mathrm{dom}\,(\mathcal{A}_K).
\end{aligned}$$

2) The linear manifold $\mathfrak{M}$ is the orthogonal complement to $\operatorname{dom}(L_1)$ in $\operatorname{dom}(L_2)$ w.r.t. the inner product

$$(f,g)_{L_2} := (f,g)_H + (L_2 f, L_2 g)_{\mathfrak{H}}.$$

Actually, the relation

$$(f,g)_{L_2} = 0$$

for all $f \in \operatorname{dom}(L_1)$ yields that $L_2 g \in \operatorname{dom}(L_1^*)$ and $L_1^* L_2 g = -g$, i.e., $g \in \mathfrak{M}$. On the other hand, if $g \in \mathfrak{M}$, then $(f,g)_{L_2} = 0$ for all $f \in \operatorname{dom}(L_1)$.

Clearly, $\mathfrak{M} \subseteq \mathfrak{N}_{-1}(\mathcal{A})$. Taking into account that $\mathcal{A}_F = L_1^* L_1$, we get that (see (2.3))

$$\mathfrak{M} = \operatorname{dom}(L_2) \cap \mathfrak{N}_{-1}(\mathcal{A}).$$

Because $\operatorname{dom}(L_2^* L_1) = \operatorname{dom}(L_1^* L_1) \cap \operatorname{dom}(L_2^* L_2)$, the selfadjoint extensions $L_1^* L_1$ and $L_2^* L_2$ of $\mathcal{A} = L_2^* L_1$ are disjoint. Hence, $\mathfrak{M}$ is at least dense in $\mathfrak{N}_{-1}(\mathcal{A})$ [6]. But $\mathfrak{M}$ is a subspace in H. Therefore $\mathfrak{N}_{-1}(\mathcal{A}) = \mathfrak{M}$. This means that $L_1^* L_1$ and $L_2^* L_2$ are transversal. It follows $\mathcal{A}^* = L_1^* L_2$. □

Remark 3.2. From (3.2) it follows that even in the case $\dim(\operatorname{dom}(L_2)/\operatorname{dom}(L_1)) = \infty$ (under condition (3.1)) the operator $P_{\overline{\operatorname{ran}}(L_1)} L_2$ is closed. The latter is equivalent to

$$\|P_{\ker(L_1^*)} L_2 f\|^2_{\mathfrak{H}} \le C\left(\|f\|^2_H + \|P_{\overline{\operatorname{ran}}(L_1)} L_2 f\|^2_{\mathfrak{H}}\right), \quad f \in \operatorname{dom}(L_2)$$

with some $C > 0$.

Proposition 3.3. *Let $\mathcal{L}_0$ be a densely defined closed symmetric operator in H. The following conditions are equivalent:*

(i) $(\mathcal{L}_0^2)_F = \mathcal{L}_0^* \mathcal{L}_0$,
(ii) $\operatorname{dom}(\mathcal{L}_0) \cap \operatorname{ran}(\mathcal{L}_0 - \lambda I)$ *is dense in* $\operatorname{ran}(\mathcal{L}_0 - \lambda I)$ *for at least one non-real* λ.

Proof. Clearly,

$$(\mathcal{L}_0 f, \mathcal{L}_0 g) + (f,g) = ((\mathcal{L}_0 + iI)f, (\mathcal{L}_0 + iI)g) = ((\mathcal{L}_0 - iI)f, (\mathcal{L}_0 - iI)g), \quad f, g \in \operatorname{dom}(\mathcal{L}_0)$$

One can easily proof that

$$\operatorname{dom}(\mathcal{L}_0^2) = (\mathcal{L}_0 - \lambda I)^{-1}\left(\operatorname{ran}(\mathcal{L}_0 - \lambda I) \cap \operatorname{dom}(\mathcal{L}_0)\right), \quad \operatorname{Im}\lambda \neq 0. \tag{3.4}$$

The equality $(\mathcal{L}_0^2)_F = \mathcal{L}_0^* \mathcal{L}_0$ is equivalent to the condition: $\operatorname{dom}(\mathcal{L}_0^2)$ is dense in $\operatorname{dom}(\mathcal{L}_0)$ w.r.t. graph norm in $\operatorname{dom}(\mathcal{L}_0)$. The latter is equivalent to that there is no nontrivial vector $g \in \operatorname{dom}(\mathcal{L}_0)$ such that $(\mathcal{L}_0 f, \mathcal{L}_0 g) + (f,g) = 0$ for all $f \in \operatorname{dom}(\mathcal{L}_0^2)$. From (3.4) it follows the equivalence of (i) and (ii). □

Theorem 3.4. *Let $\mathcal{L}_0$ be a densely defined closed symmetric operator with equal deficiency indices in H. Suppose $\mathcal{L}_0^2$ is densely defined and $(\mathcal{L}_0^2)^* = \mathcal{L}_0^{*2}$. Then for an arbitrary selfadjoint extension $\mathcal{L}$ of $\mathcal{L}_0$ the equalities*

$$(\mathcal{L}\mathcal{L}_0)^* = \mathcal{L}_0^* \mathcal{L} \tag{3.5}$$

and

$$(\mathcal{L}_0\mathcal{L})^* = \mathcal{L}\mathcal{L}_0^* \tag{3.6}$$

hold.

Proof. Denote by $\mathfrak{N}_\lambda(\mathcal{L}_0)$ the deficiency subspace of $\mathcal{L}_0$. Since $\mathcal{L}_0^{*2}+I = (\mathcal{L}_0^*+iI)(\mathcal{L}_0^*-iI)$, we have the equality

$$\ker(\mathcal{L}_0^{*2} + I) = \mathfrak{N}_i(\mathcal{L}_0)\dot{+}\mathfrak{N}_{-i}(\mathcal{L}_0).$$

Taking into account that $(\mathcal{L}_0^2)^* = \mathcal{L}_0^{*2}$ and applying statement 2) of Theorem 3.1 to the pair $\mathcal{L}_0 \subset \mathcal{L}_0^*$, we get that $\mathfrak{N}_i(\mathcal{L}_0)\dot{+}\mathfrak{N}_{-i}(\mathcal{L}_0)$ is a subspace in H. Let a selfadjoint extension $\mathcal{L}$ is given by

$$\operatorname{dom}(\mathcal{L}) = \operatorname{dom}(\mathcal{L}_0)\dot{+}(I + \mathcal{U})\mathfrak{N}_i(\mathcal{L}_0),$$

where $\mathcal{U}$ is an isometric mapping of $\mathfrak{N}_i(\mathcal{L}_0)$ onto $\mathfrak{N}_{-i}(\mathcal{L}_0)$.

Let us show that $(\mathcal{L}\mathcal{L}_0)^* = \mathcal{L}_0^*\mathcal{L}$. Statement 1) of Theorem 3.1 for the pair $\mathcal{L}_0 \subset \mathcal{L}_0^*$ and the equality $(\mathcal{L}_0^2)^* = \mathcal{L}_0^{*2}$ imply that the operator $\mathcal{L}_0^*\mathcal{L}_0$ is the Friedrichs extension of the operator $\mathcal{L}_0^2$. In addition, because $\mathfrak{N}_i(\mathcal{L}_0)\dot{+}\mathfrak{N}_{-i}(\mathcal{L}_0)$ is a subspace in H, the linear manifold $(I + \mathcal{U})\mathfrak{N}_i(\mathcal{L}_0)$ is a subspace in H as well. Clearly,

$$\ker(\mathcal{L}_0^*\mathcal{L} + I) = (I + \mathcal{U})\mathfrak{N}_i(\mathcal{L}_0).$$

Since $\mathcal{L}\mathcal{L}_0 \supseteq \mathcal{L}_0^2$ and $\operatorname{dom}(\mathcal{L}_0^2)$ is dense in $\operatorname{dom}(\mathcal{L}_0)$ (w.r.t. the graph norm in $\operatorname{dom}(\mathcal{L}_0)$), the domain $\operatorname{dom}(\mathcal{L}\mathcal{L}_0)$ is also dense in $\operatorname{dom}(\mathcal{L}_0)$, i.e., $(\mathcal{L}\mathcal{L}_0)_F = \mathcal{L}_0^*\mathcal{L}_0$. Applying statement 2) of Theorem 3.1 to the pair $\mathcal{L}_0 \subset \mathcal{L}$, we obtain that $(\mathcal{L}\mathcal{L}_0)^* = \mathcal{L}_0^*\mathcal{L}$.

Next we equip $\operatorname{dom}(\mathcal{L}_0^*)$ by the inner product

$$(f, g)_+ = (f, g) + (\mathcal{L}_0^* f, \mathcal{L}_0^* g).$$

Then $\operatorname{dom}(\mathcal{L}_0^*)$ becomes a Hilbert space, which we denote by H_+. Then (+)-orthogonal decomposition

$$H_+ = \operatorname{dom}(\mathcal{L}_0) \oplus \mathfrak{N}_i(\mathcal{L}_0) \oplus \mathfrak{N}_{-i}(\mathcal{L}_0)$$

holds. Let $\mathcal{N} = (I + \mathcal{U})\mathfrak{N}_i(\mathcal{L}_0)$ and $\mathcal{M} = (I - \mathcal{U})\mathfrak{N}_i(\mathcal{L}_0)$. We have (+)-orthogonal decompositions

$$\operatorname{dom}(\mathcal{L}) = \operatorname{dom}(\mathcal{L}_0) \oplus \mathcal{N}, \quad H_+ = \operatorname{dom}(\mathcal{L}) \oplus \mathcal{M}.$$

Clearly

$$\mathcal{L}\mathcal{N} = \mathcal{M}, \quad \mathcal{L}_0^*\mathcal{M} = \mathcal{N},$$

and

$$\mathcal{L}\mathcal{L}_0^* h = -h, \quad h \in \mathcal{M},$$
$$\mathcal{L}_0^*\mathcal{L} e = -e, \quad e \in \mathcal{N}.$$

Let $\widetilde{\mathcal{L}}$ be one more selfadjoint extension of $\mathcal{L}_0$ given by

$$\operatorname{dom}(\widetilde{\mathcal{L}}) = \operatorname{dom}(\mathcal{L}_0) \oplus \mathcal{M} = \operatorname{dom}(\mathcal{L}_0)\dot{+}(I - \mathcal{U})\mathfrak{N}_i(\mathcal{L}_0), \quad \widetilde{\mathcal{L}} = \mathcal{L}_0^*\restriction \operatorname{dom}(\widetilde{\mathcal{L}}).$$

Then, considering the pair $\mathcal{L}_0 \subset \widetilde{\mathcal{L}}$, we conclude that $(\widetilde{\mathcal{L}}\mathcal{L}_0)_F = \mathcal{L}_0^*\mathcal{L}_0$, i.e., $\mathrm{dom}\,(\widetilde{\mathcal{L}}\mathcal{L}_0)$ is dense in $\mathrm{dom}\,(\mathcal{L}_0)$ in $(+)$-norm. In addition, the linear manifold

$$\ker(\widetilde{\mathcal{L}}\mathcal{L}_0^* + I) = (I - \mathcal{U})\mathfrak{N}_i(\mathcal{L}_0) = \mathcal{M}$$

is a subspace in H. In addition $\mathcal{L}\widetilde{\mathcal{L}}h = -h$ for all $h \in \mathcal{M}$, $\widetilde{\mathcal{L}}\mathcal{L}e = -e$ for all $e \in \mathcal{N}$, and

$$(\widetilde{\mathcal{L}}h, e)_+ = -(h, \mathcal{L}e)_+, \quad h \in \mathcal{M}, \quad e \in \mathcal{N}.$$

Let us describe $\mathrm{dom}\,(\mathcal{L}_0\mathcal{L})$. Denote by $P^+_{\mathcal{M}}$ the $(+)$-orthogonal projection in H_+ onto $\mathcal{M}$. Let $f \in \mathrm{dom}\,(\mathcal{L})$. Then

$$f = \varphi_0 + e,\ \varphi_0 \in \mathrm{dom}\,(\mathcal{L}_0), \quad e \in \mathcal{N}, \quad \mathcal{L}f = \mathcal{L}_0\varphi_0 + \mathcal{L}e.$$

Because $\mathcal{L}e \in \mathcal{M}$ we have that $\mathcal{L}f \in \mathrm{dom}\,(\mathcal{L}_0)$ if and only if $\mathcal{L}_0\varphi_0 = \mathcal{L}f - \mathcal{L}e \in \mathrm{dom}\,(\widetilde{\mathcal{L}})$ $\iff$ $\varphi_0 \in \mathrm{dom}\,(\widetilde{\mathcal{L}}\mathcal{L}_0)$ and $P^+_{\mathcal{M}}\mathcal{L}_0\varphi_0 = -\mathcal{L}e$. Finally,

$$\mathrm{dom}\,(\mathcal{L}_0\mathcal{L}) = (I + \widetilde{\mathcal{L}}P^+_{\mathcal{M}}\mathcal{L}_0)\mathrm{dom}\,(\widetilde{\mathcal{L}}\mathcal{L}_0).$$

Let us show now that $\mathrm{dom}\,(\mathcal{L}_0\mathcal{L})$ is dense in $\mathrm{dom}\,(\mathcal{L})$ w.r.t. $(+)$-norm. Suppose there is $g \in \mathrm{dom}\,(\mathcal{L})$ such that g is $(+)$-orthogonal to $\mathrm{dom}\,(\mathcal{L}_0\mathcal{L})$,

$$((I + \widetilde{\mathcal{L}}P^+_{\mathcal{M}}\mathcal{L}_0)h_0, g)_+ = 0 \quad \text{for all} \quad h_0 \in \mathrm{dom}\,(\widetilde{\mathcal{L}}\mathcal{L}_0).$$

In particular, taking $h_0 \in \mathrm{dom}\,(\mathcal{L}_0^2)$, we get that the vector g is $(+)$-orthogonal to $\mathrm{dom}\,(\mathcal{L}_0^2)$. But $\mathrm{dom}\,(\mathcal{L}_0^2)$ is $(+)$-dense in $\mathrm{dom}\,(\mathcal{L}_0)$. It follows that $g \in \mathcal{N}$. Since $\mathrm{dom}\,(\widetilde{\mathcal{L}}\mathcal{L}_0) \subset \mathrm{dom}\,(\mathcal{L}_0)$, we have

$$(\widetilde{\mathcal{L}}P^+_{\mathcal{M}}\mathcal{L}_0h_0, g)_+ = 0, \quad h_0 \in \mathrm{dom}\,(\widetilde{\mathcal{L}}\mathcal{L}_0).$$

Further

$$0 = (\widetilde{\mathcal{L}}P^+_{\mathcal{M}}\mathcal{L}_0h_0, g)_+ = (P^+_{\mathcal{M}}\mathcal{L}_0h_0, \mathcal{L}g)_+.$$

Let $\mathcal{L}g = x$. Then $x \in \mathcal{M}$ and

$$\begin{aligned} 0 &= (\widetilde{\mathcal{L}}P^+_{\mathcal{M}}\mathcal{L}_0h_0, g)_+ = (\mathcal{L}_0h_0, x)_+ = (\mathcal{L}_0h_0, x) + (\mathcal{L}\mathcal{L}_0h_0, \widetilde{\mathcal{L}}x) = \\ &= (h_0, \widetilde{\mathcal{L}}x) + (\mathcal{L}\mathcal{L}_0h_0, \widetilde{\mathcal{L}}x) = ((\widetilde{\mathcal{L}}\mathcal{L}_0 + I)h_0, \widetilde{\mathcal{L}}x). \end{aligned}$$

It follows that

$$\widetilde{\mathcal{L}}x \in \ker((\widetilde{\mathcal{L}}\mathcal{L}_0)^* + I).$$

Applying equality (3.5) to $\widetilde{\mathcal{L}}$ instead of $\mathcal{L}$ we get that $(\widetilde{\mathcal{L}}\mathcal{L}_0)^* = \mathcal{L}_0^*\widetilde{\mathcal{L}}$. Hence, $\widetilde{\mathcal{L}}x \in \mathcal{M}$. On the other hand $\widetilde{\mathcal{L}}x = -g \in \mathcal{N}$. Hence $g = 0$. Thus, $\mathrm{dom}\,(\mathcal{L}_0\mathcal{L})$ is $(+)$-dense in $\mathrm{dom}\,(\mathcal{L})$ and, therefore, $(\mathcal{L}_0\mathcal{L})_F = \mathcal{L}^2$. Applying statement 2) of Theorem 3.1, we arrive at (3.6). □

4. APPLICATIONS

Let Y be a finite or infinite monotonic sequence of points in $\mathbb{R}$ satisfying condition (1.6). Let A_0, $\mathring{A}$ and H_0 be given by (1.3), (1.4), (1.5), respectively. Notice that (see [1]):

$$\begin{aligned}&\operatorname{dom}(A_0^*) = W_2^1(\mathbb{R}) \cap W_2^2(\mathbb{R}\setminus Y),\ A_0^* = -\frac{d^2}{dx^2},\\ &\operatorname{dom}(\mathring{A}^*) = \{g \in W_2^2(\mathbb{R}) : g'(y+) = g'(y-),\ y \in Y\},\ \mathring{A}^* = -\frac{d^2}{dx^2},\\ &\operatorname{dom}(H_0^*) = W_2^2(\mathbb{R}\setminus Y),\ H_0^* = -\frac{d^2}{dx^2}.\end{aligned} \tag{4.1}$$

Let $\mathbb{Z}$ be the set of all integers and let

$$\mathbb{Z}_- = \{j \in \mathbb{Z},\ j \le -1\},\ \mathbb{Z}_+ = \{j \in \mathbb{Z},\ j \ge 1\}.$$

For infinite Y it is possible three cases

$$\begin{aligned}&Y = \{y_j,\ j \in \mathbb{Z}\}, &&\text{if}\quad \inf\{Y\} = -\infty \quad\text{and}\quad \sup\{Y\} = +\infty,\\ &Y = \{y_j,\ j \in \mathbb{Z}_-\}, &&\text{if}\quad y_{-1} = \sup\{Y\} < +\infty,\\ &Y = \{y_j,\ j \in \mathbb{Z}_+\}, &&\text{if}\quad y_1 = \inf\{Y\} > -\infty.\end{aligned}$$

By $\mathbb{J}$ we will denote one of the sets $\mathbb{Z}$, $\mathbb{Z}_-$, $\mathbb{Z}_+$ for infinite Y.

Consider in the Hilbert space $L_2(\mathbb{R})$ the following operators

$$\operatorname{dom}(\mathcal{L}_0) = \{f \in W_2^1(\mathbb{R}) : f(y) = 0,\ y \in Y\}, \qquad \mathcal{L}_0 = i\frac{d}{dx}, \tag{4.2}$$

$$\operatorname{dom}(\mathcal{L}) = W_2^1(\mathbb{R}), \qquad \mathcal{L} = i\frac{d}{dx}. \tag{4.3}$$

From (4.2) it follows that $\mathcal{L}_0$ is a densely defined symmetric operator and its adjoint $\mathcal{L}_0^*$ is given by

$$\operatorname{dom}(\mathcal{L}_0^*) = W_2^1(\mathbb{R}\setminus Y),\ \mathcal{L}_0^* = i\frac{d}{dx}. \tag{4.4}$$

The operator $\mathcal{L}$ is a selfadjoint extension of $\mathcal{L}_0$. So, we have

$$\mathcal{L}_0 \subset \mathcal{L} \subset \mathcal{L}_0^*.$$

In addition $A_0 \supset H_0$, $\mathring{A} \supset H_0$. If Y consists of N points, then the deficiency indices of $\mathcal{L}_0$ are $\langle N, N\rangle$, and the deficiency indices of H_0, A_0, $\mathring{A}$ are $\langle 2N, 2N\rangle$, $\langle N, N\rangle$, and $\langle N, N\rangle$, respectively.

Let $d_k = |y_k - y_{k+1}|$, $k \in \mathbb{J}$,

$$\mathcal{L}_{0k} = i\frac{d}{dx},\ \operatorname{dom}(\mathcal{L}_{0k}) = \{f \in W_2^1([y_k, y_{k+1}]) : f(y_k) = f(y_{k+1}) = 0\}, \quad k \in \mathbb{J}.$$

The operator $\mathcal{L}_{0k}$ is symmetric with deficiency indices $(1,1)$ in the Hilbert space $L_2[y_k, y_{k+1}]$. Hence, $(\mathcal{L}_{0k}^2)^* = \mathcal{L}_{0k}^{*2}$ (see Theorem 2.2). Clearly,

$$\operatorname{dom}(\mathcal{L}_0) = \bigoplus_k \operatorname{dom}(\mathcal{L}_{0k}), \quad \mathcal{L}_0 = \bigoplus_k \mathcal{L}_{0k},$$
$$\operatorname{dom}(\mathcal{L}_0^*) = \bigoplus_k \operatorname{dom}(\mathcal{L}_{0k}^*), \quad \mathcal{L}_0^* = \bigoplus_k \mathcal{L}_{0k}^*.$$

Hence,

$$\ker(\mathcal{L}_{0k}^*) = \{f(x) = \text{const},\ x \in [y_k, y_{k+1}]\},$$

and

$$\ker(\mathcal{L}_0^*) = \bigoplus_k \ker(\mathcal{L}_{0k}^*).$$

Observe that

$$\begin{aligned} \operatorname{dom}(H_0) &= \bigoplus_k \operatorname{dom}(\mathcal{L}_{0k}^2), \quad H_0 = \mathcal{L}_0^2 = \bigoplus_k \mathcal{L}_{0k}^2, \\ \operatorname{dom}(H_0^*) &= \bigoplus_k \operatorname{dom}(\mathcal{L}_{0k}^{*2}), \quad H_0^* = \mathcal{L}_0^{*2} = \bigoplus_k \mathcal{L}_{0k}^{*2}. \end{aligned} \tag{4.5}$$

From Theorem 3.4 (and also from (4.1), (4.3) (1.3), (1.4), (1.5)) it follows that

$$A_0 = \mathcal{L}\mathcal{L}_0, \quad \mathring{A} = \mathcal{L}_0\mathcal{L}, \quad H_0 = \mathcal{L}_0^2, \quad A_0^* = \mathcal{L}_0^*\mathcal{L}, \quad \mathring{A}^* = \mathcal{L}\mathcal{L}_0^*, \quad H_0^* = \mathcal{L}_0^{*2}. \tag{4.6}$$

Denote by χ_k the characteristic function of the interval $[y_k, y_{k+1}]$. Then the functions

$$\left\{\frac{\chi_k}{\sqrt{d_k}}\right\}_{k\in\mathbb{J}}$$

form an orthonormal basis of $\ker(\mathcal{L}_0^*)$. Therefore,

$$\begin{aligned} P_{\ker(\mathcal{L}_0^*)}\mathcal{L}_0^* f &= \sum_k \frac{1}{d_k}\left(\int\limits_{y_k}^{y_{k+1}} if'(x)dx\right)\chi_k = \\ &= i\sum_k \frac{1}{d_k}(f(y_{k+1}-0) - f(y_k+0))\chi_k, \quad f \in \operatorname{dom}(\mathcal{L}_0^*), \end{aligned} \tag{4.7}$$

and

$$P_{\overline{\operatorname{ran}}(\mathcal{L}_0)}\mathcal{L}_0^* f = if' - i\sum_k \frac{1}{d_k}(f(y_{k+1}-0) - f(y_k+0))\chi_k, \quad f \in \operatorname{dom}(\mathcal{L}_0^*). \tag{4.8}$$

If $f \in W_2^1(\mathbb{R})$, then $f(y \pm 0) = f(y)$, $y \in Y$.

From (4.6) it follows that conditions (1.1) and (3.1) are fulfilled for the pairs $\langle \mathcal{L}_0, \mathcal{L}\rangle$, $\langle \mathcal{L}, \mathcal{L}_0^*\rangle$, and $\langle \mathcal{L}_0, \mathcal{L}_0^*\rangle$ and we can apply Theorem 3.1.

4.1. THE FRIEDRICHS AND KREĬN EXTENSIONS OF THE OPERATOR A_0

Let A_0 be given by (1.3). Then as has been mentioned above one has

$$A_0 = \mathcal{L}\mathcal{L}_0, \quad A_0^* = \mathcal{L}_0^*\mathcal{L},$$

where $\mathcal{L}_0$, $\mathcal{L}$, and $\mathcal{L}_0^*$ are given by (4.2), (4.3), and (4.4), respectively. Since $\mathcal{L}$ is a selfadjoint extension of $\mathcal{L}_0$ we can apply Theorem 3.1 by setting $L_1 = \mathcal{L}_0$, $L_2 = \mathcal{L}$. Hence, the Friedrichs extension A_{0F} is the operator

$$A_{0F} = \mathcal{L}_0^*\mathcal{L}_0,$$

i.e.,

$$\operatorname{dom}(A_{0F}) = \left\{f \in W_2^1(\mathbb{R}) : f' \in W_2^1(\mathbb{R}\setminus Y),\ f(y) = 0,\ y \in Y\right\}, \quad A_{0F}f = -\frac{d^2 f}{dx^2}.$$

From Theorem 3.1 and (4.8) we get

$$\begin{aligned}\operatorname{dom}(A_{0K}) &= \left\{f \in \operatorname{dom}(\mathcal{L}) : P_{\overline{\operatorname{ran}}(\mathcal{L}_0)}\mathcal{L}f \in \operatorname{dom}(\mathcal{L})\right\} = \\ &= \left\{f \in W_2^1(\mathbb{R}) : f' - \sum_k \frac{1}{d_k}(f(y_{k+1}) - f(y_k))\chi_k \in W_2^1(\mathbb{R})\right\},\end{aligned}$$

and

$$A_{0K}f = -\frac{d^2 f}{dx^2}, \quad f \in \operatorname{dom}(A_{0K}).$$

It follows that the boundary conditions for $f \in \operatorname{dom}(A_{0K})$ are

$$f'(y_k - 0) - \frac{1}{d_{k-1}}(f(y_k) - f(y_{k-1})) = f'(y_k + 0) - \frac{1}{d_k}(f(y_{k+1}) - f(y_k)), \quad k \in \mathbb{J},$$

or in equivalent form

$$f'(y_k + 0) - f'(y_k - 0) = \frac{1}{d_{k-1}}f(y_{k-1}) - \left(\frac{1}{d_{k-1}} + \frac{1}{d_k}\right)f(y_k) + \frac{1}{d_k}f(y_{k+1}), \quad k \in \mathbb{J}.$$

Additional conditions arise in the cases $\inf\{Y\} > -\infty$, $\sup\{Y\} < +\infty$. In particular, if Y is an infinite set, $-\infty < y_1 = \inf\{Y\}$, then

$$f'(y_1 - 0) - f'(y_1 + 0) = \frac{1}{d_1}(f(y_1) - f(y_2)),$$

and if $+\infty > y_{-1} = \sup\{Y\}$, then

$$f'(y_{-1} + 0) - f'(y_{-1} - 0) = \frac{1}{d_{-1}}(f(y_{-1}) - f(y_0)).$$

For a finite $Y = \{y_1, y_2, \dots, y_N\}$ we get

$$\begin{aligned} f'(y_1-0) - f'(y_1+0) &= \frac{1}{d_1}(f(y_1) - f(y_2)),\\ f'(y_N+0) - f'(y_N-0) &= \frac{1}{d_{N-1}}(f(y_N) - f(y_{N-1})),\\ f'(y_k+0) - f'(y_k-0) &= \frac{1}{d_{k-1}}f(y_{k-1}) - \left(\frac{1}{d_{k-1}} + \frac{1}{d_k}\right) f(y_k) +\\ &\quad + \frac{1}{d_k} f(y_{k+1}), k = 2, \dots, N-1. \end{aligned}$$

For $A_{0K}[f,g]$ we get $\mathcal{D}[A_{0K}] = W_2^1(\mathbb{R})$ and

$$A_{0K}[f,g] = \int\limits_{\mathbb{R}} f'(x)\overline{g'(x)}dx - \sum_k \frac{1}{d_k}(f(y_{k+1}) - f(y_k))\left(\overline{g(y_{k+1})} - \overline{g(y_k)}\right), \; f, g \in W_2^1(\mathbb{R}).$$

4.2. THE FRIEDRICHS AND KREĬN EXTENSIONS OF THE OPERATOR $\mathring{A}$

Now we consider the operator $\mathring{A}$ given by (1.4). Then $\mathring{A} = \mathcal{L}_0\mathcal{L}$, $\mathring{A}^* = \mathcal{L}\mathcal{L}_0^*$. Put $L_1 = \mathcal{L}$, $L_2 = \mathcal{L}_0^*$. Applying Theorem 3.1 we get that

$$\operatorname{dom}(\mathring{A}_F) = \operatorname{dom}(\mathcal{L}^2) = W_2^2(\mathbb{R}),\; \mathring{A}_F f = \mathcal{L}^2 f = -\frac{d^2 f}{dx^2},\; f \in W_2^2(\mathbb{R}).$$

Since $\ker(\mathcal{L}) = \{0\}$ we get

$$\mathring{A}_K = \mathcal{L}_0\mathcal{L}_0^*,$$

i.e.,

$$\operatorname{dom}(\mathring{A}_K) = \left\{f \in W_2^1(\mathbb{R}\setminus Y) : f' \in W_2^1(\mathbb{R}),\; f'(y) = 0,\; y \in Y\right\},\; \mathring{A}_K f = -\frac{d^2 f}{dx^2}.$$

In addition $\mathcal{D}[\mathring{A}_K] = W_2^1(\mathbb{R}\setminus Y)$ and

$$\mathring{A}_K[f,g] = \int\limits_{\mathbb{R}} f'(x)\overline{g'(x)}dx,\; f, g \in W_2^1(\mathbb{R}\setminus Y).$$

4.3. THE FRIEDRICHS AND KREĬN EXTENSIONS OF THE OPERATOR H_0

Let H_0 be given by (1.5), then $H_0 = \mathcal{L}_0^2$, $H_0^* = \mathcal{L}_0^{*2}$. Put $L_1 = \mathcal{L}_0$, $L_2 = \mathcal{L}_0^*$. Applying Theorem 3.1 we obtain the Friedrichs extension

$$\begin{aligned} \operatorname{dom}(H_{0F}) &= \{f \in \operatorname{dom}(\mathcal{L}_0) : \mathcal{L}_0 f \in \operatorname{dom}(\mathcal{L}_0^*)\} =\\ &= \{f \in W_2^1(\mathbb{R}),\; f' \in W_2^1(\mathbb{R}\setminus Y),\; f(y) = 0,\; y \in Y\}. \end{aligned}$$

Notice that $A_{0F} = H_{0F}$. For $\mathrm{dom}\,(H_{0K})$ we have

$$\mathrm{dom}\,(H_{0K}) = \{f \in \mathrm{dom}\,(\mathcal{L}_0^*) : P_{\overline{\mathrm{ran}}\,(\mathcal{L}_0)}\mathcal{L}_0^* f \in \mathrm{dom}\,(\mathcal{L}_0)\} =$$
$$= \Big\{ f \in W_2^1(\mathbb{R}\setminus Y) : g = f' - \sum_k \frac{1}{d_k}(f(y_{k+1}-0) - f(y_k+0))\chi_k \in W_2^1(\mathbb{R}),$$
$$g(y) = 0,\ y \in Y \Big\}.$$

The boundary conditions for $f \in \mathrm{dom}\,(H_{0K})$ we can write in the form:

$$f'(y_k+0) = \frac{1}{d_k}\left(f(y_{k+1}-0) - f(y_k+0)\right),$$
$$f'(y_k-0) = \frac{1}{d_{k-1}}\left(f(y_k-0) - f(y_{k-1}+0)\right) \quad \text{for all} \quad y_k \in Y,$$

and additionally

$$f'(y_{-1}+0) = 0 \quad \text{if} \quad +\infty > y_{-1} = \sup\{Y\},$$

or

$$f'(y_1-0) = 0 \quad \text{if} \quad -\infty < y_1 = \inf\{Y\},$$

and if $Y = \{y_1, \ldots, y_N\}$, then

$$f'(y_1-0) = 0,\ f'(y_N+0) = 0,$$
$$f'(y_k+0) = \frac{1}{d_k}\left(f(y_{k+1}-0) - f(y_k+0)\right), \qquad k = 1, \ldots, N-1,$$
$$f'(y_k-0) = \frac{1}{d_{k-1}}\left(f(y_k-0) - f(y_{k-1}+0)\right), \quad k = 2, \ldots, N.$$

Clearly, $\mathcal{D}[H_{0K}] = W_2^1(\mathbb{R}\setminus Y)$ and

$$H_{0K}[f,g] = \int_{\mathbb{R}} f'(x)\overline{g'(x)}dx -$$
$$- \sum_k \frac{1}{d_k}\left(f(y_{k+1}-0) - f(y_k+0)\right)\left(\overline{g(y_{k+1}-0)} - \overline{g(y_k+0)}\right),$$

$f, g \in W_2^1(\mathbb{R}\setminus Y)$.

Notice that due to (4.5) and according to [25, Corollary 5.5] we have

$$H_{0F} = \bigoplus_k \left(\mathcal{L}_{0k}^2\right)_F, \quad H_{0K} = \bigoplus_k \left(\mathcal{L}_{0k}^2\right)_K.$$

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